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ORTHOGONAL TWO-DIRECTION REFINABLE FUNCTION OF ORDER 3[†]

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Abstract. In this paper we construct orthogonal two-direction scaling function of order 3 and corresponding wavelet function. In this paper we propose a different approach using orthogonal symmetric/antisymmetric multiwavelets of order 3. An example for constructing orthogonal two-direction scaling function of order 3 and corresponding wavelet function is given.

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1. Introduction

Two-direction refinable vector functions are of interest to be studied and are investigated in [2, 3, 4, 5, 6, 7].

A method for constructing orthogonal two-direction scaling function of order 2 and corresponding wavelet function was proposed by Yang in [5]. Another way of constructing orthogonal two-direction scaling function of order 2 and corresponding wavelet function was proposed by Kwon in [4]. To the best of our knowledge, examples for orthogonal two-direction scaling functions of order higher than 2 and corresponding wavelet functions have not been shown in the literature. Motivated by [4], we propose a method for constructing two-direction scaling function ϕ of order 3 and wavelet function ψ in this paper. In our method we employ orthogonal symmetric/antisymmetric multiscaling function $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ of order 3 and multiwavelet $\boldsymbol{\psi} = [\psi_1, \psi_2]^T$.

A standard (one-direction) scaling function of dilation factor 2 is a real-valued function ϕ which satisfies a recursion relation of the form

$$\phi(x) = \sum_{k\in\mathbb{Z}} p_k \, \phi(2x - k) \tag{1.1}$$

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and generates a multiresolution approximation (MRA) of $L^2(\mathbb{R})$. The recursion coefficients p_k are scalars. We assume that the summation in (1.1) is finite.

A *two-direction refinable function* of dilation factor 2 is a real-valued function $\phi(x)$ which satisfies a recursion relation

$$\phi(x) = \sum_{k\in\mathbb{Z}} \left[p_k^+\, \phi(2x-k) + p_k^-\, \phi(k-2x)\right] \tag{1.2}$$

and generates a multiresolution approximation of $L^2(\mathbb{R})$. The p_k^+, p_k^- are called positive- and negative-direction recursion coefficients for ϕ, respectively.

The *two-direction wavelet function* ψ associated with ϕ satisfy

$$\psi(x) = \sum_{k\in\mathbb{Z}} \left[q_k^+\, \phi(2x-k) + q_k^-\, \phi(k-2x)\right]. \tag{1.3}$$

The q_k^+, q_k^- are called positive- and negative-direction recursion coefficients for ψ, respectively. The two-direction scaling function and wavelet function together will be called a *two-direction wavelet*.

In this paper we only consider real recursion coefficients p_k^+, p_k^-, q_k^+, and q_k^- in $\mathbb{R}$ for $k \in \mathbb{Z}$.

This paper is organized as follows. A method for constructing two-direction scaling functions of order 3 and wavelet function from orthogonal symmetric/antisymmetric multiscaling functions of order 3 and multiwavelets is introduced in section 2. One example for illustrating the general theory in sections 1 and 2 is given in section 3.

2. Two-direction wavelets of order 3

2.1. Construction of two-direction scaling function of order 3. Orthogonal symmetric/antisymmetric multiscaling function $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ of order 3 supported on $[0, 3]$ is given as

$$\begin{aligned}\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\end{bmatrix} &= \begin{bmatrix}a_0 & b_0\\ c_0 & d_0\end{bmatrix}\begin{bmatrix}\phi_1(2x)\\ \phi_2(2x)\end{bmatrix} + \begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}\begin{bmatrix}\phi_1(2x-1)\\ \phi_2(2x-1)\end{bmatrix}\\ &\quad + \begin{bmatrix}a_1 & -b_1\\ -c_1 & d_1\end{bmatrix}\begin{bmatrix}\phi_1(2x-2)\\ \phi_2(2x-2)\end{bmatrix} + \begin{bmatrix}a_0 & -b_0\\ -c_0 & d_0\end{bmatrix}\begin{bmatrix}\phi_1(2x-3)\\ \phi_2(2x-3)\end{bmatrix}.\end{aligned} \tag{2.1}$$

where $a_0, a_1, b_0, b_1, c_0, c_1, d_0$, and d_1 are constants.

