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ON THE COMPLETE CONVERGENCE FOR WEIGHTED SUMS OF NEGATIVELY SUPERADDITIVE DEPENDENT RANDOM VARIABLES†

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Abstract. We are presented of several basic properties for negatively superadditive dependent(NSD) random variables. By using this concept we are obtained complete convergence for maximum partial sums of rowwise NSD random variables. These results obtained in this paper generalize a corresponding ones for independent random variables and negatively associated random variables.

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1. Introduction

Alam and Saxena(1) introduced the concepts of negatively associated property and Joag-Dev and Proschan(6), Block et al.(2), Seo and Baek(3) have carefully studied a number of well known multivariate distributions posesses the NA property.

Definition 1.1. (1) A finite family $\{X_i : 1 \leq i \leq n\}$ is said to be negatively associated (NA) if, for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, 3, \cdots, n\}$,

$$\mathrm{Cov}(f(X_i, i \in A_1), g(X_j, j \in A_2)) \leq 0,$$

whenever f and g are coordinatewise nondecreasing functions such that this covariance exists. An infinite sequence $\{X_n : n \geq 1\}$is NA if every finite subcollection is NA.

Hu(7) was introduced the concept of negatively superadditive dependent(NSD) random variables which is based on the class of superadditive functions

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Definition 1.2. (7) A random vector $(X_1, X_2, \cdots, X_n)$is said to be NSD if

$$E\phi(X_1, X_2, ..., X_n) \le E\phi(X_1^*, X_2^*, ..., X_n^*)$$

where ${X_1}^*, X_2^*, ..., X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in the above equation exist.

Definition 1.3. (7) A sequence $\{X_n : n \ge 1\}$ of random variables is said to be NSD if for all $n \ge 1, (X_1, X_2, ..., X_n)$ is NSD.

Hu gave an example illustrating that NSD does not imply NA and Christofides and Vaggelatou(4) indicated that negatively association implies negatively superadditive dependence. Negatively superadditive dependence structure is an extension of negatively associated structure and sometimes more useful than negatively associated structure(see Joag-Dev and Proschan(6)).
Since concepts of NSD is weaker than independent and NA, the studying of the limit behavior of the NSD random variable is of interest. The main purpose of this paper is to study the complete convergence for weighted sums of NSD random variables.

The following concept of stochastic domination will be used in this paper.

Definition 1.4. A sequence $\{X_n : n \ge 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \le CP(|X| > x)$$

for all $x \ge 0$ and $n \ge 1$.

Finally, in section 2 we study some preliminary results for NSD random variables and the main results of this paper is discuss complete convergence for maximum partial sums of rowwise NSD random variables in section 3.

2. Preliminaries

Throughout this paper, $a = O(b)$ means $a \le Cb$ and C will represent positive constants which their value may change from one place to another. For $x \ge 0$ the symbol $[x]$ denotes the greatest integer in x. In this section, we will introduce some important lemmas which will be need to prove our main results of this paper.

Lemma 2.1. *(7) (a) Let $(X_1, X_2, X_3, \cdots, X_n)$ be an NSD random vector, then $(-X_1, -X_2, -X_3, \cdots, -X_n)$ is NSD.*
(b) Let $(X_1, X_2, X_3, \cdots, X_n)$ be an NSD random vector and $f_1, f_2, \cdots, f_n$ are non-decreasing functions, then $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ are NSD.

Lemma 2.2. *(7) Let $(X_1, X_2, X_3, \cdots, X_n)$and $(Z_1, Z_2, Z_3, \cdots, Z_n)$ be independent random vectors. If $(X_1, X_2, X_3, \cdots, X_n)$ and $(Z_1, Z_2, Z_3, \cdots, Z_n)$ are both NSD, then for any $\alpha \in R$, $(X_1 + \alpha Z_1, X_2 + \alpha Z_2, \cdots, X_n + \alpha Z_n)$ is NSD.*

Lemma 2.3. *Let $(X_1, X_2, X_3, \cdots, X_n)$ be an NSD random vector. Then for each $n \geq 1$ and $t > 0$,*

$$Ee^{\sum tX_i} \leq \prod_{i=1}^{n} Ee^{tX_i}.$$

Proof. By Definition 1.2, Lemma 2.1(b) and Lemma 2.2, we obtain that

$$\begin{aligned} Ee^{\sum tX_i} &\leq E\Big(e^{\sum tX_i{}^*}\Big) \\ &= Ee^{tX_1}Ee^{tX_2}\cdots Ee^{tX_n} \\ &= \prod_{i=1}^{n} Ee^{tX_i}. \end{aligned}$$

