East Asian Math. J.
Vol. 35 (2019), No. 3, pp. 277–283
http://dx.doi.org/10.7858/eamj.2019.023

A REMARK ON NILPOTENTS

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Abstract. In this article we show that direct limits do not preserve the property that the Wedderburn radical contains all nilpotents, comparing the fact that the NI property is preserved by direct limits.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. A nilpotent element is also said to be a *nilpotent* for short. We use $N(R)$, $N^*(R)$, and $W(R)$ to denote the set of all nilpotents and the upper nilradical (i.e., the sum of all nil ideals), and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. $W(R) \subseteq N^*(R) \subseteq N(R)$ can be shown easily. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $T_n(R)$); and write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use e_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere.

1. Direct limits

A ring is usually called *reduced* if it has no nonzero nilpotents. Following Marks [2], a ring R is called *NI* if $N(R) = N^*(R)$. It is obvious that a ring R is NI if and only if $R/N^*(R)$ is reduced.

The proof of the following fact is simply stated in [1, Proposition 1.1]. But, for our purpose, we need a concrete one as is written in the following.

Proposition 1.1. *The direct limit preserves the NI property.*

Proof. Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of NI rings R_i for $i \in I$, and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$.

Received January 6, 2019; Accepted February 11, 2019.
2010 *Mathematics Subject Classification.* 16N40, 16S50.
Key words and phrases. NWR ring, NI ring, Wedderburn radical, nilpotent.
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(pISSN 1226-6973, eISSN 2287-2833)

We will prove that R is an NI ring. Take $x, y \in R$. Then $x = \iota_i(x_i)$, $y = \iota_j(y_j)$ for some $i, j \in I$; and moreover there is $k \in I$ such that $i \leq k$, $j \leq k$, and $x = \iota_k(\alpha_{ik}(x_i))$, $y = \iota_k(\alpha_{jk}(y_j))$.

Define

$$x + y = \iota_k(\alpha_{ik}(x_i) + \alpha_{jk}(y_j)) \text{ and } xy = \iota_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j)),$$

where $\alpha_{ik}(x_i)$ and $\alpha_{jk}(y_j)$ are in R_k. Then R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$.

Suppose $x, y \in N(R)$. Then $x^n = 0 = y^m$ for some positive integers m, n. There are $i, j, k \in I$ such that $x = \iota_i(x_i)$, $y = \iota_j(y_j)$, $i \leq k$, $j \leq k$. Moreover

$$\alpha_{ik}(x_i)^n = \alpha_{ik}(x_i^n) = 0 \text{ and } \alpha_{jk}(y_j^m) = \alpha_{jk}(y_j)^m = 0,$$

entailing $\alpha_{ik}(x_i), \alpha_{jk}(y_j) \in N(R_k)$. But R_k is NI and so $\alpha_{ik}(x_i) - \alpha_{jk}(y_j) \in N(R_k)$. Then $(\alpha_{ik}(x_i) - \alpha_{jk}(y_j))^t = 0$ for some positive integer t; hence we get

$$(x - y)^t = \iota_k(\alpha_{ik}(x_i) - \alpha_{jk}(y_j))^t = \iota_k((\alpha_{ik}(x_i) - \alpha_{jk}(y_j))^t) = \iota_k(0) = 0,$$

entailing $x - y \in N(R)$. Take $r \in R$. Then there exist $s, h \in I$ such that $r = \iota_s(r_s)$, $i \leq h$, $s \leq h$. Note $rx = \iota_h(r_h x_h)$ and $xr = \iota_h(x_h r_h)$ with $x_h = \alpha_{ih}(x_i)$ and $r_h = \alpha_{sh}(r_s)$. Since $x_h \in N(R_h)$ as above and R_h is NI, $(r_h x_h)^{t_1} = 0 = (x_h r_h)^{t_2}$ for some positive integers t_1, t_2. Then we obtain

$$(rx)^{t_1} = \iota_h(r_h x_h)^{t_1} = \iota_h((r_h x_h)^{t_1}) = \iota_h(0) = 0$$

and similarly $(xr)^{t_2} = 0$. Therefore rx, xr are contained in $N(R)$, proving that R is an NI ring. □

A ring R is usually called *directly finite* (or *Dedekind finite*) if $ab = 1$ for $a, b \in R$ implies $ba = 1$. NI rings are directly finite by [1, Proposition 2.7].

