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Curvature Properties of η-Ricci Solitons on Para-Kenmotsu Manifolds

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Abstract. In the present paper, we study curvature properties of η-Ricci solitons on para-Kenmotsu manifolds. We obtain some results of η-Ricci solitons on para-Kenmotsu manifolds satisfying $R(\xi, X).C = 0$, $R(\xi, X).\widetilde{M} = 0$, $R(\xi, X).P = 0$, $R(\xi, X).\widetilde{C} = 0$ and $R(\xi, X).H = 0$, where C, $\widetilde{M}$, P, $\widetilde{C}$ and H are a quasi-conformal curvature tensor, a M-projective curvature tensor, a pseudo-projective curvature tensor, and a concircular curvature tensor and conharmonic curvature tensor, respectively.

1. Introduction

In 1982, Hamilton [12] introduced the notion of the Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij.}$$

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g). A Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0,$$

where S is a Ricci tensor of M and L_V denotes the Lie derivative operator along the vector field V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive, respectively [10]. Ricci solitons have been studied in many contexts: on Kähler manifolds [11], on contact and Lorentzian

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manifolds [2, 14, 15, 17, 18], on Sasakian [13], α-Sasakian [1] and K-contact manifolds [19, 7], on Kenmotsu [3] and f-Kenmotsu manifolds [8] etc. In paracontact geometry, Ricci solitons firstly appeared in the paper of G. Calvaruso and D. Perrone [16]. Recently, C. L. Bejan and M. Crasmareanu dealed with Ricci solitons on 3-dimensional normal paracontact manifolds [4]. A more general notion is that of η-Ricci soliton introduced by J. T. Cho and M. Kimura [9], which was treated by C. Calin and M. Crasmareanu on Hopf hypersurfaces in complex space forms [8]. η-Ricci solitons on para-Kenmotsu manifolds were studied by A. M. Blaga [5] and η-Ricci solitons on Lorentzian Para-Sasakian Manifolds were also studied by A. M. Blaga [6]. Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞ and ∇ be its Levi-Civita connection. The Riemannian-Christoffel curvature tensor R, the quasi-conformal curvature tensor C; the M-projective curvature tensor $\widetilde{M}$; pseudo-projective curvature tensor P; the concircular curvature tensor $\widetilde{C}$ and the conharmonic curvature tensor H of (M, g) are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

$$\begin{aligned} C(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

$$\begin{aligned} M(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY], \end{aligned}$$

$$\begin{aligned} P(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad - \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y],$$

and

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY], \end{aligned}$$

respectively, where Q is the Ricci operator, defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, $r = tr(S)$ is the scalar curvature and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

The paper is organized as follows:

In the present paper, we studied curvature properties of η-Ricci solitons on para-Kenmotsu manifolds. In section 2, we recall some well known basic formulas and properties of para-Kenmotsu manifolds. Section 3 contains a brief review of Ricci and η-Ricci solitons. In sections $4-8$, we obtained some interesting results on η-Ricci solitons in para-Kenmotsu manifolds satisfying $R(\xi, X).C = 0$, $R(\xi, X).\widetilde{M} = 0$, $R(\xi, X).P = 0$, $R(\xi, X).\widetilde{C} = 0$ and $R(\xi, X).H = 0$, where C, $\widetilde{M}$, P, $\widetilde{C}$ and H are quasi-conformal curvature tensor ; M-projective curvature tensor; pseudo-projective curvature tensor; concircular curvature tensor and conharmonic curvature tensor, respectively.

2. Para-Kenmotsu Manifolds

Let $(M, \varphi, \eta, \xi, g)$ be a n-dimensional smooth manifold, where φ is a tensor field of $(1,1)$-type, η a 1-form, ξ a vector field and g a pseudo-Riemannian metric on M. We say that (φ, η, ξ, g) is an almost paracontact metric structure on M, if satisfies the conditions [5]:

$$\nabla_X \xi = \varphi^2 X = X - \eta(X)\xi, \tag{2.1}$$

$$\varphi^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1, \tag{2.2}$$

$$\varphi\xi = 0, \eta \circ \varphi = 0 \text{ and } rank(\varphi) = n - 1, \tag{2.3}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

for any vector fields X and Y on M.

