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New Methods of Construction for Biharmonic Maps

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Abstract. In this paper we study some properties of Riemannian manifolds, we construct a new example of non-harmonic biharmonic maps, and we characterize the biharmonicity of some curves on Riemannian manifolds.

1. Preliminaries and Notations

Let (M, g) be a Riemannian manifold. By R, Ric and Ricci we denote respectively the Riemannian curvature tensor, the Ricci curvature and the Ricci tensor of (M, g). Thus R, Ric and Ricci are defined by:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.1}$$

$$\mathrm{Ric}(X,Y) = g(R(X,e_i)e_i, Y), \quad \mathrm{Ricci}\, X = R(X,e_i)e_i, \tag{1.2}$$

where ∇ is the Levi-Civita connection with respect to g, $\{e_i\}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. Given a smooth function f on M, the gradient of f is defined by

$$g(\mathrm{grad}\, f, X) = X(f), \quad \mathrm{grad}\, f = e_i(f)e_i, \tag{1.3}$$

the Hessian of f is defined by

$$\mathrm{Hess}_f(X,Y) = g(\nabla_X \mathrm{grad}\, f, Y), \tag{1.4}$$

where $X, Y \in \Gamma(TM)$, the Laplacian of f is defined by

$$\Delta f = \mathrm{trace}\, \mathrm{Hess}_f = g(\nabla_{e_i} \mathrm{grad}\, f, e_i). \tag{1.5}$$

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(For more details, see for example [6]).
Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds, the tension field of φ is given by

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi = \nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i), \tag{1.6}$$

where $\{e_i\}$ is an orthonormal frame on (M, g), and ∇^{φ} denote the pull-back connection on $\varphi^{-1}TN$. Then, φ is called harmonic map if the tension field vanishes, i.e. $\tau(\varphi) = 0$ (for more details on the concept of harmonic maps see [2, 3, 4]). We define the index form for harmonic maps by (see [8]):

$$I(v, w) = \int_M h(J_{\varphi}(v), w)v^g, \quad v, w \in \Gamma(\varphi^{-1}TN) \tag{1.7}$$

(or over any compact subset $D \subset M$), where:

$$\begin{aligned} J_{\varphi}(v) &= -\operatorname{trace} R^N(v, d\varphi)d\varphi - \operatorname{trace}(\nabla^{\varphi})^2 v \\ &= -R^N(v, d\varphi(e_i))d\varphi(e_i) - \nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} v + \nabla^{\varphi}_{\nabla_{e_i} e_i} v, \end{aligned} \tag{1.8}$$

R^N is the curvature tensor of (N, h), ∇^N is the Levi-Civita connection of (N, h), and v^g is the volume form of (M, g). If $\tau_2(\varphi) \equiv J_{\varphi}(\tau(\varphi))$ is null on M, then φ is called a biharmonic map (see [3], [5]).

2. The Riemannian Manifold $(M, \widetilde{g})$

Definition 2.1. Let M be a Riemannian manifold equipped with Riemannian metric g, and let $f \in C^{\infty}(M)$. We define on M a Riemannian metric, denoted $\widetilde{g}$, by $\widetilde{g} = g + df \otimes df$. For $X, Y \in \Gamma(TM)$, we have the following

$$\widetilde{g}(X, Y)_x = g(X, Y)_x + X(f)_x Y(f)_x, \quad \forall x \in M. \tag{2.1}$$

The Levi-Civita connection of $(M, \widetilde{g})$ can now be related to those of (M, g) as follows.

Theorem 2.2. *Let (M, g) be a Riemannian manifold, if $\widetilde{\nabla}$ denote the Levi-Civita connection of $(M, \widetilde{g})$, then*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{\operatorname{Hess}_f(X, Y)}{1 + \|\operatorname{grad} f\|^2} \operatorname{grad} f, \tag{2.2}$$

where ∇ is the Levi-Civita connection of (M, g), Hess_f (resp. $\operatorname{grad} f$) is the Hessian (resp. the gradient vector) of f with respect to g, and $\|\operatorname{grad} f\|^2 = g(\operatorname{grad} f, \operatorname{grad} f)$.

Proof. Let $X, Y, Z \in \Gamma(TM)$, from the Koszul formula (see [6]), we have

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_X Y, Z) &= 2g(\nabla_X Y, Z) + X(Y(f)Z(f)) + Y(Z(f)X(f)) \\ &\quad - Z(X(f)Y(f)) + Z(f)[X, Y](f) + Y(f)[Z, X](f) \\ &\quad - X(f)[Y, Z](f), \end{aligned} \tag{2.3}$$

let $\{E_i\}$ be a geodesic frame on (M,g) at $x\in M$, by (2.3) we obtain

$$\begin{aligned}2\widetilde{g}(\widetilde{\nabla}_XY,E_i) &= 2g(\nabla_XY,E_i)+X(Y(f)g(E_i,\operatorname{grad} f))\\ &\quad +Y(X(f)g(E_i,\operatorname{grad} f))-E_i(g(X,\operatorname{grad} f)g(Y,\operatorname{grad} f))\\ &\quad +E_i(f)[X,Y](f)+Y(f)(\nabla_{E_i}X)(f)+X(f)(\nabla_{E_i}Y)(f), \end{aligned} \tag{2.4}$$

