

Existence and Uniqueness of Solutions of Fractional Differential Equations with Deviating Arguments under Integral Boundary Conditions

DNYANOBA DHAIGUDE AND BAKR RIZQAN*

*Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University,
Aurangabad - 431 004, India*

e-mail : dnyanraja@gmail.com and bakeralhaaiti@yahoo.com

ABSTRACT. The aim of this paper is to develop a monotone iterative technique by introducing upper and lower solutions to Riemann-Liouville fractional differential equations with deviating arguments and integral boundary conditions. As an application of this technique, existence and uniqueness results are obtained.

1. Introduction

Differential equations with deviating arguments arise in various branches of science, engineering, economics and so on (see [4, 8] and the references therein). Many researchers have studied the existence, uniqueness, continuous dependence, and stability of solutions of nonlinear fractional differential equations (see [1, 2, 3, 5, 6, 7, 10, 11, 12, 15, 16, 17, 19, 20, 25, 30, 33, 34]). The monotone iterative technique [23] combined with the method of upper and lower solutions provides an effective mechanism to prove constructive existence results for nonlinear differential equations. The monotone technique is an interesting and powerful tool to deal with existence results for fractional differential equations. In 2008, the monotone technique for fractional differential equations with initial conditions was first developed by Lakshmikantham and Vatsala [25]. Later, a series of papers appeared in the literature to prove existence and uniqueness of solution of various problems with initial conditions, boundary conditions, integral boundary conditions, nonlinear boundary conditions, and periodic boundary conditions for fractional differential equations, (see, for ex-

* Corresponding Author.

Received April 12, 2017; revised August 07, 2018; accepted December 19, 2018.

2010 Mathematics Subject Classification: 34A08, 34A45, 34B15, 34B37, 34C60.

Key words and phrases: fractional differential equations with deviating arguments, Riemann-Liouville fractional derivatives, existence and uniqueness, monotone iterative technique, integral boundary conditions.

ample [13, 14, 18, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 35, 37, 38, 39, 40, 41, 42, 43] and the references therein). However, work on fractional differential equations with deviating argument is rare. In this paper, we study the following problem for the Riemann-Liouville fractional differential equation with a deviating argument and integral boundary conditions:

$$(1.1) \quad \begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t), u(\theta(t))), & t \in J = [0, T], \\ u(0) = \lambda \int_0^T u(s) ds + d, & d \in \mathbb{R}, \end{cases}$$

where $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\theta \in C(J, J)$, $\theta(t) \leq t$, $t \in J$, $\lambda \geq 0$, $0 < \alpha < 1$. The paper is organized as follows. In Section 2, we introduce some useful definitions and basic lemmas. In Section 3, we study the uniqueness of a solution for the problem (1.1) using the Banach fixed point theorem. In Section 4, we develop the monotone method and apply it to obtain existence and uniqueness results for Riemann-Liouville fractional differential equations with deviating arguments and integral boundary conditions.

2. Preliminaries

For the reader's convenience, we present some necessary definitions and lemmas from the theory of fractional calculus. In addition, we prove some basic results which are useful for further discussion.

Definition 2.1. ([21, 36]) For $\alpha > 0$, the integral

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

is called the *Riemann-Liouville fractional integral* of order α .

Definition 2.2. ([21, 36]) The Riemann-Liouville derivative of order α ($n-1 < \alpha \leq n$) can be written as

$$D_{0+}^{\alpha} u(t) = \left(\frac{d}{dt} \right)^n (I_{0+}^{n-\alpha} u(t)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \quad t > 0.$$

Lemma 2.1. ([21]) Let $u \in C^n[0, T]$, $\alpha \in (n-1, n)$, $n \in \mathbb{N}$. Then for $t \in J$,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0).$$

Consider the space $C_{1-\alpha}(J, \mathbb{R}) = \{u \in C((0, T], \mathbb{R}) : t^{1-\alpha} u \in C(J, \mathbb{R})\}$.