If, moreover

$$(\nabla_X \varphi)Y = -g(X, \varphi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the Levi-Civita connection of g, then the almost paracontact metric structure (φ, η, ξ, g) is called para-Kenmotsu manifold.

From the definition, it follows that η is the g-dual of ξ:

$$g(X, \xi) = \eta(X), \tag{2.6}$$

ξ is a unitary vector field:

$$g(\xi, \xi) = 1, \tag{2.7}$$

and φ is a g-skew-symmetric operator.

We shall further give some immediate properties of this structure.

Proposition 2.1. *On a para-Kenmotsu manifold* $(M, \varphi, \eta, \xi, g)$*, the following relations hold:*

$$\nabla\xi = I - \eta \otimes \xi, \tag{2.8}$$

$$\eta(\nabla_X \xi) = 0, \nabla_\xi \xi = 0, \tag{2.9}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.10}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.11}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.12}$$

$$\eta(R(X,Y)Z) = -\eta(X)g(Y,Z) + \eta(Y)g(X,Z), \qquad \eta(R(X,Y)\xi) = 0, \tag{2.13}$$

$$\nabla\eta = g - \eta \otimes \eta, \qquad \nabla_\xi \eta = 0, \tag{2.14}$$

$$L_\xi \varphi = 0, \quad L_\xi \eta = 0, \quad L_\xi(\eta \otimes \eta) = 0, \quad L_\xi g = 2(g - \eta \otimes \eta) \tag{2.15}$$

where R *is the Riemann curvature tensor field and* ∇ *is the Levi-Civita connection associated to* g*.*

3. Ricci and η-Ricci Solitons on $(M, \varphi, \xi, \eta, g)$

Let $(M, \varphi, \xi, \eta, g)$ be a paracontact metric manifold. Consider the equation

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{3.1}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, and λ and μ are real constants. Writing $L_\xi g$ in terms of the Levi-Civita connection ∇, we get

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{3.2}$$

for any $X, Y \in \chi(M)$, or equivalent:

$$S(X,Y) = -(\lambda + 1)g(X,Y) - (\mu - 1)\eta(X)\eta(Y), \tag{3.3}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (3.1) is said to be an η-Ricci soliton on M [8]; in particular, if $\mu = 0$, (g, ξ, λ) is a Ricci soliton [18] and it is called shrinking, steady or expanding according as λ is negative, zero or positive, respectively [19].

Taking $Y=\xi$ in (3.3), we get

$$S(X,\xi)=S(\xi,X)=-(\lambda+\mu)\eta(X). \tag{3.4}$$

On a n-dimensional paracontact manifold M, we have

$$S(X,\xi)=-(\dim(M)-1)\eta(X)=-(n-1)\eta(X), \tag{3.5}$$

so:

$$\lambda+\mu=n-1.$$

In this case, the Ricci operator Q defined by $g(QX,Y)=S(X,Y)$ has the expression:

$$QX=-(\lambda+1)X-(\mu-1)\eta(X)\xi. \tag{3.6}$$

The above equation yields that

$$r=-n(\lambda+1)-(\mu-1). \tag{3.7}$$

4. η-Ricci Solitons on Para-Kenmotsu Manifolds satisfying $R(\xi,X).C=0$

The Quasi-conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y)Z &= aR(X,Y)Z+b[S(Y,Z)X-S(X,Z)Y \\ &\quad +g(Y,Z)QX-g(X,Z)QY] \\ &\quad -\frac{r}{n}\left(\frac{a}{n-1}+2b\right)[g(Y,Z)X-g(X,Z)Y], \end{aligned} \tag{4.1}$$

where $a,b\neq 0$ are constants. Putting $Z=\xi$ in (4.1) and using (2.12), (3.3), (3.6), we obtain

$$C(X,Y)\xi=\left[a+b(2\lambda+\mu+1)+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)\right][\eta(X)Y-\eta(Y)X]. \tag{4.2}$$

Similarly using (2.13), (3.3), (3.4) and (3.6) in (4.1), we obtain

$$\begin{aligned} \eta(C(X,Y)Z) &= \left[a+b(2\lambda+\mu+1)+\frac{r}{n}\left(\frac{a}{n-1}+2b\right)\right] \\ &\quad [g(X,Z)\eta(Y)-g(Y,Z)\eta(X)]. \end{aligned} \tag{4.3}$$