from equation (2.4), and the definition of Hessian (1.4), we get

$$\begin{aligned}\widetilde{g}(\widetilde{\nabla}_XY,E_i) &= g(\nabla_XY,E_i)+g(\nabla_XY,\operatorname{grad} f)g(E_i,\operatorname{grad} f)\\ &\quad +\operatorname{Hess}_f(X,Y)g(E_i,\operatorname{grad} f),\end{aligned} \tag{2.5}$$

from equation (2.5), we obtain

$$\begin{aligned}\widetilde{g}(\widetilde{\nabla}_XY,Z) &= g(\nabla_XY,Z)+g(\nabla_XY,\operatorname{grad} f)g(Z,\operatorname{grad} f)\\ &\quad +\operatorname{Hess}_f(X,Y)g(Z,\operatorname{grad} f),\end{aligned} \tag{2.6}$$

so that

$$\widetilde{g}(\widetilde{\nabla}_XY,Z) = \widetilde{g}(\nabla_XY,Z)+\operatorname{Hess}_f(X,Y)Z(f). \tag{2.7}$$

Hence Theorem 2.2 follows from (2.7), with

$$Z(f)=\frac{1}{1+\|\operatorname{grad} f\|^2}\widetilde{g}(Z,\operatorname{grad} f). \tag{2.8}$$ □

Now consider the curvature tensor $\widetilde{R}$ of $(M,\widetilde{g})$, writing R for the curvature tensor of (M,g). We have the following result:

Theorem 2.3. *Let (M,g) be a Riemannian manifold, and let $f\in C^\infty(M)$. Then, for all $X,Y,Z\in\Gamma(TM)$, we have*

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z+\frac{g(R(X,Y)\operatorname{grad} f,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f\\ &\quad -\frac{\operatorname{Hess}_f(X,\operatorname{grad} f)\operatorname{Hess}_f(Y,Z)}{(1+\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f\\ &\quad +\frac{\operatorname{Hess}_f(Y,\operatorname{grad} f)\operatorname{Hess}_f(X,Z)}{(1+\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f\\ &\quad +\frac{\operatorname{Hess}_f(Y,Z)}{1+\|\operatorname{grad} f\|^2}\nabla_X\operatorname{grad} f-\frac{\operatorname{Hess}_f(X,Z)}{1+\|\operatorname{grad} f\|^2}\nabla_Y\operatorname{grad} f.\end{aligned} \tag{2.9}$$

Proof. By the definition of the curvature tensor $\widetilde{R}$,

$$\widetilde{R}(X,Y)Z=\widetilde{\nabla}_X\widetilde{\nabla}_YZ-\widetilde{\nabla}_Y\widetilde{\nabla}_XZ-\widetilde{\nabla}_{[X,Y]}Z, \tag{2.10}$$

and Theorem 2.2, we obtain

$$\begin{aligned}\widetilde{R}(X,Y)Z &= \widetilde{\nabla}_X(\nabla_Y Z + \frac{\mathrm{Hess}_f(Y,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f)\\ &\quad -\widetilde{\nabla}_Y(\nabla_X Z + \frac{\mathrm{Hess}_f(X,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f)\\ &\quad -(\nabla_{[X,Y]} Z + \frac{\mathrm{Hess}_f([X,Y],Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f), \qquad (2.11)\end{aligned}$$

the first term of (2.11) is given by

$$\begin{aligned}\widetilde{\nabla}_X(\nabla_Y Z &+ \frac{\mathrm{Hess}_f(Y,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f)\\ &= \nabla_X(\nabla_Y Z + \frac{\mathrm{Hess}_f(Y,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f)\\ &\quad + \frac{\mathrm{Hess}_f(X, \nabla_Y Z + \frac{\mathrm{Hess}_f(Y,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f, \qquad (2.12)\end{aligned}$$

by equation (2.12), and the definition of Hessian (1.4), we obtain

$$\begin{aligned}\widetilde{\nabla}_X(\nabla_Y Z &+ \frac{\mathrm{Hess}_f(Y,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f)\\ &= \nabla_X\nabla_Y Z + \frac{g(\nabla_X\nabla_Y \operatorname{grad} f, Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f\\ &\quad + \frac{\mathrm{Hess}_f(Y,\nabla_X Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f - \frac{\mathrm{Hess}_f(X,\operatorname{grad} f)\,\mathrm{Hess}_f(Y,Z)}{(1+\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f\\ &\quad + \frac{\mathrm{Hess}_f(Y,Z)}{1+\|\operatorname{grad} f\|^2}\nabla_X \operatorname{grad} f + \frac{\mathrm{Hess}_f(X,\nabla_Y Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f, \qquad (2.13)\end{aligned}$$