□

Lemma 2.4. *Let $(X_1, X_2, X_3, \cdots, X_n)$ be an NSD random vector with mean zero and $0 < C_n = \sum_{l=1}^{n} EX_l^2 < \infty$. Then*

$$P\Big(S_n \geq x\Big) \leq \sum_{l=1}^{n} P\Big(|X_l| \geq y\Big) + 2e^{\left(\frac{x}{y} - \frac{x}{y} log(1 + \frac{xy}{C_n})\right)} \tag{2.1}$$

where $S_n = \sum_{l=1}^{n} X_l$.

Proof. The proof is similar to that of Theorem 2 in Fuk and Nagaev(5). Let $Y_i = X_i I\Big(X_i \leq y\Big) + yI\Big(X_i > y\Big)$ and $Z_n = \sum_{i=1}^{n} Y_i$ and note that $Y_i \leq X_i, EY_i \leq 0$ and $EY_i^2 \leq EX_i^2$. By Lemma 2.1, for $t > 0, e^{tY_1}, e^{tY_2}, \cdots, e^{tY_n}$ are NSD. Thus, by Lemma 2.3,

$$Ee^{tZ_n} = Ee^{t\sum Y_i} \leq \prod_{i=1}^{n} Ee^{tY_i} \tag{2.2}$$

Let $F_i(x) = P(X_i \leq x_i)$. Then, we obtain that for $t > 0$.

$$\begin{aligned} Ee^{tY_i} &= \int_{-\infty}^{y} e^{tx} dF_i(x) + e^{ty} P\Big(X_i \geq x\Big) \\ &= 1 + tEY_i + \int_{-\infty}^{y} \Big(e^{tx} - 1 - tx\Big) dF_i(x) + \Big(e^{ty} - 1 - ty\Big) P\Big(X_i \geq y\Big) \\ &\leq 1 + \int_{-\infty}^{y} \Big(e^{tx} - 1 - tx) dF_i(x) + \Big(e^{ty} - 1 - ty\Big) P\Big(X_i \geq y\Big). \quad (2.3) \end{aligned}$$

Since $f(x) = \frac{e^{tx} - 1 - tx}{x^2}$ is increasing function for all $x, t \geq 0$ and $1 + x \leq e^x$ for all real number x, it follows from (2.3) that

$$e^{tY_i} \leq \frac{1 + e^{ty - 1 - ty}}{y^2\Big(\int_{-\infty}^{y} x^2 dF_i(x) + y^2 P(X_i \geq y)\Big)}$$

$$\leq \quad \frac{1+(e^{ty-1-ty})EX_i^2}{y^2}$$
$$\leq \quad e^{\left(\frac{(e^{ty}-1-ty)EX_i^2}{y^2}\right)} \tag{2.4}$$

Thus, by (2.2) and (2.4) we obtain that for all $x > 0$ and $t > 0$

$$e^{(-tx)}Ee^{tZ_n} \leq e^{\left(-tx+EX_i^2\frac{e^{ty}-1-ty}{y^2}\right)}$$

Taking $t = \log(1+xy/C_n)/y$, we obtain that

$$\begin{aligned} e^{-tx}Ee^{tZ_n} &\leq e^{\left(\frac{x}{y}-\frac{x}{y}\log(1+\frac{xy}{C_n})-\frac{C_n}{y^2}\log(1+\frac{xy}{C_n}))\right)} \\ &\leq e^{\left(\frac{x}{y}-\frac{x}{y}\log(1+\frac{xy}{C_n})\right)} \end{aligned}$$

Clearly, the events $\{S_n \geq x\} \subset \{Z_n \neq S_n\} \cup \{Z_n \geq x\}$. Then we obtain that

$$\begin{aligned} P(S_n \geq x) &\leq P(S_n \neq Z_n) + P(Z_n \geq x) \\ &\leq \sum_{l=1}^{n} P(X_l \geq y) + e^{-tx}Ee^{tZ_n} \\ &\leq \sum_{l=1}^{n} P(X_l \geq y) + e^{\left(\frac{x}{y}-\frac{x}{y}\log(1+\frac{xy}{C_n})\right)}. \end{aligned} \tag{2.5}$$