The condition that must be satisfied by R is:

$$\begin{aligned} & R(\xi,X)C(U,V)W-C(R(\xi,X)U,V)W \\ & -C(U,R(\xi,X)V)W-C(U,V)R(\xi,X)W \\ = \;& 0. \end{aligned} \tag{4.4}$$

By virtue of (2.11) and (4.4), we get

$$
\begin{aligned}
(4.5) \qquad & \eta(C(U,V)W)X - g(X, C(U,V)W)\xi - \eta(U)C(X,V)W \\
& + g(X,U)C(\xi,V)W - \eta(V)C(U,X)W + g(X,V)C(U,\xi)W \\
& - \eta(W)C(U,V)X + g(X,W)C(U,V)\xi \\
= \; & 0.
\end{aligned}
$$

Taking the inner product with ξ , the relation (4.5) becomes:

$$
\begin{aligned}
(4.6) \qquad & \eta(C(U,V)W)\eta(X) - g(X, C(U,V)W) - \eta(U)\eta(C(X,V)W) \\
& + g(X,U)\eta(C(\xi,V)W) - \eta(V)\eta(C(U,X)W) + g(X,V)\eta(C(U,\xi)W) \\
& - \eta(W)\eta(C(U,V)X) + g(X,W)\eta(C(U,V)\xi) \\
= \; & 0.
\end{aligned}
$$

By virtue of $(4.2), (4.3)$ and (4.6), we get

$$
\begin{aligned}
(4.7) \qquad g(X, C(U,V)W) &= \left[a + b(2\lambda + \mu + 1) + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right] \\
& \quad [g(X,V)g(U,W) - g(X,U)g(V,W)].
\end{aligned}
$$

By using (4.1) in (4.7) and putting $X = U = e_i$, summing over $i = 1, 2, ..., n$ and on simplification, we have

$$
\begin{aligned}
(4.8) \qquad [a + b(n-2)]\, S(V,W) &= (1-n)(a + b(2\lambda + \mu + 1))g(V,W) \\
& \quad - rbg(V,W).
\end{aligned}
$$

Taking $V = W = \xi$ in (4.8) and using (2.4), (3.7), we find the following equation

$$
(4.9) \qquad \lambda = -\mu + n - 1.
$$

Thus, we can state the following theorem:

Theorem 4.1. *If (φ, ξ, η, g) is a para-Kenmotsu structure on the n-dimensional manifold M, $(\varphi, \xi, \lambda, \mu)$ is an η-Ricci soliton on M and $R(\xi, X).C = 0$, then $\lambda + \mu - (n-1) = 0$ and (M, g) is an Einstein manifold.*

5. η-Ricci Solitons on Para-Kenmotsu Manifolds satisfying $R(\xi, X).\widetilde{M} = 0$

The M-projective curvature tensor $\widetilde{M}$ is defined by

$$
\begin{aligned}
\widetilde{M}(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y \\
(5.1) \qquad & \quad + g(Y,Z)QX - g(X,Z)QY].
\end{aligned}
$$

Putting $Z = \xi$ in (5.1) and using (2.12), (3.3), (3.6), we obtain

$$\widetilde{M}(X,Y)\xi = \left[1 - \frac{(2\lambda+\mu+1)}{2(n-1)}\right][\eta(X)Y - \eta(Y)X]. \tag{5.2}$$

Similarly using (2.8), (3.3), (3.4), (3.6) in (5.1), we obtain

$$\eta(\widetilde{M}(X,Y)Z) = \left[1 - \frac{(2\lambda+\mu+1)}{2(n-1)}\right][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \tag{5.3}$$

The condition that must be satisfied by R is:

$$R(\xi,X)\widetilde{M}(Y,Z)W - \widetilde{M}(R(\xi,X)Y,Z)W - \widetilde{M}(Y,R(\xi,X)Z)W - \widetilde{M}(Y,Z)R(\xi,X)W = 0. \tag{5.4}$$

By virtue of (2.11) and (5.4), we get

$$\begin{aligned} &\eta(\widetilde{M}(Y,Z)W)X - g(X,\widetilde{M}(Y,Z)W)\xi - \eta(Y)\widetilde{M}(X,Z)W \\ &+g(X,Y)\widetilde{M}(\xi,Z)W - \eta(Z)\widetilde{M}(Y,X)W + g(X,Z)\widetilde{M}(Y,\xi)W \\ &-\eta(W)\widetilde{M}(Y,Z)X + g(X,W)\widetilde{M}(Y,Z)\xi \\ = {}& 0. \end{aligned} \tag{5.5}$$