Construct a function ϕ by

$$\phi(x) = \frac{\sqrt{2}}{2}\left[\phi_1(x) - \phi_2(x)\right]. \tag{2.2}$$

Since $\phi_1(3-x) = \phi_1(x)$ and $\phi_2(3-x) = -\phi_2(x)$ by symmetric/antisymmetric property, we have

$$\phi(3-x) = \frac{\sqrt{2}}{2}\left[\phi_1(x) + \phi_2(x)\right]. \tag{2.3}$$

By solving (2.2) and (2.3) for ϕ_1 and ϕ_2, we have

$$\phi_1(x) = \frac{1}{\sqrt{2}}\left[\phi(x) + \phi(3-x)\right], \qquad \phi_2(x) = \frac{1}{\sqrt{2}}\left[\phi(3-x) - \phi(x)\right]. \tag{2.4}$$

Clearly, ϕ provides approximation order 3, since $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ provides approximation order 3. ϕ is supported on $[0,3]$, since ϕ_1 and ϕ_2 are supported on $[0,3]$. ϕ is refinable, since ϕ_1 and ϕ_2 are refinable.

Now we want to prove that ϕ is a two-direction refinable function of the form

$$\phi(x) = \sum_{k=0}^{3} p_k^+ \phi(2x-k) + \sum_{k=3}^{6} p_k^- \phi(k-2x), \tag{2.5}$$

for some p_k^+ and p_k^-.

By applying (2.2), we have

$$\begin{aligned}
\sqrt{2}\,\phi(x) &= \phi_1(x) - \phi_2(x) \\
&= (a_0 - c_0)\phi_1(2x) + (a_1 - c_1)\phi_1(2x-1) + (a_1 + c_1)\phi_1(2x-2) \\
&\quad + (a_0 + c_0)\phi_1(2x-3) + (b_1 - d_0)\phi_2(2x) + (b_0 - d_1)\phi_2(2x-1) \\
&\quad + (-b_1 - d_1)\phi_2(2x-2) + (-b_0 - d_0)\phi_2(2x-3).
\end{aligned}$$

By (2.4), we have

$$\begin{aligned}
2\,\phi(x) &= (a_0 - c_0)[\phi(2x) + \phi(3-2x)] + (a_1 - c_1)[\phi(2x-1) + \phi(4-2x)] \\
&\quad + (a_1 + c_1)[\phi(2x-2) + \phi(5-2x)] + (a_0 + c_0)[\phi(2x-3) + \phi(6-2x)] \\
&\quad + (b_0 - d_0)[\phi(3-2x) - \phi(2x)] + (b_1 - d_1)[\phi(4-2x) - \phi(2x-1)] \\
&\quad + (-b_1 - d_1)[\phi(5-2x) - \phi(2x-2)] + (-b_0 - d_0)[\phi(6-2x) - \phi(2x-3)] \\
&= (a_0 - b_0 - c_0 + d_0)\phi(2x) + (a_1 - b_1 - c_1 + d_1)\phi(2x-1) \\
&\quad + (a_1 + b_1 + c_1 + d_1)\phi(2x-2) + (a_0 + b_0 + c_0 + d_0)\phi(2x-3) \\
&\quad + (a_0 + b_0 - c_0 - d_0)\phi(3-2x) + (a_1 + b_1 - c_1 - d_1)\phi(4-2x) \\
&\quad + (a_1 - b_1 + c_1 - d_1)\phi(5-2x) + (a_0 - b_0 + c_0 - d_0)\phi(6-2x).
\end{aligned}$$

