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Paracontact Metric (k, μ)-spaces Satisfying Certain Curvature Conditions

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Abstract. The object of this paper is to classify paracontact metric (k, μ)-spaces satisfying certain curvature conditions. We show that a paracontact metric (k, μ)-space is Ricci semisymmetric if and only if the metric is Einstein, provided $k < -1$. Also we prove that a paracontact metric (k, μ)-space is ϕ-Ricci symmetric if and only if the metric is Einstein, provided $k \neq 0, -1$. Moreover, we show that in a paracontact metric (k, μ)-space with $k < -1$, a second order symmetric parallel tensor is a constant multiple of the associated metric tensor. Several consequences of these results are discussed.

1. Introduction

After being introduced by Kaneyuki and Williams [10] in 1985, a systematic study of paracontact metric manifolds and their subclasses, especially para-Sasakian manifolds, was carried out by Zamkovoy [21]. Paracontact metric manifolds have been studied by several authors such as Alekseevski et. al. [1, 2], Cortés [6], Erdem [9], Martin-Molina [12]. Recently, Cappelletti-Montano et. al. [5] introduced a new type of paracontact geometry, so-called paracontact metric (k, μ)-spaces, where k and μ are real constants. It is well known [3] that in the contact case one requires $k \leq 1$, but there is no such restriction for k in the paracontact case [5]. Also, in the contact case, $k = 1$ implies the manifold is *Sasakian* but in paracontact case, $k = -1$ does not imply the manifold is para-Sasakian.

Among the geometric properties of manifolds symmetry is an important one. From the local point of view it was introduced by Shirokov [18] as a Riemannian manifold with covariant constant curvature tensor R, that is, with $\nabla R = 0$, where

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∇ is the Levi-Civita connection. An extensive theory of symmetric Riemannian manifolds was carried out by Cartan in 1927. A manifold is called semisymmetric if the curvature tensor R satisfies $R(X,Y)\cdot R=0$, where $R(X,Y)$ is considered to be a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y. Semisymmetric manifolds were locally classified by Szabó [19].
A manifold is said to be *Ricci semisymmetric* if $R(X,Y)\cdot S=0$ where S denotes the Ricci tensor of type $(0,2)$. A general classification of these manifolds has been worked out by Mirzoyan [13].

The notion of locally ϕ-symmetric was introduced by Takahashi [20] in Sasakian geometry as a weaker version of locally symmetric manifolds. In [7], De and Sarkar studied *ϕ-Ricci symmetric* Sasakian manifolds. Prakasha and Mirji [15] studied ϕ-Ricci symmetric $N(k)$-paracontact metric manifolds.

Also one of the main purposes of this paper is to study Eisenhart problem. In 1923, Eisenhart [8] proved that if a positive definite Riemannian manifold admits a second order parallel symmetric covariant tensor other than a constant multiple of the associated metric tensor, then it is reducible. In 1925, Levy [11] proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the associated metric tensor. Sharma [16, 17] extended the result in contact geometry. Recently, Mondal et. al. [14] studied second order parallel tensors on (k,μ)-contact metric manifolds. Here we consider second order parallel symmetric covariant tensors on paracontact metric (k,μ)-spaces.

A paracontact metric (k,μ)-manifold is said to be an Einstein manifold if the Ricci tensor satisfies $S=\lambda g$, where λ is some constant.

The paper is organized as follows:
In Section 2, we provide some basic results of paracontact metric (k,μ)-manifolds. Sections 3 and 4 are devoted to study Ricci semisymmetric and ϕ-Ricci symmetric paracontact metric (k,μ)-manifolds, respectively. In Section 5, we study the existence of symmetric parallel covariant tensors on paracontact metric (k,μ)-spaces. Several consequences of these results are discussed.

2. Preliminaries

A smooth manifold M^{2n+1} is said to admit an almost paracontact structure (ϕ,ξ,η) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying [10]

(i) $\phi^2X=X-\eta(X)\xi$, for any vector field $X\in\chi(M)$, the set of all differential vector fields on M,

(ii) $\phi(\xi)=0,\ \eta\circ\phi=0,\ \eta(\xi)=1$,

(iii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $\mathcal{D}=ker(\eta)$, that is, the eigendistributions $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n.

