KYUNGPOOK Math. J. 59(2019), 101-123
https://doi.org/10.5666/KMJ.2019.59.1.101
pISSN 1225-6951 eISSN 0454-8124

Approximation Solvability for a System of Nonlinear Variational Type Inclusions in Banach Spaces

Salahuddin
Department of Mathematics, Jazan University, Jazan-45142, Kingdom of Saudi Arabia
e-mail : drsalah12@hotmail.com

Abstract. In this paper, we consider a system of nonlinear variational type inclusions involving (H, η, φ)-monotone operators in real Banach spaces. Further, we define a proximal operator associated with an (H, η, φ)-monotone operator and show that it is single valued and Lipschitz continuous. Using proximal point operator techniques, we prove the existence and uniqueness of a solution and suggest an iterative algorithm for the system of nonlinear variational type inclusions. Furthermore, we discuss the convergence of the iterative sequences generated by the algorithms.

1. Introduction

Variational inequalities, variational inclusions, complementarity problems and equilibrium problems are among the most interesting, novel, innovative and important mathematical problems. They have been widely studied by many authors in the recent years due to their wide applications in mechanics, physics, optimizations and controls, nonlinear programming, economics and transportation, equilibriums and engineering sciences, *see,* [6, 10, 13, 22, 32, 33, 34].

Very recently, Luo and Huang [28] and Kim *et al.* [25] introduced and studied a new class of (H, η, φ)-monotone operators, *respectively* (H, ϕ, ψ)-η-monotone operators in Banach spaces. This provides a unified framework for a class of maximal monotone operators, maximal η-monotone operators, H-monotone operators and (H, η)-monotone operators. Using proximal point operator techniques, they studied the convergence analysis of the iterative algorithms for some classes of variational inclusions, *see,* [11, 12, 14, 15, 16, 23, 26, 36].

Motivated and inspired by such works as [2, 8, 9, 18, 19, 21, 24, 27, 30, 31, 35],

Received March 9, 2017; revised November 3, 2018; accepted December 6, 2018.
2010 Mathematics Subject Classification: 49J40, 47H09, 47J20.
Key words and phrases: system of nonlinear variational type inclusions, algorithms, (H, φ, η)-monotone operators, iterative sequences, Banach spaces.

in this paper we further generalize these proximal point operator techniques and define a new iterative algorithm for solving a system of nonlinear variational type inclusions. Moreover, we discuss the convergence for the approximate solutions of a system of nonlinear variational type inclusions in real Banach spaces.

2. Preliminaries

Let X be a real Banach space endowed with the topological dual space X^*, the norm $\|\cdot\|$ and the dual pairing $\langle\cdot,\cdot\rangle$ between X and X^*, 2^X denotes the family of all nonempty subsets of X. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle f, x\rangle = \|f\|\|x\|, \|f\| = \|x\|\}, \forall x \in X.$$

Let $\mathfrak{U} = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in \mathfrak{U}$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \|\frac{x+y}{2}\| \leq 1 - \delta.$$

We note that, the L_p, ℓ_p and Sobolev spaces W^p_m $(1 < p < \infty)$ are uniformly convex.

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be smooth if the limit,

$$(*) \qquad \lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in \mathfrak{U}$. It is also said to be uniformly smooth if the limit $(*)$ is attained uniformly for $x, y \in \mathfrak{U}$. Again, we note that if X is uniformly convex and uniformly smooth, then J is invertible and $J^{-1} = J^*$ (the duality mapping from X^* to X^{**}), hence J is a single valued and if $X = H$ (a Hilbert space) then J is an identity mapping.

We define a function $\rho_X : [0, +\infty) \to [0, +\infty)$ called the modulus of smoothness of X as follows:

$$\rho_X(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

It is known that a Banach space X is uniformly smooth if and only if

$$\lim_{t\to\infty} \frac{\rho_X(t)}{t} = 0.$$

Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant c such that

$$\rho_X(t) \leq ct^q, \quad \forall t > 0.$$

Let $CB(X)$ be a family of all nonempty closed and bounded subsets of X; $\mathfrak{H}(\cdot,\cdot)$ be a Hausdorff metric on $CB(X)$ defined by

$$\mathfrak{H}(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\},$$

where $A, B \in CB(X), d(x,B) = \inf_{y\in B} d(x,y)$ and $d(A,y) = \inf_{x\in A} d(x,y)$.

The following concepts and results are needed in the sequel.

Definition 2.1. Let $T : X \to X^*$ and $g : X \to X$ be two single valued mappings. Then

(i) T is *monotone*, if

$$\langle T(x) - T(y), x - y\rangle \geq 0, \forall x, y \in X;$$

(ii) T is *strictly monotone*, if

$$\langle T(x) - T(y), x - y\rangle > 0, \forall x, y \in X,$$

and equality holds if and only if $x = y$;

(iii) T is *α-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq \alpha\|x - y\|^2, \forall x, y \in X;$$

(iv) T is *β-Lipschitz continuous* if there exists a constant $\beta > 0$ such that

$$\|T(x) - T(y)\| \leq \beta\|x - y\|, \forall x, y \in X;$$

(v) g is *κ-strongly accretive* if there exists $j(x - y) \in J(x - y)$ and a constant $\kappa > 0$ such that

$$\langle g(x) - g(y), j(x - y)\rangle \geq \kappa\|x - y\|^2, \forall x, y \in X.$$

Definition 2.2. Let $M : X \to 2^{X^*}$ be a multivalued mapping, $H : X \to X^*$ and $\eta : X \times X \to X$ be the single valued mappings. Then

(i) M is *monotone*, if

$$\langle u - v, x - y\rangle \geq 0, \forall x, y \in X, u \in M(x), v \in M(y);$$

(ii) M is *η-monotone*, if

$$\langle u - v, \eta(x, y)\rangle \geq 0, \forall x, y \in X, u \in M(x), v \in M(y);$$

(iii) M is *strictly η-monotone*, if

$$\langle u - v, \eta(x, y)\rangle > 0, \forall x, y \in X, u \in M(x), v \in M(y)$$

and equality holds if and only if $x = y$;

(iv) M is *λ-strongly η-monotone* if there exists a constant $\lambda > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \geq \lambda \|x - y\|^2, \forall x, y \in X, u \in M(x), v \in M(y);$$

(v) M is *maximal monotone*, if M is monotone and

$$(J + \lambda M)(X) = X^*, \forall \lambda > 0,$$

where J is a normalized duality mapping;

(vi) M is *maximal η-monotone*, if M is monotone and

$$(J + \lambda M)(X) = X^*, \forall \lambda > 0;$$

(vii) M is *H-monotone*, if M is monotone and

$$(H + \lambda M)(X) = X^*, \forall \lambda > 0;$$

(viii) M is *(H, η)-monotone*, if M is η-monotone and

$$(H + \lambda M)(X) = X^*, \forall \lambda > 0;$$

(ix) η is *τ-Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \forall x, y \in X.$$

Definition 2.3. For all $x, y \in X$, a mapping $N : X \times X \to X^*$ is said to be

(i) *β_1-Lipschitz continuous with respect to first argument* if there exists a constant $\beta_1 > 0$ such that

$$\|N(u, \cdot) - N(v, \cdot)\| \leq \beta_1 \|u - v\|;$$

(ii) *β_2-Lipschitz continuous with respect to second argument* if there exists a constant $\beta_2 > 0$ such that

$$\|N(\cdot, u) - N(\cdot, v)\| \leq \beta_2 \|u - v\|.$$

Definition 2.4.([29]) Let X be a complete metric space, $T : X \to CB(X)$ be a set valued mapping. Then for any $\epsilon > 0$ and for any $x, y \in X, u \in T(x)$ there exists $v \in T(y)$ such that

$$d(u, v) \leq (1 + \epsilon)\mathfrak{H}(T(x), T(y)).$$

Lemma 2.5. *Let X be a uniformly smooth Banach space and $J : X \to 2^{X^*}$ be a normalized duality mapping. Then for all $x, y \in X$,*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$;

(ii)
$$\langle x-y, j(x)-j(y)\rangle \leq 2d^2\rho_X\left[\frac{4\|x-y\|}{d}\right]$$

where
$$d=\sqrt{\frac{\|x\|^2+\|y\|^2}{2}}.$$

Definition 2.6. Let X be a Banach space with the dual space X^*. Let $H : X \to X^*, \varphi : X^* \to X^*, \eta : X \times X \to X$ be single valued mappings and $M : X \to 2^{X^*}$ be a multivalued mapping. The mapping M is said to be (H, η, φ)*-monotone* if $\varphi \circ M$ is η-monotone and
$$(H+\varphi\circ M)(X)=X^*.$$

Definition 2.7. Let X be a Banach space with the dual space X^*. Let $H : X \to X^*, \varphi : X^* \to X^*, \eta : X \times X \to X$ be single valued mappings and $M : X \to 2^{X^*}$ be a multivalued mapping. The mapping M is said to be

(i) (H,φ)*-monotone*, if $(\varphi\circ M)$ is monotone and
$$(H+\varphi\circ M)(X)=X^*;$$

(ii) *maximal* (φ,η)*-monotone*, if $(\varphi\circ M)$ is η-monotone and
$$(H+\varphi\circ M)(X)=X^*;$$

(iii) *maximal* φ*-monotone*, if $(\varphi\circ M)$ is monotone and
$$(H+\varphi\circ M)(X)=X^*.$$

Examples 2.8. Let $X = R = (-\infty,+\infty), M(x) = x^2, H(x) = e^x, \forall x \in R$ and $\varphi(x)=\sqrt{x}, \forall x \geq 0$. Let $\eta(x,y)=x^4-y^4, \forall x,y \in R$. Then
$$\langle M(x)-M(y),\eta(x,y)\rangle = \langle x^2-y^2, x^4-y^4\rangle = (x^2+y^2)(x^2-y^2)^2 \geq 0$$
and
$$(I+M)(x)=x+x^2=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}\geq -\frac{1}{4}.$$
This shows that $(I+M)$ is not surjective. Since
$$(H+M)(x)=e^x+x^2>0.$$

We know that $(H+M)$ is not surjective. Thus, M is neither maximal η-monotone nor (H,η)-monotone.
Moreover, we have

$$\langle \varphi \circ M(x) - \varphi \circ M(y), x-y \rangle = \langle x-y, x-y\rangle = (x-y)^2 \geq 0$$

and

$$(I + \varphi \circ M)(x) = x + x - 2x.$$

It follows that $(I+\varphi\circ M)$ is surjective. It is easy to see that

$$(H + \varphi \circ M)(x) = e^x + x$$

and so $(H+\varphi\circ M)$ is surjective. Therefore, M is both maximal φ-monotone and (H,φ)-monotone.