Lemma 2.2. ([9]) Let $m \in C_{1-\alpha}(J, \mathbb{R})$ where for some $t_1 \in (0, T]$,

$$m(t_1) = 0 \text{ and } m(t) \leq 0 \text{ for } 0 \leq t \leq t_1.$$

Then it follows that

$$D^\alpha m(t_1) \geq 0.$$

Lemma 2.3. Let $f \in C(J \times \mathbb{R}^2, \mathbb{R})$. A function $u \in C_{1-\alpha}(J, \mathbb{R})$ is a solution of the problem (1.1) if and only if u is a solution of the integral equation

$$(2.1) \quad u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u(\theta(s))) ds + \lambda \int_0^T u(s) ds + d.$$

Proof. Assume that u satisfies the problem (1.1). From the first equation of the problem (1.1) and Lemma 2.1, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u(\theta(s))) ds + \lambda \int_0^T u(s) ds + d.$$

Conversely, assume that $u \in C_{1-\alpha}(J, \mathbb{R})$ satisfies the integral equation (2.1). Applying the Riemann-Liouville operator D_{0+}^α to both sides of the integral equation (2.1), we have

$$\begin{aligned} D_{0+}^\alpha u(t) &= D_{0+}^\alpha \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u(\theta(s))) ds + \lambda \int_0^T u(s) ds + d \right) \\ D_{0+}^\alpha u(t) &= f(t, u(t), u(\theta(t))). \end{aligned}$$

In addition, we have $u(0) = \lambda \int_0^T u(s) ds + d$ from the integral equation (2.1). The proof is complete. \square

Lemma 2.4. Suppose that $\{u_\epsilon\}$ is a family of continuous functions defined on J , for each $\epsilon > 0$, which satisfies

$$(2.2) \quad \begin{cases} D_{0+}^\alpha u_\epsilon(t) = f(t, u_\epsilon(t), u_\epsilon(\theta(t))), \\ u_\epsilon(0) = \lambda \int_0^T u_\epsilon(s) ds + d, \end{cases}$$

where $|f(t, u_\epsilon(t), u_\epsilon(\theta(t)))| \leq M$ for $t \in J$. Then the family $\{u_\epsilon\}$ is equicontinuous on J .

Proof. For $0 \leq t_1 < t_2 \leq T$, consider

$$\begin{aligned} |u_\epsilon(t_1) - u_\epsilon(t_2)| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, u_\epsilon(s), u_\epsilon(\theta(s))) ds - \right. \\ &\quad \left. \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, u_\epsilon(s), u_\epsilon(\theta(s))) ds \right| \\ &\leq \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}] ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{M}{\Gamma(\alpha+1)} [t_1^\alpha - t_2^\alpha + 2(t_2 - t_1)^\alpha] \\ &\leq \frac{2M}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha < \epsilon, \end{aligned}$$

provided that $|t_2 - t_1| < \delta = \left[\frac{\epsilon \Gamma(\alpha+1)}{2M} \right]^{\frac{1}{\alpha}}$, proving the result. \square

3. Uniqueness of Solution

In this section, we obtain the uniqueness of solution of the problem (1.1) for Riemann-Liouville fractional differential equations with deviating argument and integral boundary conditions.

Theorem 3.1. *Assume that*

- (i) $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\theta(t) \in C(J, J)$, $\theta \leq t$, $t \in J$,
- (ii) *there exists nonnegative constants M and N such that function f satisfies*

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq M |u_1 - v_1| + N |u_2 - v_2|,$$

for all $t \in J$, $u_i, v_i \in \mathbb{R}$, $i = 1, 2$. If $\lambda < \frac{\Gamma(\alpha+1) - T^\alpha(M+N)}{T\Gamma(\alpha+1)}$, then the problem (1.1) has a unique solution.