Taking the inner product with ξ, the relation (5.5) becomes:

$$\begin{aligned} &\eta(\widetilde{M}(Y,Z)W)\eta(X) - g(X,\widetilde{M}(Y,Z)W) - \eta(Y)\eta(\widetilde{M}(X,Z)W) \\ &+g(X,Y)\eta(\widetilde{M}(\xi,Z)W) - \eta(Z)\eta(\widetilde{M}(Y,X)W) + g(X,Z)\eta(\widetilde{M}(Y,\xi)W) \\ &-\eta(W)\eta(\widetilde{M}(Y,Z)X) + g(X,W)\eta(\widetilde{M}(Y,Z)\xi) \\ = {}& 0. \end{aligned} \tag{5.6}$$

By virtue (5.2), (5.3) and (5.6), we have

$$g(X,\widetilde{M}(Y,Z)W) = \left[1 - \frac{(2\lambda+\mu+1)}{2(n-1)}\right][g(X,Z)g(Y,W) - g(X,Y)g(Z,W)]. \tag{5.7}$$

By using (5.1) in (5.7) and Putting $X = Y = e_i$, summing over $i = 1, 2, ..., n$ and on simplification, we have

$$S(Z,W) = \left[1 - 2\lambda - \mu + \frac{r-1}{(n-1)}\right] g(Z,W). \tag{5.8}$$

Taking $V = W = \xi$ in (5.8) and by virtue of (3.6), (3.7), we find the following equation

$$(2n-1)\lambda + \mu + 1 = 0. \tag{5.9}$$

Thus, we can state the following theorem:

Theorem 5.1. *If* (φ, ξ, η, g) *is a para-Kenmotsu structure on the* n*-dimensional manifold* M*,* $(\varphi, \xi, \lambda, \mu)$ *is an* η*-Ricci soliton on* M *and* $R(\xi, X).\widetilde{M} = 0$*, then* $(2n-1)\lambda + \mu + 1 = 0$ *and* (M, g) *is an Einstein manifold.*

6. η-Ricci Solitons on Para-Kenmotsu Manifolds satisfying $R(\xi, X).P = 0$

The Pseudo-projective curvature tensor P is defined by

$$\begin{aligned} P(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ &\quad -\frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{6.1}$$

where $a, b \neq 0$ are constants. Putting $Z = \xi$ in (6.1) and using $(2.12), (3.3), (3.6)$, we obtain

$$P(X,Y)\xi = \left[a + (\lambda+\mu)b + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right][\eta(X)Y - \eta(Y)X]. \tag{6.2}$$

Similarly using $(2.13), (3.3), (3.4), (3.6)$ in (6.1), we obtain

$$\begin{aligned} \eta(P(X,Y)Z) &= \left[a + (\lambda+\mu)b + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right] \\ &\quad [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \end{aligned} \tag{6.3}$$

The condition that must be satisfied by R is:

$$\begin{aligned} & R(\xi,X)P(U,V)W - P(R(\xi,X)U,V)W \\ & -P(U,R(\xi,X)V)W - P(U,V)R(\xi,X)W \\ = {} & 0. \end{aligned} \tag{6.4}$$

By virtue of (2.11) and (6.4), we get

$$\begin{aligned} & \eta(P(U,V)W)X - g(X,P(U,V)W)\xi - \eta(U)P(X,V)W \\ & +g(X,U)P(\xi,V)W - \eta(V)P(U,X)W + g(X,V)P(U,\xi)W \\ & -\eta(W)P(U,V)X + g(X,W)P(U,V)\xi \\ = {} & 0. \end{aligned} \tag{6.5}$$

Taking the inner product with ξ, the relation (6.5) becomes:

$$\begin{aligned} &\eta(P(U,V)W)\eta(X) - g(X,P(U,V)W) - \eta(U)\eta(P(X,V)W) \\ &+g(X,U)\eta(P(\xi,V)W) - \eta(V)\eta(P(U,X)W) + g(X,V)\eta(P(U,\xi)W) \\ &-\eta(W)\eta(P(U,V)X) + g(X,W)\eta(P(U,V)\xi) \\ = \;& 0. \end{aligned} \tag{6.6}$$