Theorem 2.9. *Let X be a Banach space with the dual space X^*. Let $\varphi : X^* \to X^*, \eta : X \times X \to X$ be single valued mappings, $H : X \to X^*$ be a strictly η-monotone mapping and $M : X \to 2^{X^*}$ be a (H,η,φ)-monotone mapping if $\langle u - v, \eta(x,y)\rangle \geq 0, \forall (y,v) \in$ Graph$(\varphi\circ M)$. Then $u \in \varphi \circ M(x)$, where Graph$(\varphi \circ M) = \{(x,x^*) \in X \times X^* : x^* \in (\varphi\circ M)(x)\}$.*

Proof. Suppose that there exists (x_0,u_0) such that

$$\langle u_0 - v, \eta(x_0,y)\rangle \geq 0, \forall (y,v) \in \mathrm{Graph}(\varphi\circ M).$$

Since M is (H,η,φ)-monotone, we know that $(H+\varphi\circ M)(X) = X^*$ and so there exists $(x_1,u_1) \in \mathrm{Graph}(\varphi\circ M)$ such that

$$H(x_1) + u_1 = H(x_0) + u_0.$$

Thus, we have

$$\langle u_0 - u_1, \eta(x_0,x_1)\rangle = -\langle H(x_0) - H(x_1), \eta(x_0,x_1)\rangle \geq 0.$$

The strict η-monotonicity of H implies that $x_1 = x_0$. Thus from above, we have $u_1 = u_0$. Hence $(x_0,u_0) \in \mathrm{Graph}(\varphi\circ M)$, *i.e.*, $u_0 \in (\varphi\circ M)(x_0)$. □

Theorem 2.10. *Let X be a Banach space with the dual space X^*. Let $\varphi : X^* \to X^*, \eta : X \times X \to X$ be single valued mappings, $H : X \to X^*$ be a strictly η-monotone mapping and $M : X \to 2^{X^*}$ be a (H,η,φ)-monotone mapping. Then $(H+\rho\varphi\circ M)^{-1}$ is a single valued mapping.*

Proof. For any given $x^* \in X^*$, let $x, y \in (H+\rho\varphi\circ M)^{-1}(x^*)$. It follows that

$$\frac{1}{\rho}\{x^* - H(x)\} \in (\varphi\circ M)(x),$$
$$\frac{1}{\rho}\{x^* - H(y)\} \in (\varphi\circ M)(y).$$

From the η-monotonicity of $(\varphi \circ M)$ implies that

$$\frac{1}{\rho}\langle x^* - H(x) - (x^* - H(y)), \eta(x,y)\rangle = \frac{1}{\rho}\langle -H(x) + H(y), \eta(x,y)\rangle \geq 0.$$

This implies that $x = y$ and $(H + \rho\varphi \circ M)^{-1}$ is a single valued mapping. □

Definition 2.11. Let X be a reflexive Banach space with dual space X^*. Let $\varphi : X^* \to X^*$, $\eta : X \times X \to X$ be single valued mappings, $H : X \to X^*$ be a strictly η-monotone mapping and $M : X \to 2^{X^*}$ be a (H, η, φ)-monotone mapping. A proximal mapping $R_M^{H,\eta,\varphi} : X^* \to X$ is defined by

$$R_M^{H,\eta,\varphi}(x^*) = (H + \rho\varphi \circ M)^{-1}(x^*),\ \forall x^* \in X^*.$$

Theorem 2.12. *Let X be a reflexive Banach space with dual space X^*. Let $\varphi : X^* \to X^*$ be a single valued mapping, $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping, $H : X \to X^*$ be a γ-strongly η-monotone mapping and $M : X \to 2^{X^*}$ be a (H, η, φ)-monotone mapping. Then the proximal mapping $R_M^{H,\eta,\varphi} : X^* \to X$ is $\frac{\tau}{\gamma}$-Lipschitz continuous, i.e.,*

$$\|R_M^{H,\eta,\varphi}(x^*) - R_M^{H,\eta,\varphi}(y^*)\| \leq \frac{\tau}{\gamma}\|x^* - y^*\|,\ \forall x^*, y^* \in X^*.$$

Proof. Let $x^*, y^* \in X^*$. It follows that

$$\begin{aligned} R_M^{H,\eta,\varphi}(x^*) &= (H + \rho\varphi \circ M)^{-1}(x^*); \\ R_M^{H,\eta,\varphi}(y^*) &= (H + \rho\varphi \circ M)^{-1}(y^*) \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{\rho}\{x^* - H(R_M^{H,\eta,\varphi}(x^*))\} \in (\varphi \circ M)(R_M^{H,\eta,\varphi}(x^*)) \\ &\frac{1}{\rho}\{y^* - H(R_M^{H,\eta,\varphi}(y^*))\} \in (\varphi \circ M)(R_M^{H,\eta,\varphi}(y^*)). \end{aligned}$$

Since $(\varphi \circ M)$ is η-monotone, we have

$$\frac{1}{\rho}\langle x^* - H(R_M^{H,\eta,\varphi}(x^*)) - (y^* - H(R_M^{H,\eta,\varphi}(y^*))), \eta(R_M^{H,\eta,\varphi}(x^*), R_M^{H,\eta,\varphi}(y^*))\rangle \geq 0.$$

It follows that

$$\begin{aligned} &\tau\|x^* - y^*\|\|R_M^{H,\eta,\varphi}(x^*) - R_M^{H,\eta,\varphi}(y^*)\| \geq \|x^* - y^*\|\|\eta(R_M^{H,\eta,\varphi}(x^*), R_M^{H,\eta,\varphi}(y^*))\| \\ &\geq \langle x^* - y^*, \eta(R_M^{H,\eta,\varphi}(x^*), R_M^{H,\eta,\varphi}(y^*))\rangle \\ &\geq \langle H(R_M^{H,\eta,\varphi}(x^*)) - H(R_M^{H,\eta,\varphi}(y^*)), \eta(R_M^{H,\eta,\varphi}(x^*), R_M^{H,\eta,\varphi}(y^*))\rangle \\ &\geq \gamma\|R_M^{H,\eta,\varphi}(x^*) - R_M^{H,\eta,\varphi}(y^*)\|^2. \end{aligned}$$

Thus

$$\|R_M^{H,\eta,\varphi}(x^*) - R_M^{H,\eta,\varphi}(y^*)\| \le \frac{\tau}{\gamma}\|x^* - y^*\|, \ \forall x^*, y^* \in X^*. \qquad \square$$

Now, we formulate our main problems.

Let $N_i : X \times X \to X^*$, $\varphi : X^* \to X^*$, $\eta : X \times X \to X, g : X \to X$ be single valued mappings, and $P_i, G_i, T_i : X \to CB(X)(i = 1, \cdots, \ell)$ be set valued mappings. Let $H : X \to X^*$ be a strictly η-monotone mapping and $M : X \to 2^{X^*}$ be a (H, η, φ)-monotone mapping. Then the system of nonlinear variational type inclusions is to find $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)$ such that

$$\begin{aligned}
&0 \in H(g(x_1)) - H(g(x_2)) + \rho_1[N_1(u_1, v_1) + M(g(x_1), w_1)],\\
&0 \in H(g(x_2)) - H(g(x_3)) + \rho_2[N_2(u_2, v_2) + M(g(x_2), w_2)],\\
&\vdots\\
&0 \in H(g(x_{\ell-1})) - H(g(x_\ell)) + \rho_{\ell-1}[N_{\ell-1}(u_{\ell-1}, v_{\ell-1}) + M(g(x_{\ell-1}), w_{\ell-1})],\\
&0 \in H(g(x_\ell)) - H(g(x_1)) + \rho_\ell[N_\ell(u_\ell, v_\ell) + M(g(x_\ell), w_\ell)],
\end{aligned} \tag{2.1}$$

where ρ_i $(i = 1, 2, \cdots, \ell)$ is a positive constant.

Remarks 2.13.

(i) We note that if $i = \overline{1, 2, 3}$, then (2.1) reduces to finding $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)$ such that

$$\begin{aligned}
&0 \in H(g(x_1)) - H(g(x_2)) + \rho_1[N_1(u_1, v_1) + M(g(x_1), w_1)],\\
&0 \in H(g(x_2)) - H(g(x_3)) + \rho_2[N_2(u_2, v_2) + M(g(x_2), w_2)],\\
&0 \in H(g(x_3)) - H(g(x_1)) + \rho_3[N_3(u_3, v_3) + M(g(x_3), w_3)],
\end{aligned} \tag{2.2}$$

where ρ_i $(i = \overline{1, 2, 3})$ is a positive constant.

(ii) We note that if $i = \overline{1, 2}$, then (2.2) reduces to finding $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)$ such that

$$\begin{aligned}
&0 \in H(g(x_1)) - H(g(x_2)) + \rho_1[N_1(u_1, v_1) + M(g(x_1), w_1)],\\
&0 \in H(g(x_2)) - H(g(x_1)) + \rho_2[N_2(u_2, v_2) + M(g(x_2), w_2)],
\end{aligned} \tag{2.3}$$

where ρ_i $(i = \overline{1, 2})$ is a positive constant.

(iii) We note that if $i = \overline{1}$, then $x_1 = x, u_1 = u, v_1 = v, w_1 = w$ and (2.3) reduces to finding $x \in X, u \in T(x), v \in G(x), w \in P(x)$ such that

$$0 \in H(g(x)) - H(g(x^*)) + \rho[N(u, v) + M(g(x), w)], \ \forall x^* \in X, \tag{2.4}$$

where ρ is a positive constant.

