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Pathway Fractional Integral Formulas Involving Extended Mittag-Leffler Functions in the Kernel

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Abstract. Since the Mittag-Leffler function was introduced in 1903, a variety of extensions and generalizations with diverse applications have been presented and investigated. In this paper, we aim to introduce some presumably new and remarkably different extensions of the Mittag-Leffler function, and use these to present the pathway fractional integral formulas. We point out relevant connections of some particular cases of our main results with known results.

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1. Introduction and Preliminaries

The Swedish mathematician Gosta Mittag-Leffler [19] introduced the so-called Mittag-Leffler function

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\quad (z\in\mathbb{C};\ \Re(\alpha)>0), \tag{1.1}$$

where Γ is the familiar gamma function whose Euler's integral is given by (see, e.g., [34, Section 1.1])

$$\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}\,dt\quad (\Re(z)>0). \tag{1.2}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}_0^-$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, non-positive integers, and positive integers, respectively, and let $\mathbb{R}_0^+:=\mathbb{R}^+\cup\{0\}$. Wiman [39] generalized the Mittag-Leffler function (1.1) as follows:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\quad (z\in\mathbb{C};\ \min\{\Re(\alpha),\,\Re(\beta)\}>0). \tag{1.3}$$

The Mittag-Leffler function E_α (1.1) and the extended function $E_{\alpha,\beta}$ (1.3) have been extended in a number of ways and, together with their extensions, applied in various research areas. For those extensions and applications, we refer the reader, for example, to [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 21, 25, 27, 30, 31, 32, 33, 35, 36, 37].

Here, for an easier reference, we give a brief history of some chosen extensions of the Mittag-Leffler function E_α (1.1) and the extended function $E_{\alpha,\beta}$ (1.3). Prabhakar [25] introduced an extension of the function $E_{\alpha,\beta}$ (1.3)

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(\alpha n+\beta)}z^n \tag{1.4}$$

$$(z\in\mathbb{C};\ \min\{\Re(\alpha),\,\Re(\beta),\,\Re(\gamma)\}>0),$$

where the familiar Pochhammer symbol $(\lambda)_\nu$ is defined (for λ, $\nu\in\mathbb{C}$) by

$$\begin{aligned}(\lambda)_\nu :&=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}\quad \left(\lambda+\nu\in\mathbb{C}\setminus\mathbb{Z}_0^-\right)\\ &=\begin{cases}1 & (\nu=0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N}).\end{cases}\end{aligned} \tag{1.5}$$

Shukla and Prajapati [31] (see also [37]) defined and investigated the following extension

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{n!\Gamma(\alpha n+\beta)}z^n \tag{1.6}$$

$$(z \in \mathbb{C};\ \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0;\ q \in (0,1) \cup \mathbb{N}).$$

Salim [28] introduced

$$E_{\alpha,\beta}^{\gamma,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_n} \tag{1.7}$$

$$(z \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0).$$

Salim and Faraj [29] generalized the function (1.7)

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}} \tag{1.8}$$

$$\left(z \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q \in \mathbb{R}^+\right).$$

Özarslan and Yilmaz [23] presented the following extension

$$E_{\alpha,\beta}^{\gamma;c}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{1.9}$$

$$\left(z \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta)\} > 0; \Re(c) > \Re(\gamma) > 0; p \in \mathbb{R}_0^+\right).$$

Here $B_p(x,y)$ is the extended beta function (see [4, 15])

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt \tag{1.10}$$

$$\left(p \in \mathbb{R}_0^+; \min\{\Re(x), \Re(y)\} > 0\right),$$

whose particular case when $p = 0$ reduces to the well-known beta function (see, e.g., [34, Section 1.1])

$$\begin{aligned} B(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad (\min\{\Re(x), \Re(y)\} > 0) \\ &= \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad \left(x, y \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \end{aligned} \tag{1.11}$$

By using the pathway idea in [16] (see also [17, 18]), Nair [20] introduced the following pathway fractional integral operator

$$\left(P_{0+}^{\mu,\lambda} f\right)(x) = x^{\mu} \int_0^{\left[\frac{x}{\alpha(1-\lambda)}\right]} \left[1 - \frac{\alpha(1-\lambda)\tau}{x}\right]^{\frac{\mu}{1-\lambda}} f(\tau)\, d\tau \tag{1.12}$$

$$\left(\Re(\mu) > 0;\ \alpha \in \mathbb{R}^+;\ \lambda < 1\right),$$

where $f \in L(a,b)$ (the set of measurable real or complex valued functions) and λ is a pathway parameter.