using the similar method, the second term of (2.11) is given by

$$\begin{aligned}-\widetilde{\nabla}_Y(\nabla_X Z &+ \frac{\mathrm{Hess}_f(X,Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f)\\ &= -\nabla_Y\nabla_X Z - \frac{g(\nabla_Y\nabla_X \operatorname{grad} f, Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f\\ &\quad - \frac{\mathrm{Hess}_f(X,\nabla_Y Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f + \frac{\mathrm{Hess}_f(Y,\operatorname{grad} f)\,\mathrm{Hess}_f(X,Z)}{(1+\|\operatorname{grad} f\|^2)^2}\operatorname{grad} f\\ &\quad - \frac{\mathrm{Hess}_f(X,Z)}{1+\|\operatorname{grad} f\|^2}\nabla_Y \operatorname{grad} f - \frac{\mathrm{Hess}_f(Y,\nabla_X Z)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f. \qquad (2.14)\end{aligned}$$

Theorem 2.3 follows from equations (2.11), (2.13) and (2.14). □

3. The Biharmonicity of the Identity Map

Let (M,g) be a Riemannian manifold, $f \in C^\infty(M)$, and denote by

$$\begin{aligned}\widetilde{I} : (M,g) &\longrightarrow (M,\widetilde{g}),\\ x &\longmapsto x\end{aligned}$$

the identity map.

Theorem 3.1. *If $\|\operatorname{grad} f\| = 1$, then the identity map $\widetilde{I}$ is a proper biharmonic if and only if the function f is non-harmonic on M, and satisfying the following*

$$\begin{aligned}(\Delta f)\operatorname{Ricci}(\operatorname{grad} f) &= -(\Delta^2 f)\operatorname{grad} f - \nabla_{\operatorname{grad}(\Delta f)}\operatorname{grad} f\\ &\quad -\frac{\Delta f}{2}\operatorname{grad}(\Delta f),\end{aligned}$$

where Δf is the Laplacian of f with respect to g, and $\Delta^2 f = \Delta(\Delta f)$.

Proof. Let $\{E_i\}$ be a normal orthonormal frame on (M,g) at x, we have

$$\begin{aligned}\tau(\widetilde{I}) &= \nabla^{\widetilde{I}}_{E_i} d\widetilde{I}(E_i) - d\widetilde{I}(\nabla_{E_i}E_i)\\ &= \widetilde{\nabla}_{E_i}E_i\\ &= \frac{\operatorname{Hess}_f(E_i,E_i)}{1+\|\operatorname{grad} f\|^2}\operatorname{grad} f\\ &= \frac{\Delta f}{2}\operatorname{grad} f, \qquad (3.1)\end{aligned}$$

note that $\widetilde{I}$ is harmonic if and only if $\Delta f = 0$, i.e. the function f is harmonic on (M,g). We compute the bitension field of the identity $\widetilde{I}$, we have

$$\begin{aligned}\widetilde{R}(\tau(\widetilde{I}), d\widetilde{I}(E_i))d\widetilde{I}(E_i) &= \frac{\Delta f}{2}\widetilde{R}(\operatorname{grad} f, E_i)E_i\\ &= \frac{\Delta f}{2}\Big(R(\operatorname{grad} f, E_i)E_i\\ &\quad +\frac{1}{2}g(R(\operatorname{grad} f, E_i)\operatorname{grad} f, E_i)\operatorname{grad} f\\ &\quad -\frac{1}{4}\operatorname{Hess}_f(\operatorname{grad} f, \operatorname{grad} f)\operatorname{Hess}_f(E_i,E_i)\operatorname{grad} f\\ &\quad +\frac{1}{4}\operatorname{Hess}_f(E_i, \operatorname{grad} f)\operatorname{Hess}_f(\operatorname{grad} f, E_i)\operatorname{grad} f\\ &\quad +\frac{1}{2}\operatorname{Hess}_f(E_i,E_i)\nabla_{\operatorname{grad} f}\operatorname{grad} f\\ &\quad -\frac{1}{2}\operatorname{Hess}_f(\operatorname{grad} f, E_i)\nabla_{E_i}\operatorname{grad} f\Big), \qquad (3.2)\end{aligned}$$

since $\|\operatorname{grad} f\|$ is constant on M, we obtain

$$\operatorname{Hess}_f(\operatorname{grad} f, X) = 0, \quad \nabla_{\operatorname{grad} f}\operatorname{grad} f = 0,$$

for all $X \in \Gamma(TM)$, from (3.2) and the definition of Ricci curvature, we get

$$\begin{aligned}\widetilde{R}(\tau(\widetilde{I}), d\widetilde{I}(E_i))d\widetilde{I}(E_i) &= \frac{\Delta f}{2}\Big(\operatorname{Ricci}(\operatorname{grad} f)\\ &\quad -\frac{1}{2}\operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f\Big).\end{aligned} \tag{3.3}$$