Proof. Consider the operator T defined by

$$(Tu)(t) = \lambda \int_0^T u(s) ds + d + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), u(\theta(s))) ds.$$

Now, we show that $T : C_{1-\alpha}(J, \mathbb{R}) \rightarrow C_{1-\alpha}(J, \mathbb{R})$ is a contraction operator. For any $u, v \in C_{1-\alpha}(J, \mathbb{R})$, we have

$$\begin{aligned} \|Tu - Tv\|_C &= \max_{t \in J} |(Tu)(t) - (Tv)(t)| \\ &\leq \max_{t \in J} \lambda \int_0^T |u(s) - v(s)| ds + \max_{t \in J} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times |f(s, u(s), u(\theta(s))) - f(s, v(s), v(\theta(s)))| ds \\ &\leq \lambda \int_0^T ds \|u - v\|_C + \max_{t \in J} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times [|M(u(s) - v(s))| + |N(u(\theta(s)) - v(\theta(s)))|] ds \\ &\leq \lambda T \|u - v\|_C + \max_{t \in J} \frac{(M+N)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u - v\|_C ds \\ &\leq \lambda T \|u - v\|_C + \max_{t \in J} \frac{(M+N)t^\alpha}{\Gamma(\alpha)} \int_0^1 (1-\eta)^{\alpha-1} d\eta \|u - v\|_C \\ &\leq \left[\lambda T + \frac{T^\alpha}{\Gamma(\alpha+1)} (M+N) \right] \|u - v\|_C. \end{aligned}$$

Therefore, $\|Tu - Tv\|_C < \|u - v\|_C$. By the Banach fixed point theorem, the operator T has a unique fixed point, i.e. the problem (1.1) has a unique solution. The proof is complete. \square

Corollary 3.1. *Let M, N be constants, $\sigma \in C_{1-\alpha}(J, \mathbb{R})$. The linear problem*

$$(3.1) \quad \begin{cases} D_{0+}^{\alpha} u(t) + Mu(t) + Nu(\theta(t)) = \sigma(t), & 0 < \alpha < 1, t \in J, \\ u(0) = \lambda \int_0^T u(s) ds + d, & d \in \mathbb{R}, \end{cases}$$

has a unique solution.

Proof. It follows from the Theorem 3.1. \square

4. Monotone Iterative Method

In this section, we prove the existence and uniqueness of solution for the problem (1.1) by monotone iterative technique combined with the method of upper and lower solutions. Now we define the functional interval as follows:

$$[v_0, w_0] = \{u \in C_{1-\alpha}(J, \mathbb{R}) : v_0(t) \leq u(t) \leq w_0(t) \forall t \in J\}.$$

First, we prove the following comparison result, which plays an important role in our further discussion.

Lemma 4.1. *Let $\theta \in C(J, J)$ where $\theta(t) \leq t$ on J . Suppose that $p \in C_{1-\alpha}(J, \mathbb{R})$ satisfies the inequalities*

$$(4.1) \quad \begin{cases} D_{0+}^{\alpha} p(t) \leq -Mp(t) - Np(\theta(t)) \equiv Fp(t), & t \in J, \\ p(0) \leq 0, \end{cases}$$

where M and N are constants. If

$$(4.2) \quad -(1 + T^{\alpha})[M + N] < \Gamma(1 + \alpha),$$

then $p(t) \leq 0$ for all $t \in J$.

Proof. Consider $p_{\epsilon}(t) = p(t) - \epsilon(1 + t^{\alpha})$, $\epsilon > 0$. Then

$$\begin{aligned} D_{0+}^{\alpha} p_{\epsilon}(t) &= D_{0+}^{\alpha} p(t) - D_{0+}^{\alpha} \epsilon(1 + t^{\alpha}) \\ &\leq Fp(t) - \frac{\epsilon}{t^{\alpha} \Gamma(1 - \alpha)} - \epsilon \Gamma(1 + \alpha) \\ &= Fp_{\epsilon}(t) + \epsilon \left[-M(1 + t^{\alpha}) - N(1 + t^{\alpha}) - \frac{1}{t^{\alpha} \Gamma(1 - \alpha)} - \Gamma(1 + \alpha) \right] \\ &< Fp_{\epsilon}(t) + \epsilon [-(1 + t^{\alpha})(M + N) - \Gamma(1 + \alpha)] < Fp_{\epsilon}(t) \end{aligned}$$