Hence, we have

$$\phi(x) = \sum_{k=0}^{3} p_k^+ \phi(2x-k) + \sum_{k=3}^{6} p_k^- \phi(k-2x), \tag{2.6}$$

where

$$\begin{aligned}
p_0^+ &= \frac{1}{2}(a_0 - b_0 - c_0 + d_0), \qquad & p_1^+ &= \frac{1}{2}(a_1 - b_1 - c_1 + d_1), \\
p_2^+ &= \frac{1}{2}(a_1 + b_1 + c_1 + d_1), \qquad & p_3^+ &= \frac{1}{2}(a_0 + b_0 + c_0 + d_0),
\end{aligned} \tag{2.7}$$

$$\begin{aligned} p_3^- &= \frac{1}{2}(a_0+b_0-c_0-d_0), \qquad & p_4^- &= \frac{1}{2}(a_1+b_1-c_1-d_1), \\ p_5^- &= \frac{1}{2}(a_1-b_1+c_1-d_1), \qquad & p_6^- &= \frac{1}{2}(a_0-b_0+c_0-d_0). \end{aligned} \tag{2.8}$$

Hence, ϕ is a two-direction scaling function of order 3 supported on $[0,3]$.

2.2. Construction of two-direction wavelet function associated with two-direction scaling function of order 3. Orthogonal symmetric/antisymmetric multiwavelet function $\boldsymbol{\psi} = [\psi_1, \psi_2]^T$ of order 3 supported on $[0,3]$ is given as

$$\begin{aligned} \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} &= \begin{bmatrix} a_0' & b_0' \\ c_0' & d_0' \end{bmatrix} \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \end{bmatrix} + \begin{bmatrix} a_1' & b_1' \\ c_1' & d_1' \end{bmatrix} \begin{bmatrix} \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \\ &\quad + \begin{bmatrix} a_1' & -b_1' \\ -c_1' & d_1' \end{bmatrix} \begin{bmatrix} \phi_1(2x-2) \\ \phi_2(2x-2) \end{bmatrix} + \begin{bmatrix} a_0' & -b_0' \\ -c_0' & d_0' \end{bmatrix} \begin{bmatrix} \phi_1(2x-3) \\ \phi_2(2x-3) \end{bmatrix}, \end{aligned} \tag{2.9}$$

or

$$\begin{aligned} \psi_1(x) &= a_0'\phi_1(2x) + a_1'\phi_1(2x-1) + a_1'\phi_1(2x-2) + a_0'\phi_1(2x-3) \\ &\quad + b_0'\phi_2(2x) + b_1'\phi_2(2x-1) - b_1'\phi_2(2x-2) - b_0'\phi_2(2x-3), \\ \psi_2(x) &= c_0'\phi_1(2x) + c_1'\phi_1(2x-1) - c_1'\phi_1(2x-2) - c_0'\phi_1(2x-3) \\ &\quad + d_0'\phi_2(2x) + d_1'\phi_2(2x-1) + d_1'\phi_2(2x-2) + d_0'\phi_2(2x-3), \end{aligned} \tag{2.10}$$

where $a_0', a_1', b_0', b_1', c_0', c_1', d_0'$, and d_1' are constants.

Construct a function ψ by

$$\psi(x) = \frac{\sqrt{2}}{2}\left[-\psi_1(-x) + \psi_2(-x)\right]. \tag{2.11}$$

Since $\psi_1(3-x) = \psi_1(x)$ and $\psi_2(3-x) = -\psi_2(x)$ by symmetric/antisymmetric property, we have

$$\psi(x-3) = -\frac{\sqrt{2}}{2}\left[\psi_1(x) + \psi_2(x)\right]. \tag{2.12}$$

By solving (2.11) and (2.12) for ψ_1 and ψ_2, we have

$$\psi_1(x) = -\frac{1}{\sqrt{2}}\left[\psi(-x) + \psi(x-3)\right], \qquad \psi_2(x) = -\frac{1}{\sqrt{2}}\left[\psi(x-3) - \psi(-x)\right]. \tag{2.13}$$

Clearly, ψ is supported on $[-3,0]$, since ψ_1 and ψ_2 are supported on $[0,3]$. ψ is refinable, since ψ_1 and ψ_2 are refinable.

Now we want to prove that ψ is a two-direction wavelet function associated with scaling function ϕ of order 3 of the form

$$\psi(x) = \sum_{k=-6}^{-3} q_k^+ \phi(2x-k) + \sum_{k=-3}^{0} q_k^- \phi(k-2x), \tag{2.14}$$

for some q_k^+ and q_k^-.