We compute

$$\begin{aligned}\nabla^{\widetilde{I}}_{E_i}\nabla^{\widetilde{I}}_{E_i}\tau(\widetilde{I}) - \nabla^{\widetilde{I}}_{\nabla_{E_i}E_i}\tau(\widetilde{I}) &= \frac{1}{2}\widetilde{\nabla}_{E_i}\widetilde{\nabla}_{E_i}\Delta f \operatorname{grad} f\\ &= \frac{1}{2}\widetilde{\nabla}_{E_i}\nabla_{E_i}\Delta f \operatorname{grad} f\\ &= \frac{1}{2}\Big(\nabla_{E_i}\nabla_{E_i}\Delta f \operatorname{grad} f\\ &\quad +\frac{1}{2}\operatorname{Hess}_f(E_i, \nabla_{E_i}\Delta f \operatorname{grad} f)\operatorname{grad} f\Big),\end{aligned} \tag{3.4}$$

by definitions (1.5), (1.3), (1.4), we get

$$\begin{aligned}\nabla_{E_i}\nabla_{E_i}\Delta f \operatorname{grad} f &= (\Delta^2 f)\operatorname{grad} f + 2\nabla_{\operatorname{grad}(\Delta f)}\operatorname{grad} f\\ &\quad +(\Delta f)\operatorname{trace}(\nabla)^2\operatorname{grad} f,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{1}{2}\operatorname{Hess}_f(E_i, \nabla_{E_i}\Delta f \operatorname{grad} f) &= \frac{\Delta f}{2} g(\nabla_{E_i}\operatorname{grad} f, \nabla_{E_i}\operatorname{grad} f)\\ &= -\frac{\Delta f}{2} g(\operatorname{grad} f, \operatorname{trace}(\nabla)^2\operatorname{grad} f),\end{aligned} \tag{3.6}$$

from equations (3.3), (3.4), (3.5), (3.6), and the following

$$\operatorname{trace}(\nabla)^2\operatorname{grad} f = \operatorname{Ricci}(\operatorname{grad} f) + \operatorname{grad}(\Delta f),$$

the identity map $\widetilde{I}$ is a proper biharmonic map if and only if

$$\begin{aligned}&2(\Delta f)\operatorname{Ricci}(\operatorname{grad} f) - (\Delta f)\operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)\operatorname{grad} f\\ &+(\Delta^2 f)\operatorname{grad} f + 2\nabla_{\operatorname{grad}(\Delta f)}\operatorname{grad} f + (\Delta f)\operatorname{grad}(\Delta f)\\ &-\frac{\Delta f}{2} g(\operatorname{grad} f, \operatorname{grad}(\Delta f))\operatorname{grad} f = 0,\end{aligned} \tag{3.7}$$

with $\Delta f \neq 0$. From (3.7), we have

$$(\Delta f)\operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f) + \Delta^2 f + \frac{\Delta f}{2} g(\operatorname{grad} f, \operatorname{grad}(\Delta f)) = 0. \tag{3.8}$$

Theorem 3.1 follows from (3.7) and (3.8). □

Example 3.2. Let α be a non-constant smooth function on $(0, \infty)$, such that the derivative function $\alpha^{(1)} > 0$, and let $\mathbb{H}^4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_4 > 0\}$

be a 4-dimensional hyperbolic space, we set $M = (0,\infty) \times \mathbb{H}^4$ equipped with the Riemannian metric

$$g = 2[\alpha^{(1)}(t+x_4)]^2(dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2),$$

and let $f(t, x_1, x_2, x_3, x_4) = \alpha(t + x_4)$ for all $(t, x_1, x_2, x_3, x_4) \in M$. By direct computations we obtain

$$\begin{aligned}
\operatorname{grad} f &= \frac{1}{2\alpha^{(1)}(t+x_4)}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_4}),\\
\|\operatorname{grad} f\| &= 1,\\
\Delta f &= \frac{4\alpha^{(2)}(t+x_4)}{\alpha^{(1)}(t+x_4)^2},\\
\Delta^2 f &= \frac{-12\alpha^{(2)}(t+x_4)\alpha^{(3)}(t+x_4) + 4\alpha^{(4)}(t+x_4)\alpha^{(1)}(t+x_4)}{[\alpha^{(1)}(t+x_4)]^5},\\
\operatorname{grad}(\Delta f) &= \frac{-4[\alpha^{(2)}(t+x_4)]^2 + 2\alpha^{(3)}(t+x_4)\alpha^{(1)}(t+x_4)}{[\alpha^{(1)}(t+x_4)]^5}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_4}),\\
\operatorname{Ricci}(\operatorname{grad} f) &= \frac{-2\alpha^{(3)}(t+x_4)\alpha^{(1)}(t+x_4) + 2[\alpha^{(2)}(t+x_4)]^2}{[\alpha^{(1)}(t+x_4)]^5}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_4}),\\
\nabla_{\operatorname{grad}(\Delta f)} \operatorname{grad} f &= 0.
\end{aligned}$$