3. Main Results

Theorem 3.1. *Let $N_i : X \times X \to X^*, g : X \to X, \eta : X \times X \to X$ be single valued mappings. Let a single valued mapping $\varphi_i : X^* \to X^*$ satisfying $\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$ (i.e., $Ker(\varphi_i) = \{x \in X^* : \varphi_i(x) = 0\}$), $(i = 1, \cdots, \ell)$. Let $M : X \times X \to 2^{X^*}$ be a (H, η, φ_i)-monotone mapping. Then (x_i, u_i, v_i, w_i) where $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)(i = 1, \cdots, \ell)$ is a solution of (2.1) if and only if*

$$
\begin{aligned}
g(x_1) &= R^{H,\eta,\varphi_1}_{M(\cdot,w_1)}[Hg(x_2) - \rho_1\varphi_1 \circ N_1(u_1, v_1)],\\
g(x_2) &= R^{H,\eta,\varphi_2}_{M(\cdot,w_2)}[Hg(x_3) - \rho_2\varphi_2 \circ N_2(u_2, v_2)],\\
&\ \vdots\\
g(x_{\ell-1}) &= R^{H,\eta,\varphi_{\ell-1}}_{M(\cdot,w_{\ell-1})}[Hg(x_\ell) - \rho_{\ell-1}\varphi_{\ell-1} \circ N_{\ell-1}(u_{\ell-1}, v_{\ell-1})],\\
g(x_\ell) &= R^{H,\eta,\varphi_\ell}_{M(\cdot,w_\ell)}[Hg(x_1) - \rho_\ell\varphi_\ell \circ N_\ell(u_\ell, v_\ell)],
\end{aligned}
\tag{3.1}
$$

where $\rho_i > 0(i = 1, \cdots, \ell)$ is a constant and

$$
R^{H,\eta,\varphi_i}_{M(\cdot,w_i)} = [H + \rho_i\varphi_i \circ M(\cdot, w_i)]^{-1}, \quad (i = 1, \cdots, \ell)
$$

is a resolvent operator.

Proof. From the definition of $R^{H,\eta,\varphi_1}_{M(\cdot,w_1)}$, we have

$$
\begin{aligned}
&Hg(x_2) - \rho_1\varphi_1 \circ N_1(u_1, v_1) \in [H + \rho_1\varphi_1 \circ M(\cdot, w_1)]g(x_1)\\
\Leftrightarrow &Hg(x_2) - \rho_1\varphi_1 \circ N_1(u_1, v_1) \in Hg(x_1) + \rho_1(\varphi_1 \circ M(g(x_1), w_1))\\
\Leftrightarrow &0 \in Hg(x_1) - Hg(x_2) + \rho_1\varphi_1[N_1(u_1, v_1) + M(g(x_1), w_1)].
\end{aligned}
$$

Since $\varphi_1(u_1 + u_2) = \varphi_1(u_1) + \varphi_2(u_2)$ and $Ker(\varphi_1) = \{0\}$. Thus

$$
0 \in Hg(x_1) - Hg(x_2) + \rho_1[N_1(u_1, v_1) + M(g(x_1), w_1)], \rho_1 > 0.
$$

Similarly, for all $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)$,

$$
\begin{aligned}
&0 \in Hg(x_2) - Hg(x_3) + \rho_2[N_2(u_2, v_2) + M(g(x_2), w_2)], \rho_2 > 0,\\
&\ \vdots\\
&0 \in Hg(x_\ell) - Hg(x_1) + \rho_\ell[N_\ell(u_\ell, v_\ell) + M(g(x_\ell), w_\ell)], \rho_\ell > 0.
\end{aligned}
$$

Thus $(x_i, u_i, v_i, w_i) \in X$ is a solution of (2.1). □

On the basis of Theorem 3.1., we define the following algorithm:

Algorithm 3.2. Let $T_i, G_i, P_i : X \to CB(X),\ N_i : X \times X \to X^*, \varphi_i : X^* \to X^*, \eta : X \times X \to X, g : X \to X(i = 1, \cdots, \ell)$ be mappings. Let $H : X \to X^*$ be a strictly η-monotone mapping and $M : X \to 2^{X^*}$ be a (H, η, φ_i)-monotone mapping. Then for any arbitrary chosen $x_i^0 \in X, u_i^0 \in T_i(x_1^0), v_i^0 \in G_i(x_1^0), w_i^0 \in P_i(x_1^0)$ $(i = 1, \cdots, \ell)$, there exists a point $x_1^1 \in X$ such that

$$x_1^1 = x_1^0 - g(x_1^0) + R_{M(\cdot,w_1^0)}^{H,\eta,\varphi_1}[Hg(x_2^0) - \rho_1\varphi_1 \circ N_1(u_1^0, v_1^0)], \rho_1 > 0,$$

where

$$\begin{aligned} g(x_2^0) &= R_{M(\cdot,w_2^0)}^{H,\eta,\varphi_2}[Hg(x_3^0) - \rho_2\varphi_2 \circ N_2(u_2^0, v_2^0)], \rho_2 > 0, \\ &\vdots \\ g(x_\ell^0) &= R_{M(\cdot,w_\ell^0)}^{H,\eta,\varphi_\ell}[Hg(x_1^0) - \rho_\ell\varphi_\ell \circ N_\ell(u_\ell^0, v_\ell^0)], \rho_\ell > 0. \end{aligned}$$

By Nadler's Theorem [29], there exist $u_1^1 \in T_1(x_1^1), v_1^1 \in G_1(x_1^1), w_1^1 \in P_1(x_1^1)$ such that

$$\begin{aligned} \|u_1^1 - u_1^0\| &\leq (1 + (1+0)^{-1})\mathfrak{H}(T_1(x_1^1), T_1(x_1^0)), \\ \|v_1^1 - v_1^0\| &\leq (1 + (1+0)^{-1})\mathfrak{H}(G_1(x_1^1), G_1(x_1^0)), \\ \|w_1^1 - w_1^0\| &\leq (1 + (1+0)^{-1})\mathfrak{H}(P_1(x_1^1), P_1(x_1^0)), \\ &\vdots \\ \|u_\ell^1 - u_\ell^0\| &\leq (1 + (1+0)^{-1})\mathfrak{H}(T_\ell(x_1^1), T_\ell(x_1^0)), \\ \|v_\ell^1 - v_\ell^0\| &\leq (1 + (1+0)^{-1})\mathfrak{H}(G_\ell(x_1^1), G_\ell(x_1^0)), \\ \|w_\ell^1 - w_\ell^0\| &\leq (1 + (1+0)^{-1})\mathfrak{H}(P_\ell(x_1^1), P_\ell(x_1^0)). \end{aligned}$$

Similarly, for given $x_1^1 \in X, u_i^1 \in T_i(x_1^1), v_i^1 \in G_i(x_1^1), w_i^1 \in P_i(x_1^1) (i = 1, \cdots, \ell)$, there exists a point $x_1^2 \in X$ such that

$$x_1^2 = x_1^1 - g(x_1^1) + R_{M(\cdot,w_1^1)}^{H,\eta,\varphi_1}[Hg(x_2^1) - \rho_1\varphi_1 \circ N_1(u_1^1, v_1^1)], \rho_1 > 0,$$

where

$$\begin{aligned} g(x_2^1) &= R_{M(\cdot,w_2^1)}^{H,\eta,\varphi_2}[Hg(x_3^1) - \rho_2\varphi_2 \circ N_2(u_2^1, v_2^1)], \rho_2 > 0, \\ &\vdots \\ g(x_\ell^1) &= R_{M(\cdot,w_\ell^1)}^{H,\eta,\varphi_\ell}[Hg(x_1^1) - \rho_\ell\varphi_\ell \circ N_\ell(u_\ell^1, v_\ell^1)], \rho_\ell > 0. \end{aligned}$$

Again, from the Nadler's Theorem [29], there exist $u_1^2 \in T_1(x_1^2), v_1^2 \in G_1(x_1^2), w_1^2 \in P_1(x_1^2)$ such that

$$\begin{aligned} \|u_i^2 - u_i^1\| &\leq (1 + (1+1)^{-1})\mathfrak{H}(T_i(x_1^2), T_i(x_1^1)), \\ \|v_i^2 - v_i^1\| &\leq (1 + (1+1)^{-1})\mathfrak{H}(G_i(x_1^2), G_i(x_1^1)), \\ \|w_i^2 - w_i^1\| &\leq (1 + (1+1)^{-1})\mathfrak{H}(P_i(x_1^2), P_i(x_1^1)) \ (i = 1, \cdots, \ell). \end{aligned}$$

We can compute $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}, \{w_i^n\}$, by induction schemes:

$$x_1^{n+1} = x_1^n - g(x_1^n) + R^{H,\eta,\varphi_1}_{M(\cdot,w_1^n)}[Hg(x_2^n) - \rho_1\varphi_1 \circ N_1(u_1^n, v_1^n)], \rho_1 > 0, \tag{3.2}$$

where

$$\begin{aligned} g(x_2^n) &= R^{H,\eta,\varphi_2}_{M(\cdot,w_2^n)}[Hg(x_3^n) - \rho_2\varphi_2 \circ N_2(u_2^n, v_2^n)], \rho_2 > 0,\\ &\vdots\\ g(x_\ell^n) &= R^{H,\eta,\varphi_\ell}_{M(\cdot,w_\ell^n)}[Hg(x_1^n) - \rho_\ell\varphi_\ell \circ N_\ell(u_\ell^n, v_\ell^n)], \rho_\ell > 0, \end{aligned}$$

and

$$\begin{aligned} u_i^n \in T_i(x_1^n) &: \|u_i^{n+1} - u_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(T_i(x_1^{n+1}), T_i(x_1^n)),\\ v_i^n \in G_i(x_1^n) &: \|v_i^{n+1} - v_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(G_i(x_1^{n+1}), G_i(x_1^n)),\\ w_i^n \in P_i(x_1^n) &: \|w_i^{n+1} - w_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(P_i(x_1^{n+1}), P_i(x_1^n)) \ (i = 1, \cdots, \ell). \end{aligned}$$

If $i = \overline{1,2,3}$, then Algorithm 3.2 reduces to three step iterative algorithm, *see* [3, 4, 5], which are as following:

Algorithm 3.3. For any arbitrary chosen $x_1^0 \in X, u_1^0 \in T_1(x_1^0), v_1^0 \in G_1(x_1^0), w_1^0 \in P_1(x_1^0)$, compute $\{x_1^n\}$ by iterative schemes:

$$x_1^{n+1} = x_1^n - g(x_1^n) + R^{H,\eta,\varphi_1}_{M(\cdot,w_1^n)}[Hg(x_2^n) - \rho_1\varphi_1 \circ N_1(u_1^n, v_1^n)], \rho_1 > 0, \tag{3.3}$$

where

$$\begin{aligned} g(x_2^n) &= R^{H,\eta,\varphi_2}_{M(\cdot,w_2^n)}[Hg(x_3^n) - \rho_2\varphi_2 \circ N_2(u_2^n, v_2^n)], \rho_2 > 0,\\ g(x_3^n) &= R^{H,\eta,\varphi_3}_{M(\cdot,w_3^n)}[Hg(x_1^n) - \rho_3\varphi_3 \circ N_3(u_3^n, v_3^n)], \rho_3 > 0 \end{aligned}$$

and

$$\begin{aligned} u_i^n \in T_i(x_1^n) &: \|u_i^{n+1} - u_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(T_i(x_1^{n+1}), T_i(x_1^n)),\\ v_i^n \in G_i(x_1^n) &: \|v_i^{n+1} - v_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(G_i(x_1^{n+1}), G_i(x_1^n)),\\ w_i^n \in P_i(x_1^n) &: \|w_i^{n+1} - w_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(P_i(x_1^{n+1}), P_i(x_1^n)) \ (i = \overline{1,2,3}). \end{aligned}$$

We note that the Algorithm 3.3 gives the approximate solution to the variational inclusions (2.2).