Remark 1.1. For a given scalar $\lambda \in \mathbb{R}$, the pathway model for scalar random variables is represented by the following probability density function (see, e.g., [2])

$$f(x) = c|x|^{\nu-1}\Big[1-\alpha(1-\lambda)|x|^{\eta}\Big]^{\frac{\mu}{1-\lambda}} \tag{1.13}$$

$$\left(x \in \mathbb{R};\ \mu \in \mathbb{R}_0^+,\ \eta,\ \nu,\ [1-\alpha(1-\lambda)|x|^{\eta}] \in \mathbb{R}^+\right),$$

where c is the normalizing constant and λ is called the pathway parameter. For $\lambda \in \mathbb{R}$, the normalizing constant c is given as follows (see, e.g., [2]):

$$c = \begin{cases} \dfrac{\eta\,[\alpha(1-\lambda)]^{\frac{\nu}{\eta}}\,\Gamma\left(\frac{\nu}{\eta}+\frac{\mu}{1-\lambda}+1\right)}{2\,\Gamma\left(\frac{\nu}{\eta}\right)\Gamma\left(\frac{\mu}{1-\lambda}+1\right)} & (\lambda < 1), \\[2ex] \dfrac{\eta\,[\alpha(1-\lambda)]^{\frac{\nu}{\eta}}\,\Gamma\left(\frac{\mu}{\lambda-1}\right)}{2\,\Gamma\left(\frac{\nu}{\eta}\right)\Gamma\left(\frac{\mu}{\lambda-1}-\frac{\nu}{\eta}\right)} & (1 < \lambda < 1+\eta/\nu), \\[2ex] \dfrac{\eta\,(\alpha\mu)^{\frac{\nu}{\eta}}}{2\,\Gamma\left(\frac{\nu}{\eta}\right)} & (\lambda \to 1). \end{cases} \tag{1.14}$$

Setting $\lambda = 0$, $\alpha = 1$ and replacing μ by $\mu-1$ in (1.12) reduces to the well-known left-sided Riemann-Liouville fractional integral operator I_{a+}^{μ} (e.g., [3, 22, 24, 26, 38])

$$\left(P_{0+}^{\mu-1,0} f\right)(x) = x^{1-\mu}\,\Gamma(\mu)\,\left(I_{0+}^{\mu} f\right)(x) \quad (\Re(\mu) > 1), \tag{1.15}$$

where I_{a+}^{μ} is defined by

$$\left(I_{a+}^{\mu} f\right)(x) := \frac{1}{\Gamma(\mu)} \int_a^x (x-\tau)^{\mu-1} f(\tau)\,d\tau \quad (x > a;\ \Re(\mu) > 0) \tag{1.16}$$

and $[a,b]$ $(-\infty < a < b < \infty)$ is a finite interval on the real line $\mathbb{R}$.

In this paper, we aim to introduce (presumably) new and (remarkably) different extensions of the Mittag-Leffler function, which are also associated with the pathway fractional integral operator (1.12) to present their integral formulas. Relevant connections of some particular cases of the main results presented here with those earlier ones are also pointed out.