An almost paracontact structure is said to be *normal* [21] if and only if the $(1,2)$-type torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X,\phi Y] - \phi[\phi X,Y] - \phi[X,\phi Y]$. An almost paracontact manifold equipped with a pseudo-Riemannian metric g such that

$$g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X,Y \in \chi(M)$, is called *almost paracontact metric manifold*, where signature of g is $(n+1,n)$. An almost paracontact structure is said to be a *paracontact* structure if $g(X,\phi Y) = d\eta(X,Y)$ with the associated metric g [21]. For any almost paracontact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ admits (at least, locally) a ϕ-basis [21], that is, a pseudo-orthonormal basis of vector fields of the form $\{\xi, E_1, E_2, ..., E_n, \phi E_1, \phi E_2, ..., \phi E_n\}$, where $\xi, E_1, E_2, ..., E_n$ are space-like vector fields and then, by (2.1) the vector fields $\phi E_1, \phi E_2, ..., \phi E_n$ are time-like. In a paracontact metric manifold we define a symmetric, trace-free $(1,1)$-tensor $h = \frac{1}{2}\pounds_\xi \phi$ satisfying [21]

$$\phi h + h\phi = 0,\ h\xi = 0, \tag{2.2}$$

$$\nabla_X \xi = -\phi X + \phi h X,\ \text{for all } X \in \chi(M), \tag{2.3}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold. Noticing that the tensor h vanishes identically if and only if ξ is a Killing vector field and in such case (ϕ,ξ,η,g) is said to be a *K-paracontact structure*. An almost paracontact manifold is said to be *para-Sasakian* if and only if the following condition holds [21]

$$(\nabla_X \phi)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.4}$$

for any $X,Y \in \chi(M)$. A normal paracontact metric manifold is para-Sasakian and satisfies

$$R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y), \tag{2.5}$$

for any $X,Y \in \chi(M)$, but unlike contact metric geometry (2.5) is not a sufficient condition for a paracontact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact, but the converse is not always true, as it is shown in three dimensional case [4].

Finally, we recall the definition of paracontact metric (k,μ)-manifolds [5]:

Definition 2.1. A paracontact metric manifold is said to be a *paracontact (k,μ)-manifold* if the curvature tensor R satisfies

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.6}$$

for all vector fields $X,Y \in \chi(M)$ and k,μ are real constants.

This class is very wide containing the para-Sasakian manifolds [10, 21] as well as the paracontact metric manifolds satisfying $R(X,Y)\xi = 0$ for all $X,Y \in \chi(M)$ [22].

In particular, if $\mu = 0$, then the paracontact metric (k,μ)-manifold is called *paracontact metric $N(k)$-manifold*. Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{2.7}$$

for all $X, Y \in \chi(M)$. Though the geometric behavior of paracontact metric (k,μ)-spaces is different according as $k < -1$, or $k > -1$, or $k = -1$, but there are some common results for $k < -1$ and $k > -1$. In [5], Cappelletti-Montano et. al. pointed out the following result.

Lemma 2.1.([5], p.686, 692) *There does not exist any paracontact (k,μ)-manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exists such manifolds for $k < -1$.*

In a paracontact metric (k,μ)-manifold $(M^{2n+1}, \phi, \xi, \eta, g), n > 1$, the following relations hold [5]:

$$h^2 = (k+1)\phi^2, \tag{2.8}$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \qquad \text{for } k \neq -1, \tag{2.9}$$

$$\begin{aligned} QY =&[2(1-n) + n\mu]Y + [2(n-1) + \mu]hY \\ &+ [2(n-1) + n(2k-\mu)]\eta(Y)\xi, \qquad \text{for } k \neq -1, \end{aligned} \tag{2.10}$$

$$S(X,\xi) = 2nk\eta(X), \tag{2.11}$$

$$Q\xi = 2nk\xi, \tag{2.12}$$

$$\begin{aligned} (\nabla_X h)Y =& -[(1+k)g(X,\phi Y) + g(X,\phi hY)]\xi \\ &+ \eta(Y)\phi h(hX - X) - \mu\eta(X)\phi hY, \qquad \text{for } k \neq -1, \end{aligned} \tag{2.13}$$

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi, \tag{2.14}$$

for any vector fields $X, Y \in \chi(M)$, where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$. Making use of (2.3) we have

$$(\nabla_X \eta)Y = g(X,\phi Y) + g(\phi hX, Y), \tag{2.15}$$

for all vector fields $X, Y \in \chi(M)$.