and

$$p_{\epsilon}(0) = p(0) - \epsilon(1 + t^{\alpha}) < 0.$$

We prove that $p_\epsilon(t) < 0$ on J . Assume that $p_\epsilon(t) \not< 0$ on J . Thus there exists a $t_1 \in (0, T]$ such that $p_\epsilon(t_1) = 0$ and $p_\epsilon(t) < 0, t \in (0, t_1)$. In view of Lemma 2.2, we have $D_{0+}^\alpha p_\epsilon(t_1) \geq 0$. It follows that

$$0 < Fp_\epsilon(t) = -Np_\epsilon(\theta(t_1)).$$

If $N = 0$, then $0 < 0$, which is a contradiction. If $-N > 0$, then $p_\epsilon(\theta(t_1)) > 0$, which is again a contradiction. This proves that $p_\epsilon(t) < 0$ on J . So $p(t) - \epsilon(1 + t^\alpha) < 0$ on J . Taking $\epsilon \rightarrow 0$, we obtain required result. \square

Definition 4.1. A pair of functions $[v_0, w_0]$ in $C_{1-\alpha}(J, \mathbb{R})$ are called *lower and upper solutions* of the problem (1.1) if

$$(4.3) \quad D_{0+}^\alpha v_0(t) \leq f(t, v_0(t), v_0(\theta(t))), \quad v_0(0) \leq \int_0^T v_0(s)ds + d$$

and

$$(4.4) \quad D_{0+}^\alpha w_0(t) \geq f(t, w_0(t), w_0(\theta(t))), \quad w_0(0) \geq \int_0^T w_0(s)ds + d.$$

Theorem 4.1. Assume that

- (i) $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, $\theta \in C(J, J)$, $\theta(t) \leq t$, $t \in J$,
- (ii) functions v_0 and w_0 in $C_{1-\alpha}(J, \mathbb{R})$ are lower and upper solutions of the problem (1.1) such that $v_0(t) \leq w_0(t)$ on J ,
- (iii) there exists nonnegative constants M, N such that function f satisfies the condition

$$(4.5) \quad f(t, u_1, u_2) - f(t, v_1, v_2) \geq -M(u_1 - v_1) - N(u_2 - v_2),$$

$$\text{for } v_0(t) \leq v_1 \leq u_1 \leq w_0(t), v_0(\theta(t)) \leq v_2 \leq u_2 \leq w_0(\theta(t)).$$

Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_{1-\alpha}(J, \mathbb{R})$ such that

$$\{v_n(t)\} \rightarrow v(t) \text{ and } \{w_n(t)\} \rightarrow w(t) \text{ as } n \rightarrow \infty$$

for all $t \in J$, where v and w are minimal and maximal solutions of the problem (1.1) respectively and $v(t) \leq u(t) \leq w(t)$ on J .

Proof. For any $\eta \in C_{1-\alpha}(J, \mathbb{R})$ such that $\eta \in [v_0, w_0]$, we consider the following linear problem:

$$(4.6) \quad \begin{cases} D_{0+}^\alpha u(t) = f(t, \eta(t), \eta(\theta(t))) + M[\eta(t) - u(t)] + N[\eta(\theta(t)) - u(\theta(t))], \\ u(0) = \int_0^T u(s)ds + d, \end{cases}$$

By Corollary 3.1, the linear problem (4.6) has a unique solution $u(t)$.