By applying (2.11) and (2.10), we have

$$\begin{aligned}-\sqrt{2}\,\psi(-x) \quad &= \quad (a_0'-c_0')\phi_1(2x)+(a_1'-c_1')\phi_1(2x-1)+(a_1'+c_1')\phi_1(2x-2)\\ &+(a_0'+c_0')\phi_1(2x-3)+(b_0'-d_0')\phi_2(2x)+(b_1'-d_1')\phi_2(2x-1)\\ &+(-b_1'-d_1')\phi_2(2x-2)+(-b_0'-d_0')\phi_2(2x-3).\end{aligned}$$

By (2.4), we have

$$\begin{aligned}-2\,\psi(-x) \quad &= \quad (a_0'-b_0'-c_0'+d_0')\phi(2x)+(a_1'-b_1'-c_1'+d_1')\phi(2x-1)\\ &+(a_1'+b_1'+c_1'+d_1')\phi(2x-2)+(a_0'+b_0'+c_0'+d_0')\phi(2x-3)\\ &+(a_0'+b_0'-c_0'-d_0')\phi(3-2x)+(a_1'+b_1'-c_1'-d_1')\phi(4-2x)\\ &+(a_1'-b_1'+c_1'-d_1')\phi(5-2x)+(a_0'-b_0'+c_0'-d_0')\phi(6-2x).\end{aligned}$$

That is,

$$\begin{aligned}-2\,\psi(x) \quad &= \quad (a_0'-b_0'-c_0'+d_0')\phi(-2x)+(a_1'-b_1'-c_1'+d_1')\phi(-2x-1)\\ &+(a_1'+b_1'+c_1'+d_1')\phi(-2x-2)+(a_0'+b_0'+c_0'+d_0')\phi(-2x-3)\\ &+(a_0'+b_0'-c_0'-d_0')\phi(3+2x)+(a_1'+b_1'-c_1'-d_1')\phi(4+2x)\\ &+(a_1'-b_1'+c_1'-d_1')\phi(5+2x)+(a_0'-b_0'+c_0'-d_0')\phi(6+2x).\end{aligned}$$

Hence, we have

$$\psi(x)=\sum_{k=-6}^{-3} q_k^+\phi(2x-k)+\sum_{k=-3}^{0} q_k^-\phi(k-2x), \tag{2.15}$$

where

$$\begin{aligned} q_{-6}^+ &= -\frac{1}{2}(a_0'-b_0'+c_0'-d_0'), \qquad q_{-5}^+ = -\frac{1}{2}(a_1'-b_1'+c_1'-d_1'),\\ q_{-4}^+ &= -\frac{1}{2}(a_1'+b_1'-c_1'-d_1'), \qquad q_{-3}^+ = -\frac{1}{2}(a_0'+b_0'-c_0'-d_0'),\end{aligned} \tag{2.16}$$

$$\begin{aligned} q_{-3}^- &= -\frac{1}{2}(a_0'+b_0'+c_0'+d_0'), \qquad q_{-2}^- = -\frac{1}{2}(a_1'+b_1'+c_1'+d_1'),\\ q_{-1}^- &= -\frac{1}{2}(a_1'-b_1'-c_1'+d_1'), \qquad q_0^- = -\frac{1}{2}(a_0'-b_0'-c_0'+d_0').\end{aligned} \tag{2.17}$$

Hence, ψ is a two-direction wavelet function associated with ϕ supported on $[-3,0]$.