According to Takahashi [20] we have the following:

Definition 2.2. A paracontact metric (k,μ)-manifold is said to be *ϕ-symmetric* if it satisfies

$$\phi^2((\nabla_W R)(X,Y)Z) = 0,$$

for any vector fields W, X, Y and $Z \in \chi(M)$. In addition, if the vector fields W, X, Y, Z are horizontal then the manifold is called *locally ϕ-symmetric*. It is to be noted that ϕ-symmetry implies locally ϕ-symmetry, but the converse is not true, in general.

3. Ricci Semisymmetric Paracontact Metric (k,μ)-manifolds

In this section we discuss about Ricci semisymmetric paracontact metric (k,μ)-manifolds. Suppose the paracontact metric (k,μ)-manifold M be Ricci semisymmetric. Then

$$R(X,Y)\cdot S = 0,$$

for all $X, Y \in \chi(M)$. This is equivalent to

$$(R(X,Y)\cdot S)(U,V) = 0, \tag{3.1}$$

for any $U, V, X, Y \in \chi(M)$. Thus we have

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{3.2}$$

Substituting $X = U = \xi$ in (3.2) yields

$$S(R(\xi,Y)\xi,V) + S(\xi,R(\xi,Y)V) = 0. \tag{3.3}$$

Using (2.11) we infer from (3.3)

$$S(R(\xi,Y)\xi,V) + 2nk\eta(R(\xi,Y)V) = 0. \tag{3.4}$$

From (2.6) it follows that

$$R(\xi,X)Y = k(g(X,Y)\xi - \eta(Y)X) + \mu(g(hX,Y)\xi - \eta(Y)hX). \tag{3.5}$$

With the help of (3.4) and (3.5) we get

$$kS(Y,V) + \mu S(hY,V) - 2nk^2 g(Y,V) - 2nk\mu g(hY,V) = 0. \tag{3.6}$$

Putting $Y = hY$ in (3.6) and using (2.8) we obtain

$$\mu(k+1)S(Y,V) + kS(hY,V) - 2nk^2 g(hY,V) - 2nk\mu(k+1)g(Y,V) = 0. \tag{3.7}$$

Now suppose $k < -1$ and $\mu \neq 0$. Multiplying Equation (3.6) by k and Equation (3.7) by μ, then subtract the results we get

$$\{k^2 - \mu^2(k+1)\}[S(Y,V) - 2nkg(Y,V)] = 0. \tag{3.8}$$

If $k < -1$, then $k^2 - \mu^2(k+1) \neq 0$. Therefore from (3.8) it follows that $S(Y,V) = 2nkg(Y,V)$, which implies that the manifold is Einstein.

Also, if we take $k < -1$ and $\mu = 0$, then (3.6) becomes

$$k[S(Y,V) - 2nkg(Y,V)] = 0. \tag{3.9}$$

This implies $S(Y,V) = 2nkg(Y,V)$, that is, the manifold is Einstein one.

Conversely, if the manifold is an Einstein manifold, then it can be easily shown that $R \cdot S = 0$.

This leads to the following:

Theorem 3.1. *A $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k,μ)-manifold with $k < -1$ is Ricci semisymmetric if and only if the manifold is Einstein.*

Again Ricci symmetry ($\nabla S = 0$) implies Ricci semisymmetric ($R \cdot S = 0$), therefore we have the following:

Corollary 3.1. *A $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k,μ)-manifold with $k < -1$ is Ricci symmetric if and only if the manifold is Einstein.*

Taking covariant derivative of (2.10) along an arbitrary vector field X, we have

$$\begin{aligned}(\nabla_X Q)Y =&(2(n-1)+\mu)(\nabla_X h)Y + [2(n-1) \\ &+ n(2k-\mu)][(\nabla_X \eta)Y\xi + \eta(Y)\nabla_X \xi].\end{aligned} \tag{3.10}$$

Using (2.13) in the above equation gives

$$\begin{aligned}(\nabla_X Q)Y =&(2(n-1)+\mu)[-\{(1+k)g(X,\phi Y) + g(X,\phi hY)\}\xi \\ &+ \eta(Y)\phi h(hX - X) - \mu\eta(X)\phi hY] + [2(n-1) + n(2k-\mu)] \\ &[g(X,\phi Y)\xi + g(\phi hX, Y)\xi + \eta(Y)(-\phi X + \phi hX)].\end{aligned} \tag{3.11}$$