Next, we define the iterates as follows and construct the sequences $\{v_n\}$, $\{u_n\}$

$$(4.7) \quad \begin{cases} D_{0+}^{\alpha} v_{n+1}(t) = f(t, v_n(t), v_n(\theta(t))) - \\ \quad M[v_{n+1}(t) - v_n(t)] - N[v_{n+1}(\theta(t)) - v_n(\theta(t))], \\ v_{n+1}(0) = \int_0^T v_n(s) ds + d, \end{cases}$$

and

$$(4.8) \quad \begin{cases} D_{0+}^{\alpha} w_{n+1}(t) = f(t, w_n(t), w_n(\theta(t))) - \\ \quad M[w_{n+1}(t) - w_n(t)] - N[w_{n+1}(\theta(t)) - w_n(\theta(t))], \\ w_{n+1}(0) = \int_0^T w_n(s) ds + d, \end{cases}$$

Clearly, the existence of solutions v_{n+1} and w_{n+1} of the problems (4.7) and (4.8), respectively, follows from the above arguments. Further, by setting $n = 0$ in the problems (4.7), (4.8), we get the existence of solutions v_1 and w_1 , respectively. We show that $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Set $p(t) = v_1(t) - v_0(t)$. Since v_0 is the lower solution of the problem (4.7), we have

$$\begin{aligned} D_{0+}^{\alpha} p(t) &= D_{0+}^{\alpha} v_1(t) - D_{0+}^{\alpha} v_0(t) \\ &\geq f(t, v_0(t), v_0(\theta(t))) - f(t, v_0(t), v_0(\theta(t))) - \\ &\quad M[v_1(t) - v_0(t)] - N[v_1(\theta(t)) - v_0(\theta(t))] \\ &\geq -Mp(t) - Np(\theta(t)) \end{aligned}$$

and

$$p(0) = v_1(0) - v_0(0) \geq \int_0^T v_0(s) ds + d - \int_0^T v_0(s) ds - d = 0.$$

From Lemma 4.1, we obtain $p(t) \geq 0$, which implies that $v_1(t) \geq v_0(t)$ on J . Similarly, we can prove $v_1(t) \leq w_1(t)$ and $w_1(t) \leq w_0(t)$ on J . Thus $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Assume that for some $k > 1$,

$$v_{k-1}(t) \leq v_k(t) \leq w_k(t) \leq w_{k-1}(t) \text{ on } J.$$

We claim that $v_k(t) \leq v_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on J . To prove our claim, set $p(t) = v_{k+1}(t) - v_k(t)$. Then we have

$$\begin{aligned} D_{0+}^{\alpha} p(t) &= D_{0+}^{\alpha} v_{k+1}(t) - D_{0+}^{\alpha} v_k(t) \\ &= f(t, v_k(t), v_k(\theta(t))) - M[v_{k+1}(t) - v_k(t)] - \\ &\quad N[v_{k+1}(\theta(t)) - v_k(\theta(t))] - f(t, v_{k-1}(t), v_{k-1}(\theta(t))) + \\ &\quad M[v_k(t) - v_{k-1}(t)] + N[v_k(\theta(t)) - v_{k-1}(\theta(t))] \\ &\geq -M[v_{k+1}(t) - v_k(t)] - N[v_{k+1}(\theta(t)) - v_k(\theta(t))] \\ &\geq -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$\begin{aligned} p(0) &= v_{k+1}(0) - v_k(0) = \int_0^T v_k(s) ds + d - \int_0^T v_{k-1}(s) ds - d \\ &\geq \int_0^T [v_k(s) - v_{k-1}(s)] ds = 0. \end{aligned}$$

By Lemma 4.1, we obtain $p(t) \geq 0$, implying that $v_{k+1}(t) \geq v_k(t)$ for all $t \in J$. Similarly, we can prove $v_{k+1}(t) \leq w_{k+1}(t)$ and $w_{k+1}(t) \leq w_k(t)$ for all $t \in J$. From the principle of mathematical induction, we have

$$(4.9) \quad v_0 \leq v_1 \leq v_1 \leq \dots \leq v_k \leq w_k \leq \dots \leq w_2 \leq w_1 \leq w_0 \text{ on } J.$$

Clearly, the sequences $\{v_n\}, \{w_n\}$ are monotonic and uniformly bounded. Further we observe that $\{D_{0+}^\alpha v_n\}$ and $\{D_{0+}^\alpha w_n\}$ are also uniformly bounded on J , in view of the relations (4.7), (4.8). Applying Lemma 2.4 we can conclude that sequences $\{v_n\}, \{w_n\}$ are equicontinuous. Hence by the Ascoli-Arzelà theorem the sequences $\{v_n\}, \{w_n\}$ converge uniformly to v and w on J respectively.