2. Pathway Fractional Integration of an Extended Mittag-Leffler Function

By considering (1.6) and (1.9) together, we begin by defining a (presumably) new extension of the Mittag-Leffler function as follows:

$$E_{\rho,\beta,\delta}^{\gamma,q;c}(z;p)=\sum_{n=0}^{\infty}\frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{nq}}{\Gamma(\rho n+\beta)}\frac{z^n}{(\delta)_n} \tag{2.1}$$

$$\left(q\in\mathbb{R}^+;\ \min\{\Re(\rho),\,\Re(\beta),\,\Re(\delta)\}>0;\ \Re(c)>\Re(\gamma)>0;\ p\in\mathbb{R}_0^+\right),$$

where $B_p(x,y)$ is the same as in (1.10).

It is easy to see that (2.1) contains the Mittag-Leffler function and each of its extensions (or generalizations) given in Section 1 as in the following remark.

Remark 2.1.

(i) The particular case of (2.1) when $p=0$ and $q=1$ reduces to (1.7).

(ii) The particular case of (2.1) when $\delta=1$ is a generalization of (1.6) and (1.9).

We establish a pathway integration formula involving the extended Mittag-Leffler function (2.1), which is asserted in Theorem 2.1.

Theorem 2.1. *Let* $\rho,\ \beta,\ \gamma,\ \delta,\ c,\ \mu\in\mathbb{C}$ *with* $\min\{\Re(\rho),\,\Re(\beta),\,\Re(\delta),\,\Re(\mu)\}>0$ *and* $\Re(c)>\Re(\gamma)>0$. *Also, let* $\omega\in\mathbb{R}$, $\alpha,\ q\in\mathbb{R}^+$, *and* $p\in\mathbb{R}_0^+$. *Further, let* $\lambda<1$ *with* $\Re(\frac{\mu}{1-\lambda})>-1$. *Then*

$$\begin{aligned}&P_{0+}^{\mu,\lambda}\left(\tau^{\beta-1}E_{\rho,\beta,\delta}^{\gamma,q;c}(\omega\tau^{\rho};p)\right)(x)\\&\quad=\frac{\Gamma(1+\frac{\mu}{1-\lambda})\,x^{\mu+\beta}}{[\alpha(1-\lambda)]^{\beta}}E_{\rho,\beta+1+\frac{\mu}{1-\lambda},\delta}^{\gamma,q;c}\left(\omega\left(\frac{x}{\alpha(1-\lambda)}\right)^{\rho};p\right).\end{aligned} \tag{2.2}$$

Proof. Let $\mathcal{L}_1$ be the left-hand side of (2.10). By applying (2.1) to (1.12), and interchanging the order of integral and summation, which is verified under the given conditions in this theorem, we obtain

$$\begin{aligned}\mathcal{L}_1&=x^{\mu}\sum_{n=0}^{\infty}\frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)}\frac{(c)_{nq}}{\Gamma(\rho n+\beta)}\frac{(\omega)^n}{(\delta)_n}\\&\quad\times\int_0^{[\frac{x}{\alpha(1-\lambda)}]}\tau^{\beta+\rho n-1}\left[1-\frac{\alpha(1-\lambda)\tau}{x}\right]^{\frac{\mu}{1-\lambda}}d\tau.\end{aligned} \tag{2.3}$$

Setting $\frac{\alpha(1-\lambda)\tau}{x} = t$ and using (1.11), we get

(2.4)
$$\int_0^{[\frac{x}{\alpha(1-\lambda)}]} \tau^{\beta+\rho n-1}\left[1-\frac{\alpha(1-\lambda)\tau}{x}\right]^{\frac{\mu}{1-\lambda}} d\tau = \frac{x^{\beta+\rho n}}{[\alpha(1-\lambda)]^{\beta+\rho n}} \frac{\Gamma(\beta+\rho n)\,\Gamma\left(\frac{\mu}{1-\lambda}+1\right)}{\Gamma\left(\beta+\frac{\mu}{1-\lambda}+1+\rho n\right)}.$$