According to Theorem 3.1 the identity map $\widetilde{I} : (M, g) \longrightarrow (M, \widetilde{g})$, where

$$\widetilde{g} = 3[\alpha^{(1)}(t+x_4)]^2(dt^2+dx_4^2)+2[\alpha^{(1)}(t+x_4)]^2(dx_1^2+dx_2^2+dx_3^2)+2[\alpha^{(1)}(t+x_4)]^2 dt dx_4,$$

is a proper biharmonic map if and only if

$$5\alpha^{(2)}(t+x_4)\alpha^{(3)}(t+x_4) - \alpha^{(4)}(t+x_4)\alpha^{(1)}(t+x_4) = 0, \tag{3.9}$$

and $\alpha^{(2)}(t + x_4) \neq 0$. Note that, the differential equation (3.9) has solutions, for example, we set $\alpha(s) = s^2$, or $\alpha(s) = \sqrt{s}$, $\forall s \in (0,\infty)$.

Using the similar technique of Example 3.2 we have:

Example 3.3. Let $M = \mathbb{R}^3$ equipped with the Riemannian metric

$$g = e^{x+y}(dx^2 + dy^2) + e^{\frac{x+y}{2}} dz^2,$$

and let $f(x, y, z) = \sqrt{2}e^{\frac{x+y}{2}}$, $\forall (x, y, z) \in \mathbb{R}^3$. Then, the function f satisfies the conditions of Theorem 3.2 so, the identity map $\widetilde{I} : (\mathbb{R}^3, g) \longrightarrow (\mathbb{R}^3, \widetilde{g})$ is a proper biharmonic. Here, $\Delta f = \frac{3\sqrt{2}}{4}e^{-\frac{x+y}{2}}$, and the Riemannian metric $\widetilde{g}$ is given by

$$\widetilde{g} = \frac{3}{2}e^{x+y}(dx^2 + dy^2) + e^{\frac{x+y}{2}} dz^2 + e^{x+y} dx dy.$$

Remark 3.4. Let (M,g) be a Riemannian manifold, and let f be a smooth function on M such that $\|\operatorname{grad} f\| = 1$ and $\Delta f = k$, where $k \in \mathbb{R}$. Then, the identity map $\widetilde{I} : (M,g) \longrightarrow (M,\widetilde{g})$ is biharmonic if and only if it is harmonic. Indeed; from Theorem 3.1 the identity map $\widetilde{I}$ is a biharmonic map if and only if $\operatorname{Ricci}(\operatorname{grad} f) = 0$, and by Bochner-Weitzenböck formula for smooth functions (see [7])

$$\frac{1}{2}\Delta(\|\operatorname{grad} f\|^2) \;=\; \|\operatorname{Hess}_f\|^2 + g(\operatorname{grad} f, \operatorname{grad}(\Delta f)) + \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f),$$

we obtain $\|\operatorname{Hess}_f\| = 0$, so that $\Delta f = 0$, that is $\widetilde{I}$ is harmonic map.

4. Biharmonic Maps into a Product Manifolds

Definition 4.1. Let M and N be two Riemannian manifolds equipped with Riemannian metrics g and h, respectively, and let $f \in C^\infty(M)$. Consider the product manifold $M \times N$ and denote by $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ its projections. We define on $M \times N$ a Riemannian metric, denoted G_f, by

$$G_f = \pi^* g + \eta^* h + \pi^*(df \otimes df). \tag{4.1}$$

Remark 4.2.

(1) The Definition 4.1 is a natural generalization of diagonal Riemannian metrics on product Riemannian manifolds (see for example [6]).

(2) $(M\times N, G_f)$ is the product Riemannian manifold of the Riemannian manifolds $(M,\widetilde{g})$ and (N,h), where $\widetilde{g} = g + df \otimes df$. Then, the Levi-Civita connection of $(M \times N, G_f)$ can now be related to those of $(M,\widetilde{g})$ and (N,h) as follows

$$\nabla^{G_f}_X Y = \big(\widetilde{\nabla}_{X_1} Y_1, \nabla^N_{X_2} Y_2\big), \tag{4.2}$$

where $\widetilde{\nabla}$ (resp. ∇^N) is the Levi-Civita connection of $(M,\widetilde{g})$ (resp. (N,h)), the same for the Riemannian curvature tensor R^{G_f} of $(M \times N, G_f)$, we have

$$R^{G_f}(X,Y)Z = (\widetilde{R}(X_1,Y_1)Z_1, R^N(X_2,Y_2)Z_2), \tag{4.3}$$

where $\widetilde{R}$ (resp. R^N) is the Riemannian curvature tensor of $(M,\widetilde{g})$ (resp. (N,h)).

Here, $X = (X_1,X_2), Y = (Y_1,Y_2), Z = (Z_1,Z_2) \in \Gamma(TM) \times \Gamma(TN)$.