Similarly, since $\{-X_n|n \geq 1\}$ is NSD by Lemma 2.1(a) We obtain that

$$P(-S_n \geq x) \quad \leq \quad \sum_{l=1}^{n} P(-X_l \geq y) + e^{\left(\frac{x}{y}-\frac{x}{y}\log(1+\frac{xy}{C_n})\right)} \tag{2.6}$$

Thus, from (2.5) and (2.6) we obtain that

$$\begin{aligned} P(|S_n| \geq x) &\leq P(S_n \geq x) + P(-S_n \geq x) \\ &\leq \sum_{l=1}^{n} P(|X_l| \geq y) + 2e^{\left(\frac{x}{y}-\frac{x}{y}\log(1+\frac{xy}{C_n})\right)}. \end{aligned}$$

□

3. Main results and Proofs

Theorem 3.1. *Suppose that $\{X_{ni}|1 \leq i \leq k_n,\ n \geq 1\}$ be an array of mean zero rowwise NSD random variables which is stochastically dominated by a random variable X such that $EX^2 < \infty$ and let $\{a_n|\ n \geq 1\}$ be a sequence of positive real numbers with $a_n \uparrow \infty$. If $\sum_{n=1}^{\infty} k_n a_n^{-2} = O\left(n^{1-2p}\right) < \infty$ for some*

$p \geq \frac{1}{2}$ and $\{k_n|\ n \geq 1\}$ is a nondecreasing sequence of integer numbers , then $\sum_{n=1}^{\infty} \frac{1}{a_n} E\bigg(\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| - \varepsilon a_n\bigg)^+ < \infty$ for all $\varepsilon > 0$.

Proof. For all $\varepsilon > 0$ and for any $x \geq 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{a_n} E\bigg(\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| - \varepsilon a_n\bigg)^+ \\
&= \sum_{n=1}^{\infty} \frac{1}{a_n} \int_0^{\infty} P\bigg(\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni} - \varepsilon a_n\bigg| > x\bigg) dx \\
&\leq \sum_{n=1}^{\infty} \frac{1}{a_n} \int_0^{a_n} P\bigg(\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| > \varepsilon a_n + x\bigg) dx \\
&\ + \sum_{n=1}^{\infty} \frac{1}{a_n} \int_{a_n}^{\infty} P\bigg(\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| > \varepsilon a_n + x\bigg) dx \\
&\leq \sum_{n=1}^{\infty} P\bigg(\frac{1}{a_n}\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| > \varepsilon\bigg) + \sum_{n=1}^{\infty} \frac{1}{a_n} \int_{a_n}^{\infty} P\bigg(\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| > x\bigg) dx \\
&\doteq I + II(say).
\end{aligned}$$

Now we first need to prove that $I < \infty$. For any $\varepsilon > 0, p \geq \frac{1}{2}$, and enough large n, we obtain that

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} P\bigg(\frac{1}{a_n}\max_{1\leq k_n \leq n}\bigg|\sum_{i=1}^{k_n} X_{ni}\bigg| > \varepsilon\bigg) \\
&\leq C\sum_{n=1}^{\infty} \frac{1}{a_n} \sum_{i=1}^{k_n} E\bigg|X_{ni}\bigg| I\bigg(|X_{ni}| > a_n\bigg) \\
&\leq C\sum_{n=1}^{\infty} \frac{1}{a_n} k_n E|X| I\bigg(|X| > a_n\bigg) \\
&\leq CO\bigg(n^{1-2p}\bigg) E|X|^2 < \infty.
\end{aligned}$$

Next we only need to prove that $II < \infty$. To prove that $II < \infty$, for all $1 \leq i \leq k_n, n \geq 1$ and $y \geq 0$, we define as follows. Let $Y_{ni} = -yI(X_{ni} < -y) + X_{ni}I(|X_{ni}| \leq y) + yI(X_{ni} > y)$, $Z_{ni} = X_{ni} - Y_{ni} = (X_{ni} + y)I(X_{ni} < -y) + (X_{ni} - y)I(X_{ni} > y)$.