Now, we prove that v and w are the minimal and maximal solutions of the problem (1.1). Let u be any solution of the problem (1.1) different from v and w . So there exists a k such that $v_k(t) \leq u(t) \leq w_k(t)$ on J . Set $p(t) = u(t) - v_{k+1}(t)$. Then we have

$$\begin{aligned} D_{0+}^\alpha p(t) &= D_{0+}^\alpha u(t) - D_{0+}^\alpha v_{k+1}(t) \\ &= f(t, u(t), u(\theta(t))) - f(t, v_k(t), v_k(\theta(t))) + \\ &\quad M[v_{k+1}(t) - v_k(t)] + N[v_{k+1}(\theta(t)) - v_k(\theta(t))] \\ &\geq -M[u(t) - v_{k+1}(t)] - N[u(\theta(t)) - v_{k+1}(\theta(t))] \\ &\geq -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = u(0) - v_{k+1}(0) = \int_0^T [u(s) - v_k(s)] ds \geq 0.$$

By Lemma 4.1, we obtain $p(t) \geq 0$, implying that $u(t) \geq v_{k+1}(t)$ for all k on J . Similarly, we can prove $u(t) \leq w_{k+1}(t)$ for all k on J . Since $v_0(t) \leq u(t) \leq w_0(t)$ on J . By induction it follows that $v_k(t) \leq u(t)$ and $u(t) \leq w_k(t)$ for all k . Thus $v_k(t) \leq u(t) \leq w_k(t)$ on J . Taking the limit as $k \rightarrow \infty$, we obtain $v(t) \leq u(t) \leq w(t)$ on J . Thus the functions $v(t), w(t)$ are the minimal and maximal solutions of the problem (1.1). The proof is complete. \square

Next we prove the uniqueness of solution of the problem (1.1) as follows.

Theorem 4.2. *Assume that*

- (i) *all the conditions of the Theorem 4.1 hold,*
- (ii) *there exists nonnegative constants M, N such that the function f satisfies the condition*

$$(4.10) \quad f(t, u_1, u_2) - f(t, v_1, v_2) \leq M(u_1 - v_1) + N(u_2 - v_2),$$

for $v_0(t) \leq v_1 \leq u_1 \leq w_0(t)$, $v_0(\theta(t)) \leq v_2 \leq u_2 \leq w_0(\theta(t))$.

Then the problem (1.1) has a unique solution.

Proof. We know $v(t) \leq w(t)$ on J . It is sufficient to prove that $v(t) \geq w(t)$ on J . Consider $p(t) = w(t) - v(t)$. Then we have

$$\begin{aligned} D_{0+}^{\alpha} p(t) &= D_{0+}^{\alpha} w(t) - D_{0+}^{\alpha} v(t) \\ &= f(t, w(t), w(\theta(t))) - f(t, v(t), v(\theta(t))) \\ &\leq -M[v(t) - w(t)] - N[v(\theta(t)) - w(\theta(t))] \\ &= -Mp(t) - Np(\theta(t)) \end{aligned}$$

and

$$p(0) = w(0) - v(0) = \int_0^T [w(s) - v(s)] ds \leq 0.$$

By Lemma 4.1, we know $p(t) \leq 0$, implying that $v(t) \geq w(t)$, and the result follows. \square

Acknowledgements. The authors are grateful to the respected anonymous referee and the English language expert Prof. J. T. Neugebauer for their valuable comments and suggestions that greatly improved the presentation of the paper.