Next, let y_0 be an arbitrary point of a Riemannian manifold (N,h), and denote by $i_{y_0} : (M,g) \longrightarrow (M \times N, G_f)$, $x \longmapsto (x,y_0)$ the inclusion map of M at the y_0 level in $M \times N$, where (M,g) is a Riemannian manifold, and $f \in C^\infty(M)$. We note that the inclusion $i_{x_0} : (N,h) \longrightarrow (M \times N, G_f)$, defined by $i_{x_0}(y) = (x_0,y)$ is always a totally geodesic map, that is $\nabla di_{x_0} = 0$, thus harmonic for any function $f \in C^\infty(M)$. From Theorem 3.1, we get the following

Theorem 4.3. *If* $\|\operatorname{grad} f\| = 1$*, the inclusion map* i_{y_0} *is a proper biharmonic map if and only if the identity map* $\widetilde{I} : (M, g) \longrightarrow (M, \widetilde{g})$ *is a proper biharmonic.*

Theorem 4.4. *Let* $\psi : (M, g) \longrightarrow (N, h)$ *be a smooth map and* f *a harmonic function on* (M, g)*. Then, the graph map* $\varphi : (M, g) \longrightarrow (M \times N, G_f)$ *with* $\varphi(x) = (x, \psi(x))$ *is a biharmonic if and only if the map* $\psi : (M, g) \longrightarrow (N, h)$ *is a biharmonic. Furthermore, if* ψ *is proper biharmonic, then so is the graph.*

Proof. Let $\{E_i\}$ be a normal orthonormal frame on (M, g) at x, from the definition of tension field, Theorem 2.2, and (4.2), we have

$$\begin{aligned} \tau(\varphi) &= \nabla^{\varphi}_{E_i} d\varphi(E_i) - d\varphi(\nabla^M_{E_i} E_i) \\ &= \nabla^{G_f}_{(E_i, d\psi(E_i))}(E_i, d\psi(E_i))) \\ &= \Big(\frac{\operatorname{Hess}_f(E_i, E_i)}{1 + \|\operatorname{grad} f\|^2} \operatorname{grad} f, \nabla^N_{d\psi(E_i)} d\psi(E_i)\Big) \\ &= \Big(\frac{\Delta f}{1 + \|\operatorname{grad} f\|^2} \operatorname{grad} f, \tau(\psi)\Big), \end{aligned} \tag{4.4}$$

so that φ is harmonic if and only if $\Delta f = 0$ and $\tau(\psi) = 0$, i.e. the function f is harmonic on (M, g), and ψ is a harmonic map. Next, we compute the bitension field of the graph map, with $\Delta f = 0$. Let $\{E_i\}$ be an orthonormal frame on (M, g), according to (4.4) the tension field of φ is given by $\tau(\varphi) = (0, \tau(\psi))$, we compute

$$\begin{aligned} R^{G_f}(\tau(\varphi), d\varphi(E_i))d\varphi(E_i) &= R^{G_f}((0, \tau(\psi)), (E_i, d\psi(E_i)))(E_i, d\psi(E_i)) \\ &= (0, R^N(\tau(\psi), d\psi(E_i))d\psi(E_i)), \end{aligned} \tag{4.5}$$

by (4.5) and the following

$$\nabla^{\varphi}_{E_i}\nabla^{\varphi}_{E_i}\tau(\varphi) = \widetilde{\nabla}_{(E_i, d\psi(E_i))}(0, \nabla^N_{d\psi(E_i)}\tau(\psi)) = (0, \nabla^{\psi}_{E_i}\nabla^{\psi}_{E_i}\tau(\psi)), \tag{4.6}$$

we have $\tau_2(\varphi) = (0, \tau_2(\psi))$, so that the graph map φ is a biharmonic if and only if $\tau_2(\psi) = 0$. □

Remark 4.5. Using Theorem 4.4, we can construct many examples for proper biharmonic maps.

Example 4.6. The map $\varphi : \mathbb{R}^4 \backslash \{0\} \longrightarrow (\mathbb{R}^4 \times \mathbb{R}^4, G_f)$ given by $\varphi(x) = (x, x/\|x\|^2)$ is a proper biharmonic map, where f is a smooth harmonic function on $\mathbb{R}^4 \backslash \{0\}$. This follows from Theorem 4.2 and the fact that φ is the graph of the inversion $\psi : \mathbb{R}^4 \backslash \{0\} \longrightarrow \mathbb{R}^4$ defined by $\psi(x) = x/\|x\|^2$ which is known ([1]) to be a proper biharmonic map

5. Biharmonic Curve in $(M, \widetilde{g})$

Let $\gamma : I \subset \mathbb{R} \longrightarrow (M, \widetilde{g})$, $t \longmapsto \gamma(t)$ be a differentiable curve in a Riemannian manifold (M, g), where f be a smooth function on M. Suppose that

$$\|\operatorname{grad} f\| = 1, \quad \nabla_{\dot\gamma}\dot\gamma = \lambda(\operatorname{grad} f)\circ\gamma,$$

for some smooth function $\lambda : I \longrightarrow \mathbb{R}$. We have the following result:

Theorem 5.1. *The curve γ is biharmonic if and only if the function f satisfies the following*

$$\rho R((\operatorname{grad} f)\circ\gamma, \dot\gamma)\dot\gamma + 2\rho''(\operatorname{grad} f)\circ\gamma + 2\rho'\nabla_{\dot\gamma}\operatorname{grad} f + \rho\nabla_{\dot\gamma}\nabla_{\dot\gamma}\operatorname{grad} f = 0,$$

where $\rho(t) = \lambda(t) + \frac{1}{2}\operatorname{Hess}_f(\dot\gamma, \dot\gamma)$, $\forall t \in I$. Furthermore, if the function ρ is a non-null constant on I, then the curve γ is a proper biharmonic if and only if the gradient vector of f is Jacobi field along γ on (M, g), i.e.

$$R((\operatorname{grad} f)\circ\gamma, \dot\gamma)\dot\gamma + \nabla_{\dot\gamma}\nabla_{\dot\gamma}\operatorname{grad} f - \nabla_{\nabla_{\dot\gamma}\dot\gamma}\operatorname{grad} f = 0.$$

Proof. The tension field of the curve γ is given by

$$\tau(\gamma) = \nabla^{\gamma}_{\frac{d}{dt}} d\gamma(\frac{d}{dt}) = \widetilde{\nabla}_{\dot\gamma}\dot\gamma, \tag{5.1}$$

by (5.1), and Theorem 2.2, we have

$$\tau(\gamma) = \nabla_{\dot\gamma}\dot\gamma + \frac{1}{2}\operatorname{Hess}_f(\dot\gamma, \dot\gamma)(\operatorname{grad} f)\circ\gamma, \tag{5.2}$$

we set $\rho(t) = \lambda(t) + \frac{1}{2}\operatorname{Hess}_f(\dot\gamma(t), \dot\gamma(t))$, with $\nabla_{\dot\gamma}\dot\gamma = \lambda(\operatorname{grad} f)\circ\gamma$, we get

$$\tau(\gamma) = \rho(\operatorname{grad} f)\circ\gamma, \tag{5.3}$$

now, the curve γ is biharmonic if and only if

$$\widetilde{R}(\tau(\gamma), d\gamma(\frac{d}{dt}))d\gamma(\frac{d}{dt}) + \nabla^{\gamma}_{\frac{d}{dt}}\nabla^{\gamma}_{\frac{d}{dt}}\tau(\gamma) = 0, \tag{5.4}$$

from (5.3), and Theorem 2.3, with

$$\operatorname{Hess}_f(\operatorname{grad} f, X) = 0, \quad \nabla_{\operatorname{grad} f}\operatorname{grad} f = 0,$$

for all $X \in \Gamma(TM)$, the first term on the left-hand side of (5.4) is

$$\begin{aligned}\widetilde{R}(\tau(\gamma), d\gamma(\frac{d}{dt}))d\gamma(\frac{d}{dt}) &= \rho R((\operatorname{grad} f)\circ\gamma, \dot\gamma)\dot\gamma \\ &+ \frac{\rho}{2} g(R((\operatorname{grad} f)\circ\gamma, \dot\gamma)(\operatorname{grad} f)\circ\gamma, \dot\gamma)(\operatorname{grad} f)\circ\gamma, \end{aligned} \tag{5.5}$$

for the second term on the left-hand side of (5.4), we compute

$$\begin{aligned}\nabla^{\gamma}_{\frac{d}{dt}}\tau(\gamma) &= \nabla^{\gamma}_{\frac{d}{dt}}\rho(\operatorname{grad} f)\circ\gamma\\ &= \rho'(\operatorname{grad} f)\circ\gamma+\rho\widetilde{\nabla}_{\dot\gamma}\operatorname{grad} f,\end{aligned}\tag{5.6}$$

by (5.6), and Theorem 2.2, we get

$$\begin{aligned}\nabla^{\gamma}_{\frac{d}{dt}}\nabla^{\gamma}_{\frac{d}{dt}}\tau(\gamma) &= \nabla^{\gamma}_{\frac{d}{dt}}\big[\rho'(\operatorname{grad} f)\circ\gamma+\rho\nabla_{\dot\gamma}\operatorname{grad} f\big]\\ &= \rho''(\operatorname{grad} f)\circ\gamma+\rho'\nabla^{\gamma}_{\frac{d}{dt}}(\operatorname{grad} f)\circ\gamma\\ &\quad+\rho'\nabla_{\dot\gamma}\operatorname{grad} f+\rho\nabla^{\gamma}_{\frac{d}{dt}}\nabla_{\dot\gamma}\operatorname{grad} f\\ &= \rho''(\operatorname{grad} f)\circ\gamma+2\rho'\nabla_{\dot\gamma}\operatorname{grad} f\\ &\quad+\rho\nabla_{\dot\gamma}\nabla_{\dot\gamma}\operatorname{grad} f+\frac{\rho}{2}\operatorname{Hess}_f\big(\dot\gamma,\nabla_{\dot\gamma}\operatorname{grad} f\big)(\operatorname{grad} f)\circ\gamma,\end{aligned}\tag{5.7}$$