Then, for II,

$$\begin{aligned}
II &\leq \sum_{n=1}^{\infty} \frac{1}{a_n} \sum_{i=1}^{k_n} \int_{a_n}^{\infty} P\bigg(|X_{ni}| > y\bigg) dy + \sum_{n=1}^{\infty} \frac{1}{a_n} \int_{a_n}^{\infty} P\bigg(\max_{1\leq k_n \leq n} |\sum_{i=1}^{k_n} Y_{ni}| > y\bigg) dy \\
&\doteq I_1 + I_2(say)
\end{aligned}$$

To prove that $I_1 < \infty$, for any $y \geq 0, p \geq \frac{1}{2}$, by Definition 1.4 and above conditions, we obtain that

$$
\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} \frac{1}{a_n} \sum_{i=1}^{k_n} \int_{a_n}^{\infty} P(|X_{ni}| > y) dy \\
&\leq \sum_{n=1}^{\infty} \frac{1}{a_n} \sum_{i=1}^{k_n} P\biggl(|X_{ni}| I\Bigl(|X_{ni}| > a_n\Bigr) > y\biggr) dy \\
&\leq \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E|X_{ni}| I\Bigl(|X_{ni}| > y\Bigr)}{a_n} \\
&\leq O\Bigl(n^{1-2p}\Bigr) E|X|^2 < \infty.
\end{aligned}
$$

Next, to prove I_2, we will first prove that

$$
\sup_{y \geq a_n} \frac{1}{y} \max_{1 \leq k_n \leq n} \left| \sum_{i=1}^{k_n} X_{ni} \right| \to 0 \ as\ n \to \infty
$$

For $1 \leq i \leq \infty, n \geq 1$, since $EX_{ni} = 0, EY_{ni} = -EZ_{ni}$. If $X_{ni} > y$, then $0 \leq Z_{ni} = X_{ni} - y < X_{ni}$. Consequently,

$$
\begin{aligned}
& \sup_{y \geq a_n} \frac{1}{y} \max_{1 \leq k_n \leq n} \left| \sum_{i=1}^{k_n} EY_{ni} \right| \\
= & \sup_{y \geq a_n} \frac{1}{y} \max_{1 \leq k_n \leq n} \left| \sum_{i=1}^{k_n} Z_{ni} \right| \\
\leq & C \sup_{y \geq a_n} \frac{1}{y} \sum_{i=1}^{k_n} E\Bigl| Z_{ni} \Bigr| \\
\leq & C \sup_{y \geq a_n} \frac{1}{y} \sum_{i=1}^{k_n} E\Bigl| X_{ni} \Bigr| I(|X_{ni}| > y) \\
\leq & C \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E\Bigl| X_{ni} \Bigr| I(|X_{ni}| > y)}{a_n} \\
\leq & C \sum_{n=1}^{\infty} \frac{k_n E|X|^2}{a_n^2} \\
\leq & CO\Bigl(n^{1-2p}\Bigr) \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

Next, we will prove that $I_2 < \infty$. For I_2, by Lemma 2.1(b), $\{Y_{ni} - EY_{ni} : 1 \leq i \leq k_n, n \geq 1\}$ is still an array of rowwise NSD random variables with mean zero. Let $B_n = \sum_{i=1}^{k_n}(Y_{ni} - EY_{ni})^2 < \infty$. Then, by Lemma 2.4 and Markov inequality, we have that

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}P\Big(\max_{1\le k_n\le n}\Big|\sum_{i=1}^{k_n}Y_{ni}\Big| > y\Big)dy \\
&\le \sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}P\Big(\max_{1\le k_n\le n}\Big|\sum_{i=1}^{k_n}(Y_{ni} - EY_{ni})\Big| > \frac{y}{2}\Big)dy \\
&\quad +2e^2\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\Big(\frac{B_n}{B_n + \frac{y^2}{2}}\Big)^2 dy \\
&\le C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\frac{\sum_{i=1}^{k_n}E(Y_{ni} - EY_{ni})^2}{y^2}dy + 2e^2\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\Big(\frac{B_n}{B_n + \frac{y^2}{2}}\Big)^2 dy \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{1}{a_n}\int_{a_n}^{\infty}\frac{EY_{ni}^2}{y^2}dy + 2e^2\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\Big(\frac{B_n}{B_n + \frac{y^2}{2}}\Big)^2 dy \\
&= C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{1}{a_n}\int_{a_n}^{\infty}\frac{E|X_{ni}|^2 I(|X_{ni}| \le a_n)}{y^2}dy \\
&\quad +C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{1}{a_n}\int_{a_n}^{\infty}\frac{EX_{ni}^2 I(a_n < |X_{ni}| \le y)}{y^2}dy \\
&\quad +C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{1}{a_n}\int_{a_n}^{\infty}P(|X_{ni}| > y)dy + 2e^2\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\Big(\frac{B_n}{B_n + \frac{y^2}{2}}\Big)^2 dy \\
&\doteqdot I_3 + I_4 + I_5 + I_6(say).
\end{aligned}$$