2.3. Main Theorem. We have the following main Theorem of this paper.

Theorem 2.1. Let $\boldsymbol{\phi}=[\phi_1,\phi_2]^T$ be an orthogonal symmetric/antisymmetric multiscaling function of order 3 supported on $[0,3]$ with nonzero 2×2 recursion coefficient matrices h_0,h_1,h_2,h_3. Let $\boldsymbol{\psi}=[\psi_1,\psi_2]^T$ be an orthogonal symmetric/antisymmetric multiwavelet function associated with $\boldsymbol{\phi}$ supported on $[0,3]$

with nonzero 2×2 recursion coefficient matrices g_0, g_1, g_2, g_3. Construct functions ϕ and ψ by

$$\begin{aligned} \phi(x) &= \frac{\sqrt{2}}{2}\left[\phi_1(x) - \phi_2(x)\right], \\ \psi(x) &= \frac{\sqrt{2}}{2}\left[-\psi_1(-x) + \psi_2(-x)\right]. \end{aligned} \tag{2.18}$$

Then (i) ϕ is an orthogonal two-direction scaling function of order 3 supported on $[0,3]$ such that

$$\phi(x) = \sum_{k=0}^{3} p_k^+ \phi(2x-k) + \sum_{k=3}^{6} p_k^- \phi(k-2x) \tag{2.19}$$

for some p_k^+ and p_k^-;

(ii) ψ is an orthogonal two-direction wavelet function associated with ϕ supported on $[-3,0]$ such that

$$\psi(x) = \sum_{k=-6}^{-3} q_k^+ \phi(2x-k) + \sum_{k=-3}^{0} q_k^- \phi(k-2x) \tag{2.20}$$

for some q_k^+ and q_k^-.

Furthermore, $\phi(3-x) = \frac{\sqrt{2}}{2}[\phi_1(x)+\phi_2(x)]$, the flipping of $\phi(x)$ about $x = 3/2$, is also a two-direction scaling function of order 3 supported on $[0,3]$. $\psi(-3-x)$, the flipping of $\psi(x)$ about $x = -3/2$, is also a two-direction wavelet function associated with $\phi(3-x)$ supported on $[-3,0]$.

3. Example

In this section we provide an example to illustrate the general theory. For an example, we take Chui and Lian's orthogonal symmetric/antisymmetric multiscaling functions $\boldsymbol{\phi}$ of order 3 and multiwavelets $\boldsymbol{\psi}$ in [1]. We obtain two-direction scaling function ϕ of order 3 supported on $[0,3]$ and wavelet function ψ

Example 3.1. Chui-Lian's orthogonal symmetric/antisymmetric multiscaling function $\boldsymbol{\phi} = [\phi_1, \phi_2]^T$ of order 3 supported on $[0,3]$ is given in [1] as

$$\begin{aligned} \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \end{bmatrix} &= \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \begin{bmatrix} \phi_1(2x) \\ \phi_2(2x) \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} \phi_1(2x-1) \\ \phi_2(2x-1) \end{bmatrix} \\ &+ \begin{bmatrix} a_1 & -b_1 \\ -c_1 & d_1 \end{bmatrix} \begin{bmatrix} \phi_1(2x-2) \\ \phi_2(2x-3) \end{bmatrix} + \begin{bmatrix} a_0 & -b_0 \\ -c_0 & d_0 \end{bmatrix} \begin{bmatrix} \phi_1(2x-3) \\ \phi_2(2x-3) \end{bmatrix}, \end{aligned} \tag{3.1}$$

where

$$
\begin{aligned}
a_0 &= \frac{1}{40}(10 - 3\sqrt{10}), \quad a_1 = \frac{1}{40}(30 + 3\sqrt{10}),\\
b_0 &= \frac{1}{40}(5\sqrt{6} - 2\sqrt{5}), \quad b_1 = \frac{1}{40}(5\sqrt{6} - 2\sqrt{5}),\\
c_0 &= \frac{1}{40}(5\sqrt{6} - 3\sqrt{15}), \quad c_1 = \frac{1}{40}(-5\sqrt{6} - 7\sqrt{15}),\\
d_0 &= \frac{1}{40}(5 - 3\sqrt{10}), \quad d_1 = \frac{1}{40}(15 - 3\sqrt{10}).
\end{aligned}
\tag{3.2}
$$