Proposition 1.2. *The direct limit preserves the directly finite property.*

Proof. Applying the proof of Proposition 1.1, let $D = \{R_i, \alpha_{ij}\}$ be a direct system of directly finite rings R_i for $i \in I$, and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$. We will use the addition and multiplication which are defined in the proof of Proposition 1.1.

We will prove that R is a directly finite ring. Let $xy = 1$ for $x, y \in R$. Then $x = \iota_i(x_i)$, $y = \iota_j(y_j)$ for some $i, j \in I$; and moreover there is $k \in I$ such that $i \leq k$, $j \leq k$, and $x = \iota_k(\alpha_{ik}(x_i))$, $y = \iota_k(\alpha_{jk}(y_j))$, where $\alpha_{ik}(x_i)$ and $\alpha_{jk}(y_j)$ are in R_k. So $xy = 1$ implies

$$1 = \iota_k(\alpha_{ik}(x_i))\iota_k(\alpha_{jk}(y_j)) = \iota_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j)),$$

entailing $\alpha_{ik}(x_i)\alpha_{jk}(y_j) = 1$.

Since R_k are directly finite, we get $\alpha_{jk}(y_j)\alpha_{ik}(x_i) = 1$ and $\iota_k(\alpha_{jk}(y_j)\alpha_{ik}(x_i)) = 1$. Whence we now have

$$\begin{aligned} 1 = xy &= \iota_k(\alpha_{ik}(x_i))\iota_k(\alpha_{jk}(y_j)) = \iota_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j)) = \iota_k(\alpha_{jk}(y_j)\alpha_{ik}(x_i)) \\ &= \iota_k(\alpha_{jk}(y_j))\iota_k(\alpha_{ik}(x_i)) = yx. \end{aligned}$$

Therefore R is a directly finite ring. □

2. NWR property

In this section we prove that the NWR property is not preserved by direct limits, comparing with the fact that the NI property is preserved by direct limits by Proposition 1.1.

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (and hence not a zero divisor).

Lemma 2.1. *Let S be a domain and $n \geq 2$. Then every matrix in $D_n(S)$ with nonzero diagonal is regular.*

Proof. Let $A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R$ with a nonzero. Assume $AB = 0$ for $B = \begin{pmatrix} b & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \in R$. Then clearly $b = 0$. Assume $B \neq 0$. Set s be largest such that the s-th row contains a nonzero entry, and t be largest such that $b_{st} \neq 0$ in the s-th row. Then the (s, t)-entry of AB is ab_{st} but this is nonzero because S is a domain, contrary to $AB = 0$. Thus $B = 0$.

Next assume that $BA = 0$ and $B \neq 0$. Then $b = 0$ clearly. Set i be smallest such that the i-th row contains a nonzero entry, and j be smallest such that $b_{ij} \neq 0$ in the i-th row. Then the (i, j)-entry of BA is $b_{ij}a$ but this is nonzero because S is a domain, contrary to $BA = 0$. Thus $B = 0$.

Summarizing, A is regular in R. □

A ring R shall be said to be *NWR* if $N(R) = W(R)$. It is obvious that a ring R is NWR if and only if $R/W(R)$ is reduced. NWR rings are clearly NI, but the converse need not hold by Proposition 1.1 and Theorem 2.2 to follow.

Let R be a ring and $a \in R$ with $a^n = 0$ for some $n \geq 1$. Here if n is the least with respect to $a^n = 0$ then n is called the *index (of nilpotency)* of a. Let S be a nil subset of R. If $S^k \neq 0$ for all $k \geq 1$ then S is said to be not of bounded index (of nilpotency).

Theorem 2.2. *The NWR property is not preserved by direct limits.*

Proof. Let D be a division ring of characteristic zero and $n \geq 1$. We apply the construction in [1, Example 1.2]. Set $R_n = D_{2^n}(D)$. Then

$$W(R_n) = N(R_n) = \{(a_{ij}) \in D_{2^n}(D) \mid a_{ii} = 0 \text{ for all } i\},$$

and so R_n is NWR (hence NI). Next define a map $\sigma : R_n \to R_{n+1}$ with $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and set R be the direct limit of R_n's. Then R is NI by Proposition 1. Note that R_n can be considered as a subring of R_{n+1} via σ, i.e., $A = \sigma(A)$ for $A \in R_n$. This provides us with $R = \cup_{n=1}^{\infty} R_n$.