References

- [1] R. P. Agarwal, B. de Andrade and C. Cuevas, *On type of periodicity and ergodicity to a class of fractional order differential equations*, Adv. Difference Equ., (2010), Art. ID 179750, 25 pp.
- [2] R. P. Agarwal, B. de Andrade and G. Siracusa, *On fractional integro-differential equations with state-dependent delay*, Comput. Math. Appl., **62**(2011), 1143–1149.
- [3] R. P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109**(2010), 973–1033.
- [4] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive solutions of differential, difference and integral equations*, Kluwer Academic Publishers, Dordrecht, 1999.
- [5] M. A. Al-Bassam, *Some existence theorems on differential equations of generalized order*, J. Reine Angew. Math., **218**(1965), 70–78.
- [6] Z. Bai and H. Lu, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., **311**(2005), 495–505.
- [7] S. P. Bhairat and D. B. Dhaigude, *Ulam stability for system of nonlinear implicit fractional differential equations*, Prog. Nonlinear Dyn. Chaos, **6**(1)(2018), 29–38
- [8] T. A. Burton, *Differential inequalities for integral and delay differential equations*, Comparison Methods and Stability Theory, Lecture Notes in Pure and Appl. Math. **162**, Dekker, New York, 1994.

- [9] J. V. Devi, F. A. McRae and Z. Drici, *Variational Lyapunov method for fractional differential equations*, Comp. Math. Appl., **64**(2012), 2982–2989.
- [10] D. B. Dhaigude and Sandeep P. Bhairat, *Existence and uniqueness of solution of Cauchy-type problem for Hilfer fractional differential equations*, Comm. Appl. Anal., **22**(1) (2018), 121–134.
- [11] D. B. Dhaigude and Sandeep P. Bhairat, *Local Existence and Uniqueness of Solution for Hilfer-Hadamard fractional differential problem*, Nonlinear Dyn. Syst. Theory, **18**(2)(2018), 144–153.
- [12] C. P. Dhaigude-Chitalkar, Sandeep P. Bhairat and D. B. Dhaigude, *Solution of fractional differential equations involving Hilfer fractional derivative: method of successive approximations*, Bull. Marathwada Math. Soc., **18**(2)(2017), 1–13.
- [13] D. B. Dhaigude, N. B. Jadhav and J. A. Nanware, *Method of upper lower solutions for nonlinear system of fractional differential equations and applications*, Malaya J. Mat., **6**(3)(2018), 467–472.
- [14] D. B. Dhaigude, J. A. Nanware and V. R. Nikam, *Monotone Technique for System of Caputo Fractional Differential Equations with Periodic Boundary Conditions*, Dyn. Conti. Discrete Impuls. Syst. Ser. A Math. Anal., **19**(2012), 575–584.
- [15] D. B. Dhaigude and B. H. Rizqan, *Existence and uniqueness of solutions for fractional differential equations with advanced arguments*, Adv. Math. Models Appl., **2**(3)(2017), 240–250.
- [16] D. B. Dhaigude and B. H. Rizqan, *Monotone iterative technique for caputo fractional differential equations with deviating arguments*, Ann. Pure Appl. Math., **16**(1)(2018), 181–191.
- [17] D. B. Dhaigude and B. H. Rizqan, *Existence results for nonlinear fractional differential equations with deviating arguments under integral boundary conditions*, Far East J. Math. Sci., **108**(2)(2018), 273–284.
- [18] N. B. Jadhav and J. A. Nanware, *Integral boundary value problem for system of nonlinear fractional differential equations*, Bull. Marathwada Math. Soc., **18**(2)(2017), 23–31.
- [19] T. Jankowski, *Fractional differential equations with deviating arguments*, Dyn. Syst. Appl., **17**(3)(2008), 677–684.
- [20] T. Jankowski, *Existence results to delay fractional differential equations with nonlinear boundary conditions*, Appl. Math. Comput., **219**(2013), 9155–9164.
- [21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, **204**, 2006.
- [22] P. Kumar, D. N. Pandey and D. Bahuguna, *On a new class of abstract impulsive functional differential equations of fractional order*, J. Nonlinear Sci. Appl., **7**(2014), 102–114.
- [23] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone iterative techniques for nonlinear differential equations*, Pitman Pub. Co, Boston, 1985.
- [24] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal., **69**(2008), 3337–3343.