If $i = \overline{1,2}$, then Algorithm 3.2, reduces to the Ishikawa type iterative algorithm, *see* [1, 7, 17, 20], which are as following:

Algorithm 3.4. For any arbitrary chosen $x_1^0 \in X, u_1^0 \in T_1(x_1^0), v_1^0 \in G_1(x_1^0), w_1^0 \in P_1(x_1^0)$, compute $\{x_1^n\}$ by iterative schemes

$$x_1^{n+1} = x_1^n - g(x_1^n) + R^{H,\eta,\varphi_1}_{M(\cdot,w_1^n)}[Hg(x_2^n) - \rho_1\varphi_1 \circ N_1(u_1^n, v_1^n)], \rho_1 > 0 \tag{3.4}$$

where

$$g(x_2^n) = R^{H,\eta,\varphi_2}_{M(\cdot,w_2^n)}[Hg(x_1^n) - \rho_2\varphi_2 \circ N_2(u_2^n, v_2^n)], \rho_2 > 0$$

and

$$u_i^n \in T_i(x_1^n) : \|u_i^{n+1} - u_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(T_i(x_1^{n+1}), T_i(x_1^n)),$$
$$v_i^n \in G_i(x_1^n) : \|v_i^{n+1} - v_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(G_i(x_1^{n+1}), G_i(x_1^n)),$$
$$w_i^n \in P_i(x_1^n) : \|w_i^{n+1} - w_i^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(P_i(x_1^{n+1}), P_i(x_1^n)) \ (i = \overline{1,2}).$$

We note that the Algorithm 3.4 gives the approximate solution to the variational inclusions (2.3).

If $i = \overline{1}$, then $x_1 = x$ and we have the following Mann type iterative schemes:

Algorithm 3.5. For any arbitrary chosen $x^0 \in X, u^0 \in T(x^0), v^0 \in G(x^0), w^0 \in P(x^0)$, compute $\{x^n\}$ by iterative schemes

$$x^{n+1} = x^n - g(x^n) + R^{H,\eta,\varphi}_{M(\cdot,w^n)}[Hg(x^n) - \rho\varphi \circ N(u^n, v^n)], \rho > 0 \tag{3.5}$$

and

$$u^n \in T(x^n) : \|u^{n+1} - u^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(T(x^{n+1}), T(x^n)),$$
$$v^n \in G(x^n) : \|v^{n+1} - v^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(G(x^{n+1}), G(x^n)),$$
$$w^n \in P(x^n) : \|w^{n+1} - w^n\| \le (1 + (1+n)^{-1})\mathfrak{H}(P(x^{n+1}), P(x^n)).$$

We note that the Algorithm 3.5 gives the approximate solution to the variational inclusions (2.4).

Now, we prove that following theorem which ensures the convergence of the iterative sequences generated by Algorithm 3.2.

Theorem 3.6. *Let X be a uniformly smooth Banach space with modulus of smoothness $\rho_X(t) \le ct^2$ for some $c > 0$ and X^* its dual space. Let $g : X \to X$ be a κ-strongly accretive and δ-Lipschitz continuous, $H : X \to X^*$ be a γ-strongly η-monotone and s-Lipschitz continuous with respect to g and $N_i : X \times X \to X^*$ be (r_i, p_i)-Lipschitz continuous with respect to first and second argument, respectively. Let $T_i, G_i, P_i : X \to CB(X^*)$ be a $(\zeta_{T_i}, \zeta_{G_i}, \zeta_{P_i})$-Lipschitz continuous for $(i = 1, \cdots, \ell)$ respectively. Let $\varphi_i : X^* \to X^*$ be a single valued mapping satisfying $\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$ and $\varphi_i : X^* \to X^*$ be a θ_i-Lipschitz continuous with respect to $N_i, (i = 1, \cdots, \ell)$, respectively. Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping and $M : X \times X \to 2^{X^*}$ be a (H, η, φ_i)-monotone mapping where $(i = 1, \cdots, \ell)$. Assume that*

$$\|R^{H,\eta,\varphi_i}_{M(\cdot,w_i^n)}(x) - R^{H,\eta,\varphi_i}_{M(\cdot,w_i^{n-1})}(x)\| \le \xi_i\|w_i^n - w_i^{n-1}\| \tag{3.6}$$

where ξ_i is a positive constant for $(i = 1, \cdots, \ell)$ and

$$0 < \Omega + \frac{\tau}{\gamma}\bigg\{\frac{\tau^{\ell-1}s^{\ell-1}\delta^{\ell-1}\rho_\ell\theta_\ell(r_\ell\zeta_{T_\ell} + p_\ell\zeta_{G_\ell})}{\gamma^{\ell-1}\kappa^{\ell-1}} + \frac{\tau^{\ell-2}s^{\ell-1}\delta^{\ell-1}\xi_\ell\zeta_{P_\ell}}{\kappa^{\ell-1}\gamma^{\ell-2}} + \cdots$$

$$+\rho_1\theta_1(r_1\zeta_{T_1}+p_1\zeta_{G_1})\Big\}<1,$$

$$\frac{\tau^{\ell-1}s^{\ell-1}\delta^{\ell-1}\rho_\ell\theta_\ell(r_\ell\zeta_{T_\ell}+p_\ell\zeta_{G_\ell})}{\gamma^{\ell-1}\kappa^{\ell-1}}+\frac{\tau^{\ell-2}s^{\ell-1}\delta^{\ell-1}\xi_\ell\zeta_{P_\ell}}{\kappa^{\ell-1}\gamma^{\ell-2}}+\cdots+\rho_1\theta_1(r_1\zeta_{T_1}+p_1\zeta_{G_1})>0,$$

$$\Omega=\sqrt{1-2\kappa+64c\delta^2}+\xi_1\zeta_{P_1}+\frac{\tau^\ell s^\ell\delta^\ell}{\gamma^\ell\kappa^{\ell-1}}>0. \tag{3.7}$$

Then the sequences $\{x_i^n\},\{u_i^n\},\{v_i^n\},\{w_i^n\}(i=1,\cdots,\ell)$ *generated by Algorithm 3.2 converge strongly to the unique solutions* (x_i,u_i,v_i,w_i) *respectively, where* $x_i\in X,u_i\in T_i(x_1),v_i\in G_i(x_1),w_i\in P_i(x_1)$ *is a solution of* (2.1).

Proof. From Algorithm 3.2 and Theorem 2.12, we have

$$\|x_1^{n+1}-x_1^n\|=\|x_1^n-g(x_1^n)+R^{H,\eta,\varphi_1}_{M(\cdot,w_1^n)}[Hg(x_2^n)-\rho_1\varphi_1\circ N_1(u_1^n,v_1^n)]$$

$$-x_1^{n-1}+g(x_1^{n-1})-R^{H,\eta,\varphi_1}_{M(\cdot,w_1^{n-1})}[Hg(x_2^{n-1})-\rho_1\varphi_1\circ N_1(u_1^{n-1},v_1^{n-1})]\|$$

$$\le\|x_1^n-x_1^{n-1}-(g(x_1^n)-g(x_1^{n-1}))\|+\|R^{H,\eta,\varphi_1}_{M(\cdot,w_1^n)}[Hg(x_2^n)-\rho_1\varphi_1\circ N_1(u_1^n,v_1^n)]$$

$$-R^{H,\eta,\varphi_1}_{M(\cdot,w_1^n)}[Hg(x_2^{n-1})-\rho_1\varphi_1\circ N_1(u_1^{n-1},v_1^{n-1})]\|+\|R^{H,\eta,\varphi_1}_{M(\cdot,w_1^n)}[Hg(x_2^{n-1})$$

$$-\rho_1\varphi_1\circ N_1(u_1^{n-1},v_1^{n-1})]-R^{H,\eta,\varphi_1}_{M(\cdot,w_1^{n-1})}[Hg(x_2^{n-1})-\rho_1\varphi_1\circ N_1(u_1^{n-1},v_1^{n-1})]\|$$

$$\le\|x_1^n-x_1^{n-1}-(g(x_1^n)-g(x_1^{n-1}))\|+\tfrac{\tau}{\gamma}\|Hg(x_2^n)-\rho_1\varphi_1\circ N_1(u_1^n,v_1^n)-Hg(x_2^{n-1})$$

$$+\rho_1\varphi_1\circ N_1(u_1^{n-1},v_1^{n-1})\|+\xi_1\|w_1^n-w_1^{n-1}\|$$

$$\le\|x_1^n-x_1^{n-1}-(g(x_1^n)-g(x_1^{n-1}))\|+\frac{\tau}{\gamma}\|Hg(x_2^n)-Hg(x_2^{n-1})\|$$

$$+\frac{\tau\rho_1}{\gamma}\|\varphi_1\circ N_1(u_1^n,v_1^n)-\varphi_1\circ N_1(u_1^{n-1},v_1^{n-1})\|+\xi_1\|w_1^n-w_1^{n-1}\|. \tag{3.8}$$