For I_3, by Definition 1.4 and above conditions, we have that for $p \ge \frac{1}{2}$ and enough large n,

$$\begin{aligned}
I_3 &= C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{1}{a_n}\int_{a_n}^{\infty}\frac{E|X_{ni}|^2 I(|X_{ni}| \le a_n)}{y^2}dy \\
&\le C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{E|X|^2 I(|X| \le a_n)}{a_n^2} \\
&\le CO\Big(n^{1-2p}\Big)E|X|^2 < \infty.
\end{aligned}$$

For I_4, by Definition 1.4 and conditions, we have that for $p \ge \frac{1}{2}$ and enough large n,

$$I_4 = C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{1}{a_n}\int_{a_n}^{\infty}\frac{E|X_{ni}|^2 I(a_n < |X_{ni}| \le y)}{y^2}dy$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty}\frac{k_n}{a_n}\sum_{k=[a_n]}^{\infty}\int_k^{k+1}\frac{E|X|^2I(a_n<|X|\leq y)}{y^2}dy\\
&\leq C\sum_{n=1}^{\infty}\frac{k_n}{a_n}\sum_{k=[a_n]}^{\infty}\frac{E|X|^2I([a_n]<|X|\leq k+1)}{k^2}\\
&\leq C\sum_{n=1}^{\infty}\frac{k_n}{a_n}\sum_{k=[a_n]}^{\infty}\frac{1}{k^2}\sum_{j=[a_n]}^{k}\frac{E|X|^2I(j<|X|\leq j+1)}{k^2}\\
&\leq C\sum_{n=1}^{\infty}\frac{k_n}{a_n}\sum_{j=[a_n]}^{\infty}E|X|^2I(j<|X|\leq j+1)\sum_{k=j}^{\infty}\frac{1}{k^2}\\
&\leq C\sum_{n=1}^{\infty}\frac{k_n}{a_n}\sum_{j=[a_n]}^{\infty}\frac{1}{j}E|X|^2I(j<|X|\leq j+1)\\
&\leq C\sum_{n=1}^{\infty}\frac{k_n}{a_n}E|X|^2I(|X|>a_n)\\
&\leq CO\left(n^{1-2p}\right)E|X|^2<\infty.
\end{aligned}
$$

For I_5, by Definition 1.4 and conditions, we have that for $p\geq\frac{1}{2}$ and enough large n,

$$
\begin{aligned}
I_5 &= C\sum_{n=1}^{\infty}\sum_{i=1}^{k_n}\frac{1}{a_n}\int_{a_n}^{\infty}P(|X_{ni}|>y)dy\\
&\leq C\sum_{n=1}^{\infty}\frac{k_n}{a_n}\int_0^{\infty}P(|X|I(|X|>a_n)>y)dy\\
&\leq C\sum_{n=1}^{\infty}k_n\frac{E|X|I(|X|>a_n)}{a_n}\\
&\leq CO\left(n^{1-2p}\right)E|X|^2<\infty.
\end{aligned}
$$

For I_6, by Definition 1.4 and above conditions, we have that for $p\geq\frac{1}{2}$ and enough large n,

$$
\begin{aligned}
I_6 &= 2e^2\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\left(\frac{B_n}{B_n+\frac{y^2}{2}}\right)^2dy\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\left(\frac{\sum_{i=1}^{k_n}E(Y_{ni}-EY_{ni})^2}{y^2}\right)^2dy\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\left(\frac{\sum_{i=1}^{k_n}EY_{ni}^2}{y^2}\right)^2dy
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\Big(\sum_{i=1}^{k_n}EX_{ni}^2I(|X_{ni}|\leq a_n)\Big)^2dy\\
&+ C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\Big(\sum_{i=1}^{k_n}EX_{ni}^2I(a_n<|X_{ni}|\leq y)\Big)^2dy\\
&+ C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}\Big(\sum_{i=1}^{k_n}P(|X_{ni}|>y)\Big)^2dy\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\Big(\sum_{i=1}^{k_n}EX_{ni}^2I(|X_{ni}|\leq a_n)\Big)^2dy\\
&+ C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\Big(\sum_{i=1}^{k_n}EX_{ni}^2I(a_n<|X_{ni}|\leq y)\Big)^2dy\\
&+ C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\Big(\sum_{i=1}^{k_n}P(|X_{ni}|>y)\Big)^2dy\\
&\doteqdot I_7+I_8+I_9(say).
\end{aligned}
$$