- [25] V. Lakshmikantham and A. S. Vatsala, *General uniqueness and monotone iterative technique for fractional differential equations*, Appl. Math. Lett., **21**(2008), 828–834.
- [26] N. Li and C. Y. Wang, *New existence results of positive solution for a class of nonlinear fractional differential equations*, Acta Math. Sci. Ser. B, **33**(2013), 847–854.
- [27] L. Lin, X. Liu and H. Fang, *Method of upper and lower solutions for fractional differential equations*, Electron. J. Diff. Eq., **100**(2012), 13 pp.
- [28] X. Liu, M. Jia and B. Wu, *Existence and uniqueness of solution for fractional differential equations with integral boundary conditions*, Electron. J. Qual. Theory Differ. Equ., **69**(2009), 10 pp.
- [29] F. A. McRae, *Monotone iterative technique and existence results for fractional differential equations*, Nonlinear Anal., **71**(2009), 6093–6096.
- [30] J. A. Nanware and D. B. Dhaigude, *Existence and uniqueness of solutions of Riemann-Liouville fractional differential equation with integral boundary condition*, Int. J. Nonlinear Sci., **14**(2012), 410–415.
- [31] J. A. Nanware and D. B. Dhaigude, *Boundary Value Problems for Differential Equations of Noninteger Order Involving Caputo Fractional Derivative*, Adv. Stu. Contem. Math., **24**(2014), 369–376.
- [32] J. A. Nanware and D. B. Dhaigude, *Existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions*, J. Nonlinear Sci. Appl., **7**(2014), 246–254.
- [33] J. A. Nanware and D. B. Dhaigude, *Monotone technique for finite system of Caputo fractional differential equations with periodic boundary conditions*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **22**(1)(2015), 13–23.
- [34] J. A. Nanware, N. B. Jadhav and D. B. Dhaigude, *Monotone iterative technique for finite system of Riemann-Liouville fractional differential equations with integral boundary conditions*, Internat. Conf. Math. Sci., June, (2014), 235–238.
- [35] J. A. Nanware, N. B. Jadhav and D. B. Dhaigude, *Initial value problems for fractional differential equations involving Riemann-Liouville derivative*, Malaya J. Mat., **5** (2)(2017), 337–345.
- [36] I. Podlubny, *Fractional differential equations, mathematics in science and engineering*, Academic Press, New York, 1999.
- [37] B. H. Rizqan and D. B. Dhaigude, *Positive solutions of nonlinear fractional differential equations with advanced arguments under integral boundary value conditions*, Indian J. Math., **60**(3)(2018), 491–507.
- [38] B. H. Rizqan and D. B. Dhaigude, *Nonlinear boundary value problem of fractional differential equations with advanced arguments under integral boundary conditions*, Tamkang J. Math., (Accepted).
- [39] A. Shi and S. Zhang, *Upper and lower solutions method and a fractional differential equation boundary value problem*, Electron. J. Qual. Theory Differ. Equ., **30**(2009), 13 pp.
- [40] X. Wang, L. Wang, and Z. Qinghong, *Fractional differential equations with integral boundary conditions*, J. Nonlinear Sci. Appl., **8**(2015), 309–314.

- [41] T. Wang and F. Xie, *Existence and uniqueness of fractional differential equations with integral boundary conditions*, J. Nonlinear Sci. Appl., **1**(2008), 206–212.
- [42] S. Zhang, *Monotone iterative method for initial value problem involving Riemann–Liouville fractional derivatives*, Nonlinear Anal., **71**(2009), 2087–2093.
- [43] S. Q. Zhang and X. W. Su, *The existence of a solution for a fractional differential equation with nonlinear boundary conditions considered using upper and lower solutions in reverse order*, Comput. Math. Appl., **62**(2011), 1269–1274.