Using (2.4) in (2.3), in terms of (2.1), we have

$$\begin{aligned}\mathcal{L}_1 &= \frac{x^{\mu+\beta}\Gamma(1+\frac{\mu}{1-\lambda})}{[\alpha(1-\lambda)]^{\beta}} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}}{\Gamma(\rho n+\beta+1+\frac{\mu}{1-\lambda})} \frac{\left(\omega\left(\frac{x}{\alpha(1-\lambda)}\right)^{\rho}\right)^n}{(\delta)_n} \\ &= \frac{x^{\mu+\beta}\Gamma(1+\frac{\mu}{1-\lambda})}{[\alpha(1-\lambda)]^{\beta}} E^{\gamma,q;c}_{\rho,\beta+1+\frac{\mu}{1-\lambda},\delta}\left(\omega\left(\frac{x}{\alpha(1-\lambda)}\right)^{\rho}; p\right),\end{aligned}$$

which is the right-hand side of (2.10). This completes the proof. □

Corollary 2.1. *Let* $\rho, \beta, \gamma, c, \mu \in \mathbb{C}$ *with* $\min\{\Re(\rho), \Re(\beta), \Re(\mu)\} > 0$, $\Re(c) > \Re(\gamma) > 0$, *and* $\omega \in \mathbb{R}$. *Also, let* $\lambda < 1$ *with* $\Re(\frac{\mu}{1-\lambda}) > -1$. *Then*

(2.5)
$$\begin{aligned}&P^{\mu,\lambda}_{0+}\left(\tau^{\beta-1}E^{\gamma}_{\rho,\beta}(\omega\tau^{\rho})\right)(x) \\ &\quad = \frac{x^{\mu+\beta}\Gamma(1+\frac{\mu}{1-\lambda})}{[\alpha(1-\lambda)]^{\beta}} E^{\gamma}_{\rho,\beta+1+\frac{\mu}{1-\lambda}}\left(\omega\left(\frac{x}{\alpha(1-\lambda)}\right)^{\rho}\right).\end{aligned}$$

Proof. Setting $p = 0$, $\delta = 1$, and $q = 1$ in (2.10) together with (1.4) yields the desired result (2.5). □

Corollary 2.2. *Let* $\rho, \beta, \gamma, \mu \in \mathbb{C}$ *with* $\min\{\Re(\rho), \Re(\beta), \Re(\gamma)\} > 0$ *and* $\Re(\mu) > 1$. *Also, let* $\omega \in \mathbb{R}$. *Then*

(2.6)
$$P^{\mu-1,0}_{0+}\left(\tau^{\beta-1}\,E^{\gamma}_{\rho,\beta}\,(\omega\,\tau^{\rho})\right)(x) = \Gamma(\mu)\,x^{\mu-1+\beta}\,E^{\gamma}_{\rho,\beta}\,(\omega\,x^{\rho}).$$

Proof. Setting $p = \lambda = 0$, $q = \delta = \alpha = 1$, and replacing μ by $\mu - 1$ in (2.10), and using (1.4), we are led to (2.6). □

Corollary 2.3. *Let* $\rho, \beta, \gamma, \mu \in \mathbb{C}$ *with* $\min\{\Re(\rho), \Re(\beta), \Re(\gamma)\} > 0$ *and* $\Re(\mu) > 1$. *Also, let* $\omega \in \mathbb{R}$ *and* $q \in \mathbb{R}^{+}$. *Then*

(2.7)
$$P^{\mu-1,0}_{0+}\left(\tau^{\beta-1}\,E^{\gamma,q}_{\rho,\beta}\,(\omega\,\tau^{\rho})\right)(x) = \Gamma(\mu)\,x^{\mu-1+\beta}\,E^{\gamma,q}_{\rho,\beta}\,(\omega\,x^{\rho}).$$

Proof. Setting $p = \lambda = 0$, $\delta = \alpha = 1$, and replacing μ by $\mu - 1$ in (2.10), and using (1.6), we obtain the result (2.7). □

Corollary 2.4. *Let* $\rho, \beta, \gamma, \delta, \mu \in \mathbb{C}$ *with* $\min\{\Re(\rho), \Re(\beta), \Re(\delta), \Re(\gamma)\} > 0$ *and* $\Re(\mu) > 1$. *Also, let* $\omega \in \mathbb{R}$. *Then*