Thus, we will prove that $I_7<\infty, I_8<\infty$ and $I_9<\infty$.
So, for I_7, by Definition 1.4 and above conditions, we have that for $p\geq\frac{1}{2}$ and enough large n,

$$
\begin{aligned}
I_7 &= C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\Big(\sum_{i=1}^{k_n}EX_{ni}^2I(|X_{ni}|\leq a_n)\Big)^2dy\\
&\leq C\sum_{n=1}^{\infty}\Big(\frac{\sum_{i=1}^{k_n}EX^2I(|X|\leq a_n)}{a_n^2}\Big)^2\int_{a_n}^{\infty}y^{-4}dy\\
&\leq C\Big(\sum_{n=1}^{\infty}\frac{k_nEX^2}{a_n^2}\Big)^2\\
&\leq C\Big(O(n^{1-2p})EX^2\Big)^2<\infty.
\end{aligned}
$$

Next, for I_8, by Definition 1.4 and above conditions, we have that for $p\geq\frac{1}{2}$ and enough large n,

$$
\begin{aligned}
I_8 &= C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\Big(\sum_{i=1}^{k_n}EX_{ni}^2I(a_n<|X_{ni}|\leq y)\Big)^2dy\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\sum_{i=1}^{k_n}EX_{ni}^2I\Big(a_n<|X_{ni}|\leq y\Big)^2\int_{a_n}^{\infty}y^{-4}dy\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\Big(\sum_{i=1}^{k_n}EX_{ni}^2I(a_n<|X_{ni}|\leq y)\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\left(\sum_{i=1}^{k_n}EX^2I(a_n<|X|\leq y)\right)^2\\
&\leq C\sum_{n=1}^{\infty}\left(\frac{k_nEX^2}{a_n^2}\right)^2\\
&\leq C\left(\frac{\sum_{n=1}^{\infty}k_nEX^2}{a_n^2}\right)^2\\
&\leq C\left(O(n^{1-2p})EX^2\right)^2<\infty.
\end{aligned}
$$

Finally, for I_9, since $\sup_{n\geq a_n}\sum_{i=1}^{k_n}P(|X_{ni}|>y)\leq\sum_{i=1}^{k_n}P(|X_{ni}|>y)$, by Definition 1.4 and above conditions, we have that for $p\geq\frac{1}{2}$ and enough large n,

$$
\begin{aligned}
I_9 &= C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\left(\sum_{i=1}^{k_n}P(|X_{ni}|>y)\right)^2dy\\
&\leq C\sum_{n=1}^{\infty}\frac{1}{a_n}\int_{a_n}^{\infty}y^{-4}\left(\sum_{i=1}^{k_n}E(|X_{ni}|^2\right)^2dy\\
&= C\sum_{n=1}^{\infty}\frac{1}{a_n}\left(\sum_{i=1}^{k_n}E(|X_{ni}|^2\right)^2\int_{a_n}^{\infty}y^{-4}dy\\
&\leq C\sum_{n=1}^{\infty}\left(\frac{k_nE|X|^2}{a_n}\right)\int_{a_n}^{\infty}y^{-4}dy\\
&\leq C\left(O(n^{1-2p})E|X|^2\right)^2<\infty.
\end{aligned}
$$

□

From the Theorem 3.1 we have a the following corollary 3.2.

Corollary 3.2. *Under the assumptions of Theorem 3.1, we can obtain that*

$$\sum_{n=1}^{\infty}P\left(\frac{1}{a_n}\max_{1\leq k_n\leq n}|\sum_{i=1}^{k_n}X_{k_ni}|>\varepsilon\right)<\infty \text{ for all } \varepsilon>0.$$

Proof. Similar to the proof of Theorem 3.1, we can show that

$$\sum_{n=1}^{\infty}P\left(\frac{1}{a_n}\max_{1\leq k_n\leq n}|\sum_{i=1}^{k_n}X_{ni}|>\varepsilon\right)<\infty \text{ for all } \varepsilon>0.$$

□

Remark 3.1. Since NA random variables are the special case of NSD random variables, The results of above is an extension of NA random variables.

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