Since g is a κ-strongly accretive and δ-Lipschitz continuous and from Lemma 2.5, we have

$$\|x_1^n-x_1^{n-1}-(g(x_1^n)-g(x_1^{n-1}))\|^2=\|x_1^n-x_1^{n-1}\|^2$$

$$+2\langle j(x_1^n-x_1^{n-1}-g(x_1^n)+g(x_1^{n-1})),-(g(x_1^n)-g(x_1^{n-1}))\rangle$$

$$\le\|x_1^n-x_1^{n-1}\|^2-2\langle j(x_1^n-x_1^{n-1}),g(x_1^n)-g(x_1^{n-1})\rangle$$

$$+2\langle j(x_1^n-x_1^{n-1}-g(x_1^n)+g(x_1^{n-1}))-j(x_1^n-x_1^{n-1}),-(g(x_1^n)-g(x_1^{n-1}))\rangle$$

$$\leq \|x_1^n - x_1^{n-1}\|^2 - 2\kappa\|x_1^n - x_1^{n-1}\|^2 + 4d^2\rho_X\Big(\tfrac{4\|g(x_1^n)-g(x_1^{n-1})\|}{d}\Big)$$

$$\leq (1-2\kappa)\|x_1^n - x_1^{n-1}\|^2 + 64c\|g(x_1^n) - g(x_1^{n-1})\|^2$$

$$\leq (1-2\kappa+64c\delta^2)\|x_1^n - x_1^{n-1}\|^2. \tag{3.9}$$

Since H and g are Lipschitz continuous, we have

$$\begin{aligned}
\|Hg(x_1^n) - Hg(x_1^{n-1})\| &\leq s\delta\|x_1^n - x_1^{n-1}\|,\\
\|Hg(x_2^n) - Hg(x_2^{n-1})\| &\leq s\delta\|x_2^n - x_2^{n-1}\|,\\
&\vdots\\
\|Hg(x_\ell^n) - Hg(x_\ell^{n-1})\| &\leq s\delta\|x_\ell^n - x_\ell^{n-1}\|. \qquad (3.10)
\end{aligned}$$

Since φ_1 is a θ_1-Lipschitz continuous with respect to $N_1(\cdot,\cdot)$ and $N_1(\cdot,\cdot)$ is a (r_1,p_1)-Lipschitz continuous with respect to first and second argument, *respectively.* T_1 and G_1 are ζ_{T_1}-$\mathfrak{H}$-Lipschitz continuous and ζ_{G_1}-$\mathfrak{H}$-Lipschitz continuous *respectively*, we have

$$\begin{aligned}
\|\varphi_1\circ N_1(u_1^n,v_1^n) - \varphi_1\circ N_1(u_1^{n-1},v_1^{n-1})\| &\leq \theta_1\|N_1(u_1^n,v_1^n) - N_1(u_1^{n-1},v_1^{n-1})\|\\
&\leq \theta_1\|N_1(u_1^n,v_1^n) - N_1(u_1^{n-1},v_1^n) + N_1(u_1^{n-1},v_1^n) - N_1(u_1^{n-1},v_1^{n-1})\|\\
&\leq \theta_1\|N_1(u_1^n,v_1^n) - N_1(u_1^{n-1},v_1^n)\| + \theta_1\|N_1(u_1^{n-1},v_1^n) - N_1(u_1^{n-1},v_1^{n-1})\|\\
&\leq \theta_1 r_1\|u_1^n - u_1^{n-1}\| + \theta_1 p_1\|v_1^n - v_1^{n-1}\|\\
&\leq \theta_1 r_1(1+(1+n)^{-1})\mathfrak{H}(T_1(x_1^n),T_1(x_1^{n-1}))\\
&\quad + \theta_1 p_1(1+(1+n)^{-1})\mathfrak{H}(G_1(x_1^n),G_1(x_1^{n-1}))\\
&\leq (1+(1+n)^{-1})\theta_1(r_1\zeta_{T_1} + p_1\zeta_{G_1})\|x_1^n - x_1^{n-1}\|. \qquad (3.11)
\end{aligned}$$

Again, P_1 is a ζ_{P_1}-$\mathfrak{H}$-Lipschitz continuous, we have

$$\begin{aligned}
\|w_1^n - w_1^{n-1}\| &\leq (1+(1+n)^{-1})\mathfrak{H}(P_1(x_1^n),P_1(x_1^{n-1}))\\
&\leq (1+(1+n)^{-1})\zeta_{P_1}\|x_1^n - x_1^{n-1}\|. \qquad (3.12)
\end{aligned}$$

From (3.8)–(3.12), we have

$$\begin{aligned}
\|x_1^{n+1} - x_1^n\| \leq \{&\sqrt{1-2\kappa+64c\delta^2} + (1+(1+n)^{-1})\Big(\frac{\tau\rho_1\theta_1(r_1\zeta_{T_1}+p_1\zeta_{G_1})}{\gamma} + \xi_1\zeta_{P_1}\Big)\}\times\\
&\|x_1^n - x_1^{n-1}\| + \frac{\tau s\delta}{\gamma}\|x_2^n - x_2^{n-1}\|. \qquad (3.13)
\end{aligned}$$

Again, since $g: X\to X$ is a κ-strongly accretive and δ-Lipschitz continuous, we have

$$\begin{aligned}
\|g(x_2^n) - g(x_2^{n-1})\|\|x_2^n - x_2^{n-1}\| &\geq \langle g(x_2^n) - g(x_2^{n-1}), j(x_2^n - x_2^{n-1})\rangle\\
&\geq \kappa\|x_2^n - x_2^{n-1}\|^2. \qquad (3.14)
\end{aligned}$$

Therefore

$$\|x_2^n - x_2^{n-1}\| \le \frac{1}{\kappa}\|g(x_2^n) - g(x_2^{n-1})\|$$
$$\le \frac{1}{\kappa}\|R^{H,\eta,\varphi_2}_{M(\cdot,w_2^n)}[Hg(x_3^n) - \rho_2\varphi_2 \circ N_2(u_2^n, v_2^n)] - R^{H,\eta,\varphi_2}_{M(\cdot,w_2^{n-1})}[t]\|$$
$$\le \frac{1}{\kappa}\{\|R^{H,\eta,\varphi_2}_{M(\cdot,w_2^n)}[Hg(x_3^n) - \rho_2\varphi_2 \circ N_2(u_2^n, v_2^n)] - R^{H,\eta,\varphi_2}_{M(\cdot,w_2^n)}[t]\|$$
$$+ \|R^{H,\eta,\varphi_2}_{M(\cdot,w_2^n)}[t] - R^{H,\eta,\varphi_2}_{M(\cdot,w_2^{n-1})}[t]\|\}$$
$$\le \frac{\tau}{\kappa\gamma}\{\|Hg(x_3^n) - Hg(x_3^{n-1})\| + \rho_2\|\varphi_2 \circ N_2(u_2^n, v_2^n) - \varphi_2 \circ N_2(u_2^{n-1}, v_2^{n-1})\|\}$$
$$+ \frac{\xi_2}{\kappa}\|w_2^n - w_2^{n-1}\|$$
$$\le \frac{\tau}{\kappa\gamma}\|Hg(x_3^n) - Hg(x_3^{n-1})\| + \frac{\tau\rho_2}{\kappa\gamma}\|\varphi_2 \circ N_2(u_2^n, v_2^n) - \varphi_2 \circ N_2(u_2^{n-1}, v_2^{n-1})\|$$
$$+\frac{\xi_2}{\kappa}\|w_2^n - w_2^{n-1}\|, \tag{3.15}$$

where $t = Hg(x_3^{n-1}) - \rho_2\varphi_2 \circ N_2(u_2^{n-1}, v_2^{n-1})$.
Since φ_2 is a θ_2-Lipschitz continuous with respect to $N_2(\cdot,\cdot)$ and $N_2(\cdot,\cdot)$ is a (r_2, p_2)-Lipschitz continuous with respect to first and second argument, *respectively*, and T_2, G_2 are ζ_{T_2}-$\mathfrak{H}$-Lipschitz continuous and ζ_{G_2}-$\mathfrak{H}$-Lipschitz continuous, we have

$$\|\varphi_2 \circ N_2(u_2^n, v_2^n) - \varphi_2 \circ N_2(u_2^{n-1}, v_2^{n-1})\| \le \theta_2\|N_2(u_2^n, v_2^n) - N_2(u_2^{n-1}, v_2^{n-1})\|$$
$$\le (1 + (1+n)^{-1})\theta_2(r_2\zeta_{T_2} + p_2\zeta_{G_2})\|x_1^n - x_1^{n-1}\|. \tag{3.16}$$

Again, P_2 is a ζ_{P_2}-$\mathfrak{H}$-Lipschitz continuous, we have

$$\|w_2^n - w_2^{n-1}\| \le (1 + (1+n)^{-1})\mathfrak{H}(P_2(x_1^n), P_2(x_1^{n-1}))$$
$$\le (1 + (1+n)^{-1})\zeta_{P_2}\|x_1^n - x_1^{n-1}\|. \tag{3.17}$$

Again, from (3.10),(3.15)–(3.17), we have

$$\|x_2^n - x_2^{n-1}\| \le \tfrac{\tau s\delta}{\kappa\gamma}\|x_3^n - x_3^{n-1}\|$$
$$+(1 + (1+n)^{-1})\Big\{\frac{\tau\rho_2\theta_2(r_2\zeta_{T_2} + p_2\zeta_{G_2})}{\kappa\gamma} + \frac{\xi_2\zeta_{P_2}}{\kappa}\Big\}\|x_1^n - x_1^{n-1}\|. \tag{3.18}$$

Similarly, we have

$$\|x_3^n - x_3^{n-1}\| \le \tfrac{1}{\kappa}\|g(x_3^n) - g(x_3^{n-1})\| \le \tfrac{\tau}{\kappa\gamma}\|Hg(x_4^n) - Hg(x_4^{n-1})\|$$
$$+\frac{\tau\rho_3}{\kappa\gamma}\|\varphi_3 \circ N_3(u_3^n, v_3^n) - \varphi_3 \circ N_3(u_3^{n-1}, v_3^{n-1})\| + \frac{\xi_3}{\kappa}\|w_3^n - w_3^{n-1}\|. \tag{3.19}$$