Note first that every nilpotent matrix is of zero diagonal by Lemma 2.1. Take

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (e_{12} + e_{23} + e_{34}) \in R_2.$$

Then a has index 4 clearly. We will show $a \notin W(R)$ (i.e., R is not NWR).

Compute matrices in RaR. Notice that

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ in } R_3,$$

hence RaR contains

$$e_{12} + e_{23} + e_{34} + e_{56} + e_{67} + e_{78}.$$

Whence RaR also contains

$$b = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

by multiplying a by e_{ij}'s on the left and right. Then we have

$$b^2 = \left(\begin{array}{cccc|cccc} 0 & 0 & a_{13} & a_{14} & & & & \\ 0 & 0 & 0 & 1 & & (*) & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \hline & & & & 0 & 0 & 1 & 2 \\ & & O & & 0 & 0 & 0 & 1 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \end{array}\right) \text{ and } b^4 = \left(\begin{array}{c|c} O & (**) \\ \hline O & O \end{array}\right),$$

where O's are zero matrices. In this situation note that $(**) \neq 0$ since $a_{13} \neq 0$ and $a_{14} \neq 0$, and that the third and fourth rows of the matrix $(*)$ are both nonzero. It then follows that $b^8 = (b^4)^2 = 0$ and so the index of b is 8.

Write $a_2 = a$ (in R_3), $a_3 = b$, and proceed in this manner. Then we can obtain

$$a_3 \in Ra_2R,\ a_4 \in Ra_3R,\ \ldots,\ a_n \in Ra_{n-1}R, \ldots$$

such that

$$a_n = \left(\begin{array}{c|c} A & B \\ \hline O & A \end{array}\right) \in R_n$$

with

$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ \vdots & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & 1 \end{pmatrix} \in R_{n-1},$$

where $(2^2, 2^2+1)$-entry, $(2^3+2^2, 2^3+2^2+1)$-entry, $\ldots$, $(2^{n-2}+2^{n-3}+\cdots+2^2, 2^{n-2}+2^{n-3}+\cdots+2^2+1)$-entry in A are all zero. Note that the index of a_n is 2^n.

Next we get

$$a_{n+1} = \left(\begin{array}{c|c} a_n & C \\ \hline O & a_n \end{array}\right) \text{ with } C = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & 1 \end{pmatrix}$$

in RaR by the same method. Notice that

$$a_{n+1}^{2^n} = \left(\begin{array}{c|c} O & (***) \\ \hline O & O \end{array}\right) \text{ with } (***) \neq 0$$

since $(1, 2^n)$-entry and the 2^n-th row of $a_{n+1}^{2^{n-1}}$ are nonzero. So a_{n+1} has index 2^{n+1}, and consequently RaR is not of bouned index. Thus RaR is not contained in $W(R)$, that is, $a \notin W(R)$. Therefore R is not NWR. □

The following is an application of [1, Proposition 4.1] onto NWR rings.

Proposition 2.3. (1) *A ring R is NWR if and only if $T_n(R)$ is NWR for all $n \geq 2$.*

(2) *Let R, S be rings and ${}_RM_S$ be an (R, S)-bimodule.* $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ *is NWR if and only if R and S are both NWR.*

Proof. (1) Let $n \geq 2$. Note first

$$N(T_n(R)) = \{(a_{ij}) \in T_n(R) \mid a_{ii} \in N(R) \text{ for all } i = 1, \ldots, n\}.$$

So we get that N(R)=W(R) if and only if

$$N(T_n(R)) = \{(a_{ij}) \in T_n(R) \mid a_{ii} \in W(R) \text{ for all } i = 1, \ldots, n\}.$$

This implies that R is NWR if and only if $T_n(R)$ is NWR.

(2) can be proved similarly. □

However $Mat_n(R)$, for any ring R and $n \geq 2$, cannot be NWR because $N(Mat_(R))$ cannot form an ideal.

Acknowledgments. The authors thank the referee for very careful reading of the manuscript and many valuable suggestions that improved the paper by much.

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