Construct a refinable function ϕ by

$$\phi(x) = \frac{\sqrt{2}}{2}\left[\phi_1(x) - \phi_2(x)\right]. \tag{3.3}$$

By applying (2.6), (2.7), and (2.8), we have

$$\phi(x) = \sum_{k=0}^{3} p_k^+ \phi(2x - k) + \sum_{k=3}^{6} p_k^- \phi(k - 2x), \tag{3.4}$$

where

$$
\begin{aligned}
p_0^+ &= \frac{1}{2}(a_0 - c_0 - b_0 + d_0) = \frac{1}{2}\left(\frac{3}{8} - \frac{3\sqrt{10}}{20} - \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{8}\right) \approx -0.1138,\\
p_1^+ &= \frac{1}{2}(a_1 - c_1 - b_0 + d_1) = \frac{1}{2}\left(\frac{9}{8} + \frac{9\sqrt{15}}{40}\right) \approx 0.9982,\\
p_2^+ &= \frac{1}{2}(a_1 + c_1 + b_0 + d_1) = \frac{1}{2}\left(\frac{9}{8} - \frac{9\sqrt{15}}{40}\right) \approx 0.1268,\\
p_3^+ &= \frac{1}{2}(a_0 + c_0 + b_0 + d_0) = \frac{1}{2}\left(\frac{3}{8} - \frac{3\sqrt{10}}{20} + \frac{\sqrt{6}}{4} - \frac{\sqrt{15}}{8}\right) \approx 0.0145,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
p_3^- &= \frac{1}{2}(a_0 - c_0 + b_0 - d_0) = \frac{1}{2}\left(\frac{1}{8} + \frac{\sqrt{15}}{40}\right) \approx 0.1109,\\
p_4^- &= \frac{1}{2}(a_1 - c_1 + b_0 - d_1) = \frac{1}{2}\left(\frac{3}{8} + \frac{3\sqrt{10}}{20} + \frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{8}\right) \approx 0.9729,\\
p_5^- &= \frac{1}{2}(a_1 + c_1 - b_0 - d_1) = \frac{1}{2}\left(\frac{3}{8} + \frac{3\sqrt{10}}{20} - \frac{\sqrt{6}}{4} - \frac{\sqrt{15}}{8}\right) \approx -0.1236,\\
p_6^- &= \frac{1}{2}(a_0 + c_0 - b_0 - d_0) = \frac{1}{2}\left(\frac{1}{8} - \frac{\sqrt{15}}{40}\right) \approx 0.0141,
\end{aligned}
\tag{3.6}
$$

and

$$c=\frac{p_3^-}{p_0^+}=-\frac{p_4^-}{p_1^+}=\frac{p_5^-}{p_2^+}=-\frac{p_6^-}{p_3^+}=\frac{1}{48}\frac{\sqrt{(480-48\sqrt{10})(160+48\sqrt{10})}}{\sqrt{10}-10}\approx -0.9747. \tag{3.7}$$

$\phi(3-x)$, flipping of $\phi(x)$ about $x=3/2$, is also a two-direction scaling function of order 3 supported on $[0,3]$.

Chui-Lian's [1] orthogonal symmetric/antisymmetric multiwavelet function $\boldsymbol{\psi}=[\psi_1,\psi_2]^T$ supported on $[0,3]$ is given as

$$\begin{aligned}\begin{bmatrix}\psi_1(x)\\ \psi_2(x)\end{bmatrix}&=\begin{bmatrix}a_0' & b_0'\\ c_0' & d_0'\end{bmatrix}\begin{bmatrix}\phi_1(2x)\\ \phi_2(2x)\end{bmatrix}+\begin{bmatrix}a_1' & b_1'\\ c_1' & d_1'\end{bmatrix}\begin{bmatrix}\phi_1(2x-1)\\ \phi_2(2x-1)\end{bmatrix}\\ &\quad+\begin{bmatrix}a_1' & -b_1'\\ -c_1' & d_1'\end{bmatrix}\begin{bmatrix}\phi_1(2x-2)\\ \phi_2(2x-3)\end{bmatrix}+\begin{bmatrix}a_0' & -b_0'\\ -c_0' & d_0'\end{bmatrix}\begin{bmatrix}\phi_1(2x-3)\\ \phi_2(2x-3)\end{bmatrix},\end{aligned} \tag{3.8}$$