(2.8)
$$P^{\mu-1,0}_{0+}\left(\tau^{\beta-1}\,E^{\gamma,\delta}_{\rho,\beta}\,(\omega\,\tau^{\rho})\right)(x) = \Gamma(\mu)\,x^{\mu-1+\beta}\,E^{\gamma,\delta}_{\rho,\beta+\mu}\,(\omega\,x^{\rho}).$$

Proof. Setting $p = \lambda = 0$, $q = \alpha = 1$, and replacing μ by $\mu - 1$ in (2.10), and using (1.7), we obtain the result (2.8). □

Remark 2.2. For the results (2.5), (2.6), (2.7), and (2.8), we refer the reader, respectively, to [20, 25, 28, 31, 37]. In view of (1.15), the results (2.6), (2.7), and (2.8) can yield the corresponding ones for the left-sided Riemann-Liouville fractional integration operator I^{μ}_{a+}.

We present a further generalization of the Mittag-Leffler function, which is a slight extension of the extended Mittag-Leffler function in (2.1).

$$E^{\gamma,\delta,q;c}_{\rho,\beta,s}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}}{\Gamma(\rho n + \beta)} \frac{z^n}{(\delta)_{sn}} \tag{2.9}$$

$$\left(q,\, s \in \mathbb{R}^+;\, \min\{\Re(\rho),\, \Re(\beta),\, \Re(\delta)\} > 0;\, \Re(c) > \Re(\gamma) > 0;\, p \in \mathbb{R}^+_0\right),$$

where $B_p(x, y)$ is the same as in (1.10).

It is easy to see that the particular case $s = 1$ of (2.9) reduces to (2.1). For more particular cases, see Remark 2.1. We present a pathway integration formula involving the extended Mittag-Leffler function (2.9), which is asserted in Theorem 2.2.

Theorem 2.2. *Let* ρ, β, γ, δ, c, $\mu \in \mathbb{C}$ *with* $\min\{\Re(\rho), \Re(\beta), \Re(\delta), \Re(\mu)\} > 0$ *and* $\Re(c) > \Re(\gamma) > 0$. *Also, let* $\omega \in \mathbb{R}$, α, q, $s \in \mathbb{R}^+$, *and* $p \in \mathbb{R}^+_0$. *Further, let* $\lambda < 1$ *with* $\Re(\frac{\mu}{1-\lambda}) > -1$. *Then*

$$\begin{aligned} &P^{\mu,\lambda}_{0+}\left(\tau^{\beta-1} E^{\gamma,\delta,q;c}_{\rho,\beta,s}(\omega\tau^{\rho};p)\right)(x) \\ &\quad = \frac{\Gamma(1+\frac{\mu}{1-\lambda})\, x^{\mu+\beta}}{[\alpha(1-\lambda)]^{\beta}} E^{\gamma,\delta,q;c}_{\rho,\beta+1+\frac{\mu}{1-\lambda},s}\left(\omega\left(\frac{x}{\alpha(1-\lambda)}\right)^{\rho};p\right). \end{aligned} \tag{2.10}$$

Proof. The proof runs parallel to that of Theorem 2.1. We omit the details. □

We can also provide many particular cases of Theorem 2.2, including those results corresponding to Corollaries 2.1–2.4. The details are left to the interested reader.

3. Concluding Remarks

Among a variety of extensions (or generalizations) of the Mittag-Leffler function, the extension (2.1) (or (2.9)) seems to be a different one.

One of the Erdélyi-Kober type fractional integrals (see [14, p.105, Eq. (2.6.1)]) appears to be closely related to the pathway fractional integration operator (1.12), even though one integral cannot contain the other one as a purely special case. For generalized multi-index Mittag-Leffler functions and their applications, we refer the reader, for example, to [5] and the references cited therein.

The main results presented here, as their special cases, include many earlier ones, in particular, including some of the identities provided by Nair [20] who first introduced the pathway fractional integral operator (1.12).

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