By virtue of $(6.2), (6.3)$ and (6.6), we have

$$\begin{aligned} g(X,P(U,V)W) = & \left[a + (\lambda+\mu)b + \frac{r}{n}\left(\frac{a}{n-1} + b\right)\right] \\ & [g(X,V)g(U,W) - g(X,U)g(V,W)]. \end{aligned} \tag{6.7}$$

By using (6.1) in (6.7) and Putting $X = U = e_i$, summing over $i = 1, 2, ..., n$ and on simplification, we obtain

$$\begin{aligned} aS(V,W) = & (1-n)[a + b(\mu-1)]g(V,W) \\ & -(n-1)(\mu-1)b\eta(V)\eta(W). \end{aligned} \tag{6.8}$$

Taking $V = W = \xi$ in (6.8) and by virtue of $(3.4), (3.7)$, we find the following equation

$$\lambda + \mu - (n-1) = 0. \tag{6.9}$$

Thus, we can state the following theorem:

Theorem 6.1. *If (φ, ξ, η, g) is a para-Kenmotsu structure on the n-dimensional manifold M, $(\varphi, \xi, \lambda, \mu)$ is an η-Ricci soliton on M and $R(\xi, X).P = 0$, then $\lambda + \mu - (n-1) = 0$ and (M, g) is an η-Einstein manifold.*

7. η-Ricci Solitons on Para-Kenmotsu Manifolds satisfying $R(\xi, X).\widetilde{C} = 0$

The concircular curvature tensor $\widetilde{C}$ is defined by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{7.1}$$

Taking $Z = \xi$ in (7.1) and using $(2.12), (3.3), (3.6)$, we get

$$\widetilde{C}(X,Y)\xi = \left[1 + \frac{r}{n(n-1)}\right][\eta(X)Y - \eta(Y)X]. \tag{7.2}$$

Similarly using $(2.13), (3.3), (3.4), (3.6)$ in (7.1), we have

$$\eta(\widetilde{C}(X,Y)Z) = \left[1 + \frac{r}{n(n-1)}\right][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \tag{7.3}$$

The condition that must be satisfied by R is:

$$\begin{aligned} (7.4) \qquad & R(\xi,X)\widetilde{C}(U,V)W-\widetilde{C}(R(\xi,X)U,V)W \\ & -\widetilde{C}(U,R(\xi,X)V)W-\widetilde{C}(U,V)R(\xi,X)W \\ = \;& 0. \end{aligned}$$

By virtue of (2.11) and (7.4), we have

$$\begin{aligned} (7.5) \qquad & \eta(\widetilde{C}(U,V)W)X-g(X,\widetilde{C}(U,V)W)\xi-\eta(U)\widetilde{C}(X,V)W \\ & +g(X,U)\widetilde{C}(\xi,V)W-\eta(V)\widetilde{C}(U,X)W+g(X,V)\widetilde{C}(U,\xi)W \\ & -\eta(W)\widetilde{C}(U,V)X+g(X,W)\widetilde{C}(U,V)\xi \\ = \;& 0. \end{aligned}$$

Taking the inner product with ξ, the relation (7.5) becomes:

$$\begin{aligned} (7.6) \qquad & \eta(\widetilde{C}(U,V)W)\eta(X)-g(X,\widetilde{C}(U,V)W)-\eta(U)\eta(\widetilde{C}(X,V)W) \\ & +g(X,U)\eta(\widetilde{C}(\xi,V)W)-\eta(V)\eta(\widetilde{C}(U,X)W)+g(X,V)\eta(\widetilde{C}(U,\xi)W) \\ & -\eta(W)\eta(\widetilde{C}(U,V)X)+g(X,W)\eta(\widetilde{C}(U,V)\xi) \\ = \;& 0. \end{aligned}$$

By virtue of $(7.2), (7.3)$ and (7.6), we get

$$\begin{aligned} (7.7) \qquad g(X,\widetilde{C}(U,V)W) \;=\; & \left[1+\frac{r}{n(n-1)}\right] \\ & [g(X,V)g(U,W)-g(X,U)g(V,W)]. \end{aligned}$$