associated with $\phi(x)$, where

$$\begin{aligned} a_0'&=\frac{1}{40}(5\sqrt{6}-2\sqrt{15}), & a_1'&=\frac{1}{40}(-5\sqrt{6}+2\sqrt{15}),\\ b_0'&=\frac{1}{40}(-10+3\sqrt{10}), & b_1'&=\frac{1}{40}(30+3\sqrt{10}),\\ c_0'&=\frac{1}{40}(-5+3\sqrt{10}), & c_1'&=\frac{1}{40}(15-3\sqrt{10}),\\ d_0'&=\frac{1}{40}(5\sqrt{6}-3\sqrt{15}), & d_1'&=\frac{1}{40}(5\sqrt{6}+7\sqrt{15}).\end{aligned} \tag{3.9}$$

Construct a function ψ by

$$\psi(x)=\frac{\sqrt{2}}{2}\left[-\psi_1(-x)+\psi_2(-x)\right]. \tag{3.10}$$

By applying (2.15), (2.16), and (2.17), we have

$$\psi(x)=\sum_{k=-6}^{-3}q_k^+\phi(2x-k)+\sum_{k=-3}^{0}q_k^-\phi(k-2x), \tag{3.11}$$

associated with ϕ, where

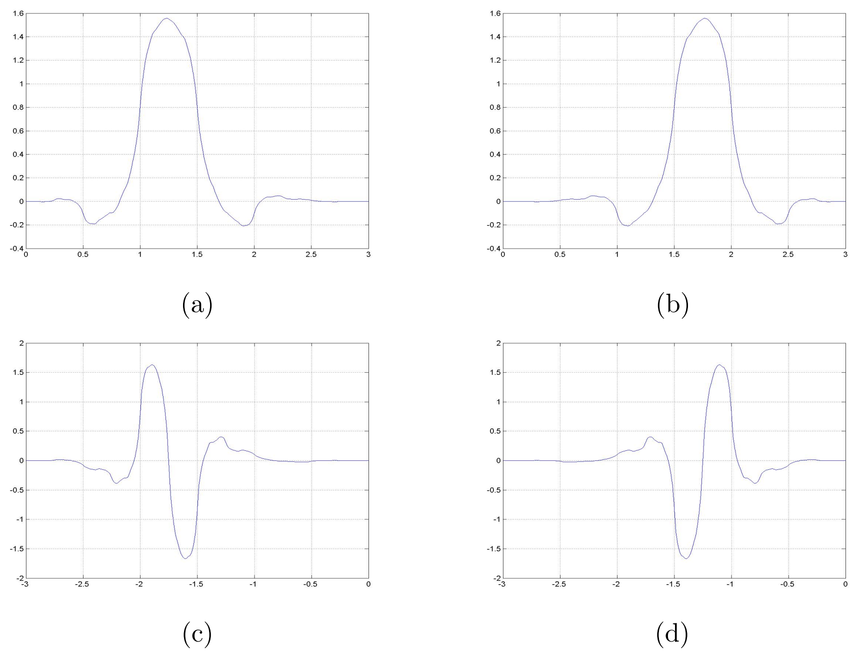
$$q_{-6}^+=-p_3^-,\quad q_{-5}^+=p_4^-,\quad q_{-4}^+=-p_5^-,\quad q_{-3}^+=p_6^-, \tag{3.12}$$

$$q_{-3}^-=p_0^+,\quad q_{-2}^-=-p_1^+,\quad q_{-1}^-=p_2^+,\quad q_0^-=-p_3^+. \tag{3.13}$$

$\psi(-3-x)$, flipping of $\psi(x)$ about $x=-3/2$, is also a two-direction wavelet function associated with $\phi(3-x)$ supported on $[-3,0]$.

For the graphs of $\phi(x)$, $\phi(3-x)$, $\psi(x)$, and $\psi(-3-x)$, see Fig. 3.1.

□



(a) (b) (c) (d)

Figure 3.1. Orthogonal two-direction scaling functions of order 3 and wavelet functions from CL3: (a) $\phi(x)$. (b) $\phi(3-x)$. (c) $\psi(x)$. (d) $\psi(-3-x)$.

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