Thus the condition $(\nabla_X Q)Y = 0$ holds if and only if $k = \frac{1}{n} - n$ and $\mu = -2(n-1)$. Hence we can state the following:

Corollary 3.2. *A $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k,μ)-manifold is Ricci symmetry ($\nabla S = 0$) if and only if $k = \frac{1}{n} - n$ and $\mu = -2(n-1)$.*

Together with Corollary 5.12 of [5] we have the following:

Corollary 3.3. *A $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k,μ)-manifold with $k < -1$ is Ricci symmetry ($\nabla S = 0$) if and only if the manifold is Einstein.*

Since semisymmetry ($R \cdot R = 0$) implies Ricci semisymmetry ($R \cdot S = 0$), we can state the following:

Corollary 3.4. *A $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k,μ)-manifold with $k < -1$ is semisymmetric if and only if the manifold is Einstein.*

Remark 3.1. If $k > -1$, then from Lemma 2.1 and Equation (3.8) yields $k^2 - \mu^2(k+1) = 0$.

4. ϕ-Ricci Symmetric Paracontact Metric (k, μ)-manifolds

In this section we characterize ϕ-Ricci symmetric paracontact metric (k, μ)-manifolds.

Definition 4.1.([7]) A paracontact metric (k, μ)-manifold is said to be *ϕ-Ricci symmetric* if it satisfies

$$\phi^2((\nabla_X Q)Y) = 0, \tag{4.1}$$

for any vector fields $X, Y \in \chi(M)$. The manifold is called *locally ϕ-Ricci symmetric* if (4.1) holds for any horizontal vector fields. It follows that ϕ-Ricci symmetry implies locally ϕ-Ricci symmetry, but the converse is not true.

Let M be a $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k, μ)-manifold. From (4.1) we have

$$(\nabla_X Q)Y - \eta((\nabla_X Q)Y)\xi = 0, \tag{4.2}$$

for any vector fields $X, Y \in \chi(M)$.
Taking inner product of (4.2) with arbitrary vector field Z we obtain

$$g((\nabla_X Q)Y, Z) - \eta((\nabla_X Q)Y)\eta(Z) = 0. \tag{4.3}$$

This implies

$$g(\nabla_X QY, Z) - S(\nabla_X Y, Z) - \eta((\nabla_X Q)Y)\eta(Z) = 0. \tag{4.4}$$

Substituting $Y = \xi$ in (4.4) gives

$$g(\nabla_X Q\xi, Z) - S(\nabla_X \xi, Z) - \eta((\nabla_X Q)\xi)\eta(Z) = 0. \tag{4.5}$$

Taking covariant derivative of (2.10) along arbitrary vector field X, we obtain

$$\begin{aligned}(\nabla_X Q)Y =& [2(n-1) + \mu](\nabla_X h)Y + [2(n-1) \\ &+ n(2k - \mu)]\{(\nabla_X \eta)(Y)\xi + \eta(Y)\nabla_X \xi\}.\end{aligned} \tag{4.6}$$

Also from (2.13) we get

$$(\nabla_X h)\xi = \phi h(hX - X). \tag{4.7}$$

Making use of (2.15), (4.7) and (4.6) one can easily obtain

$$\eta((\nabla_X Q)\xi) = 0. \tag{4.8}$$

Taking account of (2.3), (4.8) and (4.5) we have

$$2nkg(X, \phi Z) + 2nkg(\phi hX, Z) + S(\phi X, Z) - S(\phi hX, Z) = 0. \tag{4.9}$$

Replacing X by hX in (4.9) and using (2.8) gives that

$$S(\phi hX,Z)-(k+1)S(\phi X,Z)-2nkg(\phi hX,Z)+2nk(k+1)g(\phi X,Z)=0. \tag{4.10}$$

Adding (4.9) and (4.10) we obtain

$$k[S(\phi X,Z)-2nkg(\phi X,Z)]=0. \tag{4.11}$$

Since $k\neq 0$, (4.11) implies

$$S(\phi X,Z)=2nkg(\phi X,Z). \tag{4.12}$$

Putting $X=\phi X$ in (4.12) yields

$$S(X,Z)=2nkg(X,Z), \tag{4.13}$$

which shows that the manifold is an Einstein manifold.