By using (7.1) in (7.7) and Putting $X=U=e_i$, summing over $i=1,2,...,n$ and on simplification, we obtain

$$(7.8) \qquad S(V,W)=(1-n)g(V,W).$$

Taking $V=W=\xi$ in (7.8) and by virtue of $(3.4), (3.7)$, we find the following equation

$$(7.9) \qquad \lambda+\mu-(n-1)=0.$$

Thus, we can state the following theorem:

Theorem 7.1. *If (φ,ξ,η,g) is a para-Kenmotsu structure on the n-dimensional manifold M, $(\varphi,\xi,\lambda,\mu)$ is an η-Ricci soliton on M and $R(\xi,X).\widetilde{C}=0$, then $\lambda+\mu-(n-1)=0$ and (M,g) is an Einstein manifold.*

8. η-Ricci Solitons on Para-Kenmotsu Manifolds satisfying $R(\xi, X).H = 0$

The conharmonic curvature tensor H is defined by

$$\begin{aligned} H(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY]. \end{aligned} \tag{8.1}$$

Putting $Z = \xi$ in (8.1) and using (2.12), (3.3), (3.6), we obtain

$$H(X,Y)\xi = \left[1 - \frac{(2\lambda + \mu + 1)}{(n-2)}\right][\eta(X)Y - \eta(Y)X]. \tag{8.2}$$

Similarly using (2.8), (2.13), (2.14), (3.5) in (8.1), we have

$$\begin{aligned} \eta(H(X,Y)Z) &= \left[1 - \frac{(2\lambda + \mu + 1)}{(n-2)}\right] \\ &\quad [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \end{aligned} \tag{8.3}$$

The condition that must be satisfied by R is:

$$\begin{aligned} & R(\xi,X)H(Y,Z)W - H(R(\xi,X)Y,Z)W \\ & -H(Y,R(\xi,X)Z)W - H(Y,Z)R(\xi,X)W \\ = \;& 0. \end{aligned} \tag{8.4}$$

By virtue of (2.11) and (8.4), we get

$$\begin{aligned} & \eta(H(Y,Z)W)X - g(X,H(Y,Z)W)\xi - \eta(Y)H(X,Z)W \\ & +g(X,Y)H(\xi,Z)W - \eta(Z)H(Y,X)W + g(X,Z)H(Y,\xi)W \\ & -\eta(W)H(Y,Z)X + g(X,W)H(Y,Z)\xi \\ = \;& 0. \end{aligned} \tag{8.5}$$

Taking the inner product with ξ, the relation (8.5) becomes:

$$\begin{aligned} & \eta(H(Y,Z)W)\eta(X) - g(X,H(Y,Z)W) - \eta(Y)\eta(H(X,Z)W) \\ & +g(X,Y)\eta(H(\xi,Z)W) - \eta(Z)\eta(H(Y,X)W) + g(X,Z)\eta(H(Y,\xi)W) \\ & -\eta(W)\eta(H(Y,Z)X) + g(X,W)\eta(H(Y,Z)\xi) \\ = \;& 0. \end{aligned} \tag{8.6}$$

By virtue of (8.2), (8.3) and (8.6), we get

$$\begin{aligned} g(X,H(Y,Z)W) &= \left[1 - \frac{(2\lambda + \mu + 1)}{(n-2)}\right] \\ &\quad [g(X,Z)g(Y,W) - g(X,Y)g(Z,W)]. \end{aligned} \tag{8.7}$$

By using (8.1) in (8.7) and Putting $X = Y = e_i$, summing over $i = 1, 2, ..., n$ and on simplification, we obtain

$$\left[\frac{r}{(n-2)} + \left(1 - \frac{(2\lambda + \mu + 1)}{(n-2)}\right)(1-n)\right] g(Z, W) = 0, \tag{8.8}$$

where $g(Z, W) \neq 0$. Therefore, we get

$$\left[\frac{r}{(n-2)} + \left(1 - \frac{(2\lambda + \mu + 1)}{(n-2)}\right)(1-n)\right] = 0, \tag{8.9}$$

on simplification, we obtain

$$\lambda + \mu - (n-1) = 0. \tag{8.10}$$

Thus, we can state the following theorem:

Theorem 8.1. *If (φ, ξ, η, g) is a para-Kenmotsu structure on the n-dimensional manifold M, $(\varphi, \xi, \lambda, \mu)$ is an η-Ricci soliton on M and $R(\xi, X).H = 0$, then $\lambda + \mu - (n-1) = 0$.*

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