Since φ_3 is a θ_3-Lipschitz continuous with respect to $N_3(\cdot,\cdot)$ and $N_3(\cdot,\cdot)$ is a (r_3,p_3)-Lipschitz continuous with respect to first and second argument, *respectively*, and T_3, G_3 are ζ_{T_3}-$\mathfrak{H}$-Lipschitz continuous and ζ_{G_3}-$\mathfrak{H}$-Lipschitz continuous, we have

$$\begin{aligned}&\|\varphi_3\circ N_3(u_3^n,v_3^n)-\varphi_3\circ N_3(u_3^{n-1},v_3^{n-1})\|\\ &\qquad\le (1+(1+n)^{-1})\theta_3(r_3\zeta_{T_3}+p_3\zeta_{G_3})\|x_1^n-x_1^{n-1}\|.\end{aligned}\tag{3.20}$$

Again, P_3 is a ζ_{P_3}-$\mathfrak{H}$-Lipschitz continuous, we have

$$\begin{aligned}\|w_3^n-w_3^{n-1}\| &\le (1+(1+n)^{-1})\mathfrak{H}(P_3(x_1^n),P_3(x_1^{n-1}))\\ &\le (1+(1+n)^{-1})\zeta_{P_3}\|x_1^n-x_1^{n-1}\|.\end{aligned}\tag{3.21}$$

Again from (3.10),(3.18)–(3.21), we have

$$\begin{aligned}\|x_3^n-x_3^{n-1}\| &\le \frac{\tau s\delta}{\kappa\gamma}\|x_4^n-x_4^{n-1}\|\\ &\quad+(1+(1+n)^{-1})\Big\{\frac{\tau\rho_3\theta_3(r_3\zeta_{T_3}+p_3\zeta_{G_3})}{\kappa\gamma}+\frac{\xi_3\zeta_{P_3}}{\kappa}\Big\}\|x_1^n-x_1^{n-1}\|.\end{aligned}\tag{3.22}$$

Continuing this process, we have

$$\begin{aligned}\|x_{\ell-1}^n-x_{\ell-1}^{n-1}\| &\le \frac{\tau s\delta}{\kappa\gamma}\|x_\ell^n-x_\ell^{n-1}\|\\ &\quad+(1+(1+n)^{-1})\Big\{\frac{\tau\rho_{\ell-1}\theta_{\ell-1}(r_{\ell-1}\zeta_{T_{\ell-1}}+p_{\ell-1}\zeta_{G_{\ell-1}})}{\kappa\gamma}\\ &\quad+\frac{\xi_{\ell-1}\zeta_{P_{\ell-1}}}{\kappa}\Big\}\|x_1^n-x_1^{n-1}\|.\end{aligned}\tag{3.23}$$

Again,

$$\begin{aligned}\|x_\ell^n-x_\ell^{n-1}\| &\le \frac{1}{\kappa}\|g(x_\ell^n)-g(x_\ell^{n-1})\| \le \frac{1}{\kappa}\|R^{H,\eta,\varphi_\ell}_{M(\cdot,w_\ell^n)}[Hg(x_1^n)-\rho_\ell\varphi_\ell\circ N_\ell(u_\ell^n,v_\ell^n)]\\ &\qquad -R^{H,\eta,\varphi_\ell}_{M(\cdot,w_\ell^{n-1})}[Hg(x_1^{n-1})-\rho_\ell\varphi_\ell\circ N_\ell(u_\ell^{n-1},v_\ell^{n-1})]\|\\ &\le \frac{\tau}{\kappa\gamma}\|Hg(x_1^n)-Hg(x_1^{n-1})\|+\frac{\tau\rho_\ell}{\kappa\gamma}\|\varphi_\ell\circ N_\ell(u_\ell^n,v_\ell^n)-\varphi_\ell\circ N_\ell(u_\ell^{n-1},v_\ell^{n-1})\|\\ &\qquad +\frac{\xi_\ell}{\kappa}\|w_\ell^n-w_\ell^{n-1}\|.\end{aligned}\tag{3.24}$$

Since φ_ℓ is a θ_ℓ-Lipschitz continuous with respect to $N_\ell(\cdot,\cdot)$ and $N_\ell(\cdot,\cdot)$ is a (r_ℓ,p_ℓ)-Lipschitz continuous with respect to first and second argument, *respectively*, and T_ℓ, G_ℓ are ζ_{T_ℓ}-$\mathfrak{H}$-Lipschitz continuous and ζ_{G_ℓ}-$\mathfrak{H}$-Lipschitz continuous, we have

$$\|\varphi_\ell\circ N_\ell(u_\ell^n,v_\ell^n)-\varphi_\ell\circ N_\ell(u_\ell^{n-1},v_\ell^{n-1})\| \le \theta_\ell\|N_\ell(u_\ell^n,v_\ell^n)-N_\ell(u_\ell^{n-1},v_\ell^{n-1})\|$$

$$\leq (1+(1+n)^{-1})\theta_\ell(r_\ell\zeta_{T_\ell}+p_\ell\zeta_{G_\ell})\|x_1^n-x_1^{n-1}\|. \tag{3.25}$$

Again, P_ℓ is a ζ_{P_ℓ}-$\mathfrak{H}$-Lipschitz continuous, we have

$$\begin{aligned}\|w_\ell^n-w_\ell^{n-1}\| &\leq (1+(1+n)^{-1})\mathfrak{H}(P_\ell(x_1^n),P_\ell(x_1^{n-1}))\\ &\leq (1+(1+n)^{-1})\zeta_{P_\ell}\|x_1^n-x_1^{n-1}\|.\end{aligned} \tag{3.26}$$

Again from (3.10),(3.23)-(3.26), we have

$$\begin{aligned}\|x_\ell^n-x_\ell^{n-1}\| \leq{}& \frac{\tau s\delta}{\kappa\gamma}\|x_1^n-x_1^{n-1}\|+\frac{\tau\rho_\ell}{\kappa\gamma}(1+(1+n)^{-1})\theta_\ell(r_\ell\zeta_{T_\ell}+p_\ell\zeta_{G_\ell})\|x_1^n-x_1^{n-1}\|\\ &+\frac{\xi_\ell\zeta_{P_\ell}}{\kappa}(1+(1+n)^{-1})\|x_1^n-x_1^{n-1}\|\\ \leq{}& \left\{\frac{\tau s\delta}{\kappa\gamma}+\Big(\frac{\tau\rho_\ell\theta_\ell(r_\ell\zeta_{T_\ell}+p_\ell\zeta_{G_\ell})}{\kappa\gamma}+\frac{\xi_\ell\zeta_{P_\ell}}{\kappa}\Big)(1+(1+n)^{-1})\right\}\|x_1^n-x_1^{n-1}\|.\end{aligned} \tag{3.27}$$

Now,

$$\begin{aligned}\|x_{\ell-1}^n-x_{\ell-1}^{n-1}\| \leq{}& \Bigg[\frac{\tau^2s^2\delta^2}{\gamma^2\kappa^2}+(1+(1+n)^{-1})\Big(\frac{\tau^2s\delta\rho_\ell\theta_\ell(r_\ell\zeta_{T_\ell}+p_\ell\zeta_{G_\ell})}{\kappa^2\gamma^2}+\frac{\tau s\delta\xi_\ell\zeta_{P_\ell}}{\kappa^2\gamma}\Big)\\ &+(1+(1+n)^{-1})\Big(\frac{\tau\rho_{\ell-1}(r_{\ell-1}\zeta_{T_{\ell-1}}+p_{\ell-1}\zeta_{G_{\ell-1}})}{\kappa\gamma}+\frac{\xi_{\ell-1}\zeta_{P_{\ell-1}}}{\kappa}\Big)\Bigg]\|x_1^n-x_1^{n-1}\|,\end{aligned} \tag{3.28}$$

$$\vdots$$

$$\begin{aligned}\|x_2^n-x_2^{n-1}\| \leq{}& \Bigg[\frac{\tau^{\ell-1}s^{\ell-1}\delta^{\ell-1}}{\gamma^{\ell-1}\kappa^{\ell-1}}+(1+(1+n)^{-1})\bigg\{\Big(\frac{\tau^{\ell-1}s^{\ell-2}\delta^{\ell-2}\rho_\ell\theta_\ell(r_\ell\zeta_{T_\ell}+p_\ell\zeta_{G_\ell})}{\kappa^{\ell-1}\gamma^{\ell-1}}+\\ &\frac{\tau^{\ell-2}s^{\ell-2}\delta^{\ell-2}\xi_\ell\zeta_{P_\ell}}{\kappa^{\ell-1}\gamma^{\ell-2}}\Big)+\cdots+\Big(\frac{\tau\rho_2\theta_2(r_2\zeta_{T_2}+p_2\zeta_{G_2})}{\kappa\gamma}+\frac{\xi_2\zeta_{P_2}}{\kappa}\Big)\bigg\}\Bigg]\|x_1^n-x_1^{n-1}\|.\end{aligned} \tag{3.29}$$

Again,

$$\|x_1^{n+1}-x_1^n\| \leq \mu_n\|x_1^n-x_1^{n-1}\|, \tag{3.30}$$

where

$$\begin{aligned}\mu_n ={}& \Omega_n+(1+(1+n)^{-1})\bigg\{\frac{\tau\rho_1\theta_1(r_1\zeta_{T_1}+p_1\zeta_{G_1})}{\gamma}+\Big(\frac{\tau^\ell s^{\ell-1}\delta^{\ell-1}\rho_\ell\theta_\ell(r_\ell\zeta_{T_\ell}+p_\ell\zeta_{G_\ell})}{\gamma^\ell\kappa^{\ell-1}}\\ &+\frac{\tau^{\ell-1}s^{\ell-1}\delta^{\ell-1}\xi_\ell\zeta_{P_\ell}}{\kappa^{\ell-1}\gamma^{\ell-1}}\Big)+\cdots+\Big(\frac{\tau^2s\delta\theta_2\rho_2(r_2\zeta_{T_2}+p_2\zeta_{G_2})}{\kappa\gamma^2}+\frac{\xi_2\zeta_{P_2}\tau s\delta}{\kappa\gamma}\Big)\bigg\},\\ \Omega_n ={}& \sqrt{1-2\kappa+64c\delta^2}+\frac{\tau^\ell s^\ell\delta^\ell}{\gamma^\ell\kappa^{\ell-1}}+(1+(1+n)^{-1})\xi_1\zeta_{P_1},\end{aligned}$$

as $n \to \infty$, we see that $\mu_n \to \mu$, where

$$\begin{aligned}\mu = \Omega + \Big\{ \Big(& \frac{\tau^\ell s^{\ell-1}\delta^{\ell-1}\rho_\ell\theta_\ell(r_\ell\zeta_{T_\ell} + p_\ell\zeta_{G_\ell})}{\gamma^\ell\kappa^{\ell-1}} + \frac{\tau^{\ell-1}s^{\ell-1}\delta^{\ell-1}\xi_\ell\zeta_{P_\ell}}{\kappa^{\ell-1}\gamma^{\ell-1}}\Big) + \cdots \\ & + \Big(\frac{\tau^2 s\delta\theta_2\rho_2(r_2\zeta_{T_2} + p_2\zeta_{G_2})}{\kappa\gamma^2} + \frac{\xi_2\zeta_{P_2}\tau s\delta}{\kappa\gamma}\Big) + \frac{\tau\rho_1\theta_1(r_1\zeta_{T_1} + p_1\zeta_{G_1})}{\gamma}\Big\}, \\ \Omega = & \sqrt{1 - 2\kappa + 64c\delta^2} + \xi_1\zeta_{P_1} + \frac{\tau^\ell s^\ell\delta^\ell}{\gamma^\ell\kappa^{\ell-1}}.\end{aligned}$$