Conversely, suppose $S(X,Z)=2nkg(X,Z)$, which implies $QX=2nkX$. Hence $(\nabla_Y Q)X=0$, that is, $\phi^2((\nabla_Y Q)X)=0$. Therefore the manifold is ϕ-Ricci symmetric. Thus we can state the following.

Theorem 4.1. *A $(2n+1)$-dimensional $(n>1)$ paracontact metric (k,μ)-manifold is ϕ-Ricci symmetric if and only if the manifold is an Einstein manifold, provided $k\neq 0,-1$.*

By the above arguments together with $\mu=0$ we have the following:

Corollary 4.1. *A $(2n+1)$-dimensional $(n>1)$ paracontact metric $N(k)$-manifold is ϕ-Ricci symmetric if and only if the manifold is an Einstein manifold, provided $k\neq 0,-1$.*

Taking covariant differentiation of (2.10) along an arbitrary vector field X, we obtain

$$\begin{aligned}(\nabla_X Q)Y =&(2(n-1)+\mu)(\nabla_X h)Y+[2(n-1)\\ &+n(2k-\mu)][(\nabla_X\eta)Y\xi+\eta(Y)\nabla_X\xi].\end{aligned} \tag{4.14}$$

Using (2.13) in the above equation, we get

$$\begin{aligned}(\nabla_X Q)Y =&(2(n-1)+\mu)[-\{(1+k)g(X,\phi Y)+g(X,\phi hY)\}\xi\\ &+\eta(Y)\phi h(hX-X)-\mu\eta(X)\phi hY]+[2(n-1)+n(2k-\mu)]\\ &[g(X,\phi Y)\xi+g(\phi hX,Y)\xi+\eta(Y)(-\phi X+\phi hX)].\end{aligned} \tag{4.15}$$

Applying ϕ^2 on both sides of (4.15) and making use of (2.8) gives

$$\begin{aligned}\phi^2((\nabla_X Q)Y) \quad = \quad & (2(n-1)+\mu)[\eta(Y)\{(k+1)\phi X-\phi hX\}-\mu\eta(X)\phi hY]\\ &+[2(n-1)+n(2k-\mu)][\eta(Y)(-\phi X+\phi hX)].\end{aligned} \tag{4.16}$$

This is equivalent to

$$\begin{aligned} \phi^2((\nabla_X Q)Y) &= \{\mu k + \mu - 2k + n\mu\}\eta(Y)\phi X + \{2nk - n\mu - \mu\}\eta(Y)\phi hX \\ &\quad -\mu\{2(n-1) + \mu\}\eta(X)\phi hY. \end{aligned} \tag{4.17}$$

From the foregoing equation we see that $\phi^2((\nabla_X Q)Y) = 0$ if and only if $k = \frac{1}{n} - n$ and $\mu = -2(n-1)$. This leads to the following:

Theorem 4.2. *A $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k, μ)-manifold is ϕ-Ricci symmetric if and only if $k = \frac{1}{n} - n$ and $\mu = -2(n-1)$.*

Hence from the Corollary 5.12 of [5] we conclude the following:

Corollary 4.2. *A $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k, μ)-manifold with $k < -1$ is ϕ-Ricci symmetric if and only if the manifold is Einstein.*

5. Second Order Parallel Tensor

Definition 5.1.([11]) A tensor α of second order is said to be *parallel* if $\nabla\alpha = 0$, where ∇ denotes the covariant differentiation with respect to the associated metric tensor.