by definition (1.4), with $\|\operatorname{grad} f\|=1$, we have

$$\operatorname{Hess}_f\big(\dot\gamma,\nabla_{\dot\gamma}\operatorname{grad} f\big)=-g((\operatorname{grad} f)\circ\gamma,\nabla_{\dot\gamma}\nabla_{\dot\gamma}\operatorname{grad} f),\tag{5.8}$$

from (5.5), (5.7) and (5.8), the the curve γ is biharmonic if and only if

$$\begin{aligned}&\rho R((\operatorname{grad} f)\circ\gamma,\dot\gamma)\dot\gamma+\frac{\rho}{2}g(R((\operatorname{grad} f)\circ\gamma,\dot\gamma)(\operatorname{grad} f)\circ\gamma,\dot\gamma)(\operatorname{grad} f)\circ\gamma\\ &+\rho''(\operatorname{grad} f)\circ\gamma+2\rho'\nabla_{\dot\gamma}\operatorname{grad} f+\rho\nabla_{\dot\gamma}\nabla_{\dot\gamma}\operatorname{grad} f\\ &-\frac{\rho}{2}g((\operatorname{grad} f)\circ\gamma,\nabla_{\dot\gamma}\nabla_{\dot\gamma}\operatorname{grad} f)(\operatorname{grad} f)\circ\gamma=0,\end{aligned}\tag{5.9}$$

by equation (5.9) we find that

$$\begin{aligned}&-\frac{\rho}{2}g(R((\operatorname{grad} f)\circ\gamma,\dot\gamma)(\operatorname{grad} f)\circ\gamma,\dot\gamma)+\rho''\\ &+\frac{\rho}{2}g((\operatorname{grad} f)\circ\gamma,\nabla_{\dot\gamma}\nabla_{\dot\gamma}\operatorname{grad} f)=0.\end{aligned}\tag{5.10}$$

The Theorem 5.1, follows from (5.9) and (5.10). □

Remark 5.2. From equation (5.3), the curve γ is harmonic if and only if $\rho=0$.

Example 5.3. Let $\mathbb{D}=\{((x,y)\in\mathbb{R}^2|x^2+y^2<1\}$, and let $M=\mathbb{D}\times\mathbb{R}$ equipped with the Riemannian metric

$$g=dx^2+dy^2+\frac{1}{1-x^2-y^2}dz^2.$$

We consider the curve on (M,g),

$$\gamma(t)=(t,t,-t^2+2t-\ln(t+1)),\quad \frac{1}{\sqrt2}>t>-\frac{1}{\sqrt2}.$$

The tension field of the curve γ (with respect to g) is given by

$$\Big(-\frac{t}{(t+1)^2}, -\frac{t}{(t+1)^2}, \frac{-1+2\,t^2}{(t+1)^2}\Big)$$

Let $f(x,y,z) = xy + z$, $\forall(x,y,z) \in M$, we have

$$\|\operatorname{grad} f\| = 1, \quad (\operatorname{grad} f)\circ\gamma = (t, t, 1-2t^2),$$

so that, $\lambda(t) = -\frac{1}{(t+1)^2}$, and note that

$$\frac{1}{2}\operatorname{Hess}_f(\dot\gamma,\dot\gamma) = \frac{1}{(t+1)^2},$$

then the curve γ is harmonic on $(M,\tilde g)$, because $\rho(t) = 0$, with

$$\tilde g = (1+y^2)dx^2 + (1+x^2)dy^2 + \frac{x^2+y^2-2}{x^2+y^2-1}dz^2 + 2xydxdy + 2ydxdz + 2xdydz.$$

Example 5.4. Let $M = \mathbb{R}^n\backslash\{0\}$ equipped with the Riemannian metric $g = 4\|x\|^2dx_i^2$, $f(x) = \|x\|^2$, $\forall x \in M$, and consider the proper biharmonic curve on (M,g),

$$\gamma(t) = (\sqrt{\frac{t^2+1}{2}}, 0, ..., 0), \quad \forall t \in \mathbb{R}.$$

Then, $\|\operatorname{grad} f\| = 1$, the gradient vector of f is Jacobi field along γ,

$$\nabla_{\dot\gamma}\dot\gamma = (\operatorname{grad} f)\circ\gamma = \frac{1}{\sqrt{2t^2+2}}\frac{\partial}{\partial x_1},$$

and note that $\operatorname{Hess}_f(\dot\gamma,\dot\gamma) = 0$, so that $\rho(t) = 1$, $\forall t \in \mathbb{R}$. According to Theorem 5.1 the curve $\gamma : \mathbb{R} \longrightarrow (M,\widetilde g)$ is also proper biharmonic, with

$$\widetilde g = 4\|x\|^2dx_i^2 + 4x_ix_jdx_i\otimes dx_j.$$

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