Since $0 < \mu < 1$, by condition (3.7) and so $0 < \mu_n < 1$ where n is sufficiently large. It follows from (3.30) that $\{x_n\}$ is a Cauchy sequence in X. Similarly, it follows from (3.18) and (3.30) that $\{x_i^n\}(i = 1, \cdots, \ell)$ is Cauchy sequence in X. Hence there exists $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)$ such that $u_i^n \to u_i, v_i^n \to v_i$ and $w_i^n \to w_i (i = 1, \cdots, \ell)$ as $n \to \infty$. Next, we show that $u_i \in T_i(x_1), v_i \in G_i(x_1)$ and $w_i \in P_i(x_1)$, *respectively.* Since $u_i^n \in T_i(x_1)$, we have

$$\begin{aligned} d(u_1, T_1(x_1)) &\le \|u_1 - u_1^n\| + d(u_1^n, T_1(x_1)) \\ &\le \|u_1 - u_1^n\| + \mathfrak{H}(T_1(x_1^n), T_1(x_1)) \\ &\le \|u_1 - u_1^n\| + \zeta_{T_1}\|x_1^n - x_1\| \to 0, \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.31}$$

Hence $u_1 \in T_1(x_1)$, since $T_1(x_1) \in CB(X)$. Similarly, we can show that $v_1 \in G_1(x_1)$, $w_1 \in P_1(x_1)$. Thus, it follows from Algorithm 3.2 that $x_i \in X$, $u_i \in T_i(x_1)$, $v_i \in G_i(x_1)$, $w_i \in P_i(x_1)$ satisfy (3.1) and hence from Theorem 3.1, it follows that $(x_i, u_i, v_i, w_i)(i = 1, \cdots, \ell)$, where $x_i \in X$, $u_i \in T_i(x_1)$, $v_i \in G_i(x_1)$, $w_i \in P_i(x_1)$ is a solution of (2.1). Now we show that (x_i, u_i, v_i, w_i) is unique, let $(x_i^*, u_i^*, v_i^*, w_i^*)$ where $x_i^* \in X$, $u_i^* \in T_i(x_1^*)$, $v_i^* \in G_i(x_1^*)$, $w_i \in P_i(x_1^*)$ be another solution of (2.1), then from Theorem 3.1 implies that

$$g(x_1^*) = R^{H,\eta,\varphi_1}_{M(\cdot,w_1^*)}[Hg(x_2^*) - \rho_1\varphi_1 \circ N_1(u_1^*, v_1^*)], \rho_1 > 0,$$

where

$$\begin{aligned} g(x_2^*) &= R^{H,\eta,\varphi_2}_{M(\cdot,w_2^*)}[Hg(x_3^*) - \rho_2\varphi_2 \circ N_2(u_2^*, v_2^*)], \rho_2 > 0, \\ &\vdots \\ g(x_\ell^*) &= R^{H,\eta,\varphi_\ell}_{M(\cdot,w_\ell^*)}[Hg(x_1^*) - \rho_\ell\varphi_\ell \circ N_\ell(u_\ell^*, v_\ell^*)], \rho_\ell > 0. \end{aligned} \tag{3.32}$$

From (3.32) and similar argument given as above, we have

$$\begin{aligned} \|x_1 - x_1^*\| = \|x_1 - g(x_1) &+ R^{H,\eta,\varphi_1}_{M(\cdot,w_1)}[Hg(x_2) - \rho_1\varphi_1 \circ N_1(u_1, v_1)] \\ &- x_1^* + g(x_1^*) - R^{H,\eta,\varphi_1}_{M(\cdot,w_1^*)}[Hg(x_2^*) - \rho_1\varphi_1 \circ N_1(u_1^*, v_1^*)]\| \\ \le \|x_1 - x_1^*\|. \end{aligned}$$

Since $0 < \mu < 1$, thus $x_1 = x_1^*$. Similarly $x_2 = x_2^*, \cdots, x_\ell = x_\ell^*$. Therefore $(x_i^*, u_i^*, v_i^*, w_i^*)$, where $x_i \in X$, $u_i \in T_i(x_1)$, $v_i \in G_i(x_1)$, $w_i \in P_i(x_1)$ is an unique solution of (2.1). □

Corollary 3.7. *Let X be a uniformly smooth Banach space with modulus of smoothness $\rho_X(t) \le ct^2$ for some $c > 0$, and X^* be the dual space of X. Let $g : X \to X$ be a κ-strongly accretive and δ-Lipschitz continuous, $H : X \to X^*$ be a γ-strongly η-monotone and s-Lipschitz continuous with respect to g, and $N_i : X \times X \to X^*$ be a (r_i, p_i)-Lipschitz continuous with respect to first and second argument, respectively. Let $T_i, G_i, P_i : X \to CB(X^*)$ be ζ_{T_i}-$\mathfrak{H}$-Lipschitz continuous, ζ_{G_i}-$\mathfrak{H}$-Lipschitz continuous, ζ_{P_i}-$\mathfrak{H}$-Lipschitz continuous $(i = \overline{1,2,3})$, respectively. Let $\varphi_i : X^* \to X^*$ be a single valued mapping satisfying $\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$ and $\varphi_i : X^* \to X^*$ be a θ_i-Lipschitz continuous with respect to $N_i(i = \overline{1,2,3})$ respectively. Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping and $M : X \times X \to 2^{X^*}$ be a (H, η, φ_i)-monotone mapping. Assume that (3.6) holds for $(i = \overline{1,2,3})$ and*

$$0 < \Omega + \frac{\tau}{\gamma}\Big\{\frac{\tau^2 s^2\delta^2\rho_3\theta_3(r_3\zeta_{T_3} + p_3\zeta_{G_3})}{\kappa^2\gamma^2} + \frac{\tau s\delta\rho_2\theta_2(r_2\zeta_{T_2} + p_2\zeta_{G_2})}{\kappa\gamma} + \frac{\tau s^2\delta^2\xi_3\zeta_{P_3}}{\kappa^2\gamma}$$

$$+\frac{s\delta\xi_2\zeta_{P_2}}{\kappa} + \rho_1\theta_1(r_1\zeta_{T_1} + p_1\zeta_{G_1})\Big\} < 1,$$

$$\frac{\tau^2 s^2\delta^2\rho_3\theta_3(r_3\zeta_{T_3} + p_3\zeta_{G_3})}{\kappa^2\gamma^2} + \frac{\tau s\delta\rho_2\theta_2(r_2\zeta_{T_2} + p_2\zeta_{G_2})}{\kappa\gamma} + \frac{\tau s^2\delta^2\xi_3\zeta_{P_3}}{\kappa^2\gamma} + \frac{s\delta\xi_2\zeta_{P_2}}{\kappa}$$

$$+\rho_1\theta_1(r_1\zeta_{T_1} + p_1\zeta_{G_1}) > 0,$$

$$\Omega = \sqrt{1 - 2\kappa + 64c\delta^2} + \frac{\tau^3 s^3\delta^3}{\kappa^2\gamma^3} + \xi_1\zeta_{P_1} > 0. \tag{3.33}$$

Then the sequences $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}, \{w_i^n\}(i = \overline{1,2,3})$ generated by Algorithm 3.3 converge strongly to the unique solutions (x_i, u_i, v_i, w_i) respectively, where $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)$ is a solution of (2.2).

Corollary 3.8. *Let X be a uniformly smooth Banach space with modulus of smoothness $\rho_X(t) \le ct^2$ for some $c > 0$, and X^* be the dual space of X. Let $g : X \to X$ be a κ-strongly accretive and δ-Lipschitz continuous, $H : X \to X^*$ be a γ-strongly η-monotone and s-Lipschitz continuous with respect to g, and $N_i : X \times X \to X^*$ be a (r_i, p_i)-Lipschitz continuous with respect to first and second argument respectively. Let $T_i, G_i, P_i : X \to CB(X^*)$ be a ζ_{T_i}-$\mathfrak{H}$-Lipschitz continuous, ζ_{G_i}-$\mathfrak{H}$-Lipschitz continuous, ζ_{P_i}-$\mathfrak{H}$-Lipschitz continuous $(i = \overline{1,2})$, respectively. Let $\varphi_i : X^* \to X^*$ be a single valued mapping satisfying $\varphi_i(u+v) = \varphi_i(u) + \varphi_i(v)$ and $Ker(\varphi_i) = \{0\}$ and $\varphi_i : X^* \to X^*$ be a θ_i-Lipschitz continuous with respect to N_i $(i = \overline{1,2})$ respectively.*

Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping and $M : X \times X \to 2^{X^*}$ be a (H, η, φ_i)-monotone mapping. Assume that (3.6) holds for $(i = \overline{1,2})$ and

$$0 < \Omega + \frac{\tau}{\gamma}\left\{\rho_1\theta_1(r_1\zeta_{T_1} + p_1\zeta_{G_1}) + \frac{\tau s\delta\rho_2\theta_2(r_2\zeta_{T_2} + p_2\zeta_{G_2})}{\kappa\gamma} + \frac{\xi_2 s\delta\zeta_{P_2}}{\kappa}\right\} < 1,$$

$$\rho_1\theta_1(r_1\zeta_{T_1} + p_1\zeta_{G_1}) + \frac{\tau s\delta\rho_2\theta_2(r_2\zeta_{T_2} + p_2\zeta_{G_2})}{\kappa\gamma} + \frac{\xi_2 s\delta\zeta_{P_2}}{\kappa} > 0$$

$$\Omega = \sqrt{1 - 2\kappa + 64c\delta^2} + \xi_1\zeta_{P_1} + \frac{\tau^2 s^2\delta^2}{\kappa\gamma^2} > 0. \tag{3.34}$$

Then the sequences $\{x_i^n\}, \{u_i^n\}, \{v_i^n\}, \{w_i^n\}$ generated by Algorithm 3.4 converge strongly to the unique solutions $(x_i, u_i, v_i, w_i)(i = \overline{1,2})$ respectively, where $x_i \in X, u_i \in T_i(x_1), v_i \in G_i(x_1), w_i \in P_i(x_1)$ is a solution of (2.3).