Let α be a symmetric $(0, 2)$-tensor field on a paracontact metric (k, μ)-manifold M such that $\nabla\alpha = 0$. Then it follows that

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0, \tag{5.1}$$

for any vector fields $X, Y, Z, W \in \chi(M)$.
Substituting $X = Z = W = \xi$ in (5.1) and noticing α is symmetric implies

$$\alpha(R(\xi, Y)\xi, \xi) = 0. \tag{5.2}$$

Now we consider a non-empty connected open subset $\mathcal{U}$ of M and restrict our discussions to this set. Applying (2.6) in (5.2) yields

$$k\{g(Y, \xi)\alpha(\xi, \xi) - \alpha(Y, \xi)\} - \mu\alpha(hY, \xi) = 0. \tag{5.3}$$

We now consider the following cases:
Case 1. $k < -1$, $\mu = 0$,
Case 2. $k < -1$, $\mu \neq 0$.
For the Case 1, we have from (5.3)

$$g(Y, \xi)\alpha(\xi, \xi) - \alpha(Y, \xi) = 0. \tag{5.4}$$

Taking covariant differentiation of (5.4) along X, we obtain

$$\begin{aligned} \alpha(\nabla_X Y, \xi) + \alpha(Y, \nabla_X \xi) =& g(\nabla_X Y, \xi)\alpha(\xi, \xi) + g(Y, \nabla_X \xi)\alpha(\xi, \xi) \\ &+ 2g(Y, \xi)\alpha(\nabla_X \xi, \xi). \end{aligned} \tag{5.5}$$

Replacing Y by $\nabla_X Y$ in (5.4), we get

$$g(\nabla_X Y,\xi)\alpha(\xi,\xi) - \alpha(\nabla_X Y,\xi) = 0. \tag{5.6}$$

Using (5.5) and (5.6) we have

$$\alpha(Y,\nabla_X\xi) = g(Y,\nabla_X\xi)\alpha(\xi,\xi) + 2g(Y,\xi)\alpha(\nabla_X\xi,\xi). \tag{5.7}$$

Making use of (2.3) and (5.4) in (5.7) follows that

$$\alpha(Y,-\phi X) + \alpha(Y,\phi hX) = g(Y,-\phi X)\alpha(\xi,\xi) + g(Y,\phi hX)\alpha(\xi,\xi). \tag{5.8}$$

Changing X by ϕX in (5.8) and using (2.2) we have

$$\alpha(Y,X) + \alpha(Y,hX) = g(X,Y)\alpha(\xi,\xi) + g(Y,hX)\alpha(\xi,\xi). \tag{5.9}$$

Putting $X = hX$ in (5.9) and making use of (2.8) we obtain

$$\alpha(Y,hX) + (k+1)\alpha(Y,X) = g(hX,Y)\alpha(\xi,\xi) + (k+1)g(X,Y)\alpha(\xi,\xi). \tag{5.10}$$

Subtracting (5.9) from (5.10) and since $k \neq 0$ it follows that

$$\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y). \tag{5.11}$$

Since α and g are parallel tensor fields, $\alpha(\xi,\xi)$ must be constant on $\mathcal{U}$. Since $\mathcal{U}$ is an arbitrary open set of M, it follows that (5.11) holds on whole of M.

For Case 2, replacing Y by hY in (5.3) and using (2.8) we have

$$k\alpha(hY,\xi) + \mu(k+1)\{\alpha(Y,\xi) - g(Y,\xi)\alpha(\xi,\xi)\} = 0. \tag{5.12}$$

Multiplying Equation (5.3) by k and Equation (5.12) by μ (since $k < -1$ and $\mu \neq 0$), then adding the results we get

$$\{k^2 - \mu^2(k+1)\}[\alpha(Y,\xi) - g(Y,\xi)\alpha(\xi,\xi)] = 0. \tag{5.13}$$

Since $k < -1$, we see that $k^2 - \mu^2(k+1) \neq 0$. Hence, it follows from (5.13) that the relation (5.4) holds and then proceeding in the same way as in Case 1, we can show that $\alpha(X,Y) = \alpha(\xi,\xi)g(X,Y)$ for all $X, Y \in \chi(M)$.

Considering the above facts we can state the following:

Theorem 5.1. *Let M be a $(2n+1)$-dimensional $(n > 1)$ paracontact metric (k,μ)-manifold with $k < -1$. If M admits a second order symmetric parallel tensor then it is a constant multiple of the associated metric tensor.*

Application: Let us consider a paracontact metric (k,μ)-manifold which is Ricci symmetric, that is, $\nabla S = 0$. Since the Ricci tensor is symmetric $(0,2)$-tensor, thus applying Theorem 5.1, we have the following:

Corollary 5.1. *A $(2n+1)$-dimensional $(n > 1)$ Ricci symmetric $(\nabla S = 0)$ paracontact metric (k, μ)-manifold with $k < -1$ is an Einstein manifold.*

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