Corollary 3.9. *Let X be a uniformly smooth Banach space with modulus of smoothness $\rho_X(t) \le ct^2$ for some $c > 0$, and X^* be the dual space of X. Let $g : X \to X$ be a κ-strongly accretive and δ-Lipschitz continuous, $H : X \to X^*$ be a γ-strongly η-monotone and s-Lipschitz continuous with respect to g, and $N : X \times X \to X^*$ be a (r, p)-Lipschitz continuous with respect to first and second argument, respectively. Let $T, G, P : X \to CB(X^*)$ be ζ_T-$\mathfrak{H}$-Lipschitz continuous, ζ_G-$\mathfrak{H}$-Lipschitz continuous, ζ_P-$\mathfrak{H}$-Lipschitz continuous, respectively. Let $\varphi : X^* \to X^*$ be a single valued mapping satisfying $\varphi(u + v) = \varphi(u) + \varphi(v)$ and $Ker(\varphi) = \{0\}$ where $\varphi : X^* \to X^*$ be a θ-Lipschitz continuous with respect to N respectively. Let $\eta : X \times X \to X$ be a τ-Lipschitz continuous mapping and $M : X \times X \to 2^{X^*}$ be a (H, η, φ)-monotone mapping. Assume that*

$$\|R^{H,\eta,\varphi}_{M(\cdot,w^n)}(x) - R^{H,\eta,\varphi}_{M(\cdot,w^{n-1})}(x)\| \le \xi\|w^n - w^{n-1}\| \tag{3.35}$$

and

$$0 < \Omega + \frac{\tau}{\gamma}\{s\delta + \rho\theta(r\zeta_T + p\zeta_G)\} < 1,$$

$$s\delta + \rho\theta(r\zeta_T + p\zeta_G) > 0, \quad \Omega = \sqrt{1 - 2\kappa + 64c\delta^2} + \xi\zeta_P > 0. \tag{3.36}$$

Then the sequences $\{x^n\}, \{u^n\}, \{v^n\}, \{w^n\}$ generated by Algorithm 3.5 converge strongly to the unique solutions (x, u, v, w) respectively where $x \in X, u \in T(x), v \in G(x), w \in P(x)$ is a solution of (2.4).

Acknowledgements. The author wishes to express his gratitude to the Editor-in-Chief, the two anonymous referees for their detailed comments and useful suggestions that allowed to improve significantly the presentation of this paper.

References

[1] R. Ahmad, M. F. Khan and Salahuddin, *Mann and Ishikawa type perturbed iterative algorithms for generalized nonlinear variational inclusions*, Math. Comput. Appl., **6(1)**(2001), 47–52.

[2] M. K. Ahmad and Salahuddin, *Resolvent equation technique for generalized nonlinear variational inclusions*, Adv. Nonlinear Var. Inequal., **5(1)**(2002), 91–98.

[3] M. K. Ahmad and Salahuddin, *Perturbed three step approximation process with errors for a generalized implicit nonlinear quasivariational inclusions*, Int. J. Math. Math. Sci., (2006), Article ID 43818, 15 pp.

[4] M. K. Ahmad and Salahuddin, *Generalized strongly nonlinear implicit quasivariational inequalities*, J. Inequal. Appl., (2009), Art. ID 124953, 16 pp.

[5] M. K. Ahmad and Salahuddin, *A stable perturbed algorithms for a new class of generalized nonlinear implicit quasi variational inclusions in Banach spaces*, Advances Pure Math., **2(2)**(2012), 139–148.

[6] C. Baiocchi and A. Capelo, *Variational and quasi variational inequalities: applications to free boundary problems*, Wiley, New York, 1984.

[7] X. P. Ding, *Perturbed proximal point algorithms for generalized quasi variational inclusions*, J. Math. Anal. Appl., **210**(1997), 88-101.

[8] X. P. Ding and H. R. Feng, *The p-step iterative algorithm for a system of generalized mixed quasi variational inclusions with (A, η)-accretive operators in q-uniformly smooth Banach spaces*, J. Comput. Appl. Math., **220**(2008), 163–174.

[9] X. P. Ding and H. R. Feng, *Algorithm for solving a new class of generalized nonlinear implicit quasi variational inclusions in Banach spaces*, Appl. Math. Comput., **208**(2009), 547–555.

[10] X. P. Ding and Salahuddin, *A system of general nonlinear variational inclusions in Banach spaces*, Appl. Math. Mech., **36(12)**(2015), 1663–167.

[11] X. P. Ding and Salahuddin, *Strong convergence of an iterative algorithm for a class of nonlinear set valued variational inclusions*, Korean J. Math., **25(1)**(2017), 19–35.

[12] Y. P. Fang and N. J. Huang, *Approximate solutions for nonlinear operator inclusions with (H, η)-monotone operators*, Research Report Sichuan Univ., 2003.

[13] Y. P. Fang and N. J. Huang, *H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces*, Appl. Math. Lett., **17(6)**(2004), 647–653.

[14] Y. P. Fang and N. J. Huang, *Iterative algorithm for a system of variational inclusions involving H-accretive operators in Banach spaces*, Acta Math. Hungar., **108(3)**(2005), 183–195.

[15] Y. P. Fang, N. J. Huang and H. B. Thompson, *A new system of variational inclusions with (H, η)-monotone operators in Hilbert spaces*, Comput. Math. Appl., **49**(2005), 365–374.

[16] H. R. Feng and X. P. Ding, *A new system of generalized nonlinear quasi variational like inclusions with A-monotone operators in Banach spaces*, J. Comput. Appl. Math., **225**(2009), 365–373.

[17] A. Hassouni and A. Moudafi, *A perturbed algorithm for variational inclusions*, J. Math. Anal. Appl., **185(3)**(1994), 706–712.

[18] X. F. He, J. L. Lou and Z. He, *Iterative methods for solving variational inclusions in Banach spaces*, J. Comput. Appl. Math., **203**(2007), 80–86.

[19] N. J. Huang and Y. P. Fang, *Generalized m-accretive mappings in Banach spaces*, J. Sichuan Univ., **38(4)**(2001), 591–592.

[20] H. Hussain, M. F. Khan and Salahuddin, *Mann and Ishikawa type perturbed iterative algorithms for completely generalized nonlinear variational Inclusions*, Int. J. Math. Anal., **3(1)**(2006), 51–62.

[21] P. Junlouchai, S. Plubtieng and Salahuddin, *On a new system of nonlinear regularized nonconvex variational inequalities in Hilbert spaces*, J. Nonlinear Anal. Optim., **7(1)**(2016), 103–115.

[22] M. F. Khan and Salahuddin, *Generalized mixed multivalued variational inclusions involving H-accretive operators*, Adv. Nonlinear Var. Inequal., **9(2)**(2006), 29–47.

[23] M. F. Khan and Salahuddin, *Generalized co-complementarity problems in p-uniformly smooth Banach spaces*, JIPAM. J. Inequal. Pure Appl. Math., **7(2)**(2006), Article ID 66, 11 pp.

[24] M. F. Khan and Salahuddin, *Generalized multivalued nonlinear co-variational inequalities in Banach spaces*, Funct. Diff. Equ., **14(2-4)**(2007), 299–313.

[25] S. H. Kim, B. S. Lee and Salahuddin, *Fuzzy variational inclusions with (H, ϕ, ψ)-η-Monotone mappings in Banach Spaces*, J. Adv. Res. Appl. Math., **4(1)**(2012), 10–22.

[26] B. S. Lee, M. F. Khan and Salahuddin, *Generalized nonlinear quasi variational inclusions in Banach spaces*, Comput. Math. Appl. **56(5)**(2008), 1414–1422.

[27] B. S. Lee, M. F. Khan and Salahuddin, *Hybrid-type set-valued variational-like inequalities in Reflexive Banach spaces*, J. Appl. Math. Informatics, **27(5-6)**(2009), 1371–1379.

[28] X. P. Luo and N. J. Huang, *(H, ϕ)-η-monotone operators in Banach spaces with an application to variational inclusions*, Appl. Math. Comput., **216**(2010), 1131–1139.

[29] Jr. S. B. Nadler, *Multi valued contraction mappings*, Pacific J. Math., **30**(1969), 475–488.

[30] J. W. Peng and D. L. Zhu, *A system of variational inclusions with p-η-accretive operators*, J. Comput. Appl. Math., **216**(2008), 198–209.

[31] Salahuddin and S. S. Irfan, *Proximal methods for generalized nonlinear quasi-variational inclusions*, Math. Computat. Appl., **9(2)**(2004), 165–171.

[32] S. Q. Shan, Y. B. Xiao and N.J. Huang, *A new system of generalized implicit set-valued variational inclusions in Banach spaces*, Nonlinear Funct. Anal. Appl., **22(5)** (2017), 1091–1105.

[33] A. H. Siddiqi, M. K. Ahmad and Salahuddin, *Existence results for generalized nonlinear variational inclusions*, Appl. Maths. Letts., **18(8)**(2005), 859–864.

[34] Y. K. Tang, S. S. Chang and Salahuddin, *A system of nonlinear set valued variational inclusions*, SpringerPlus 2014, 3:318, Doi:10.1186/2193-180-3-318.

[35] R. U. Verma, M. F. Khan and Salahuddin, *Fuzzy generalized complementarity problems in Banach spaces*, PanAmer. Math. J., **17(4)**(2007), 71–80.

[36] F. Q. Xia and N. J. Huang, *Variational inclusions with a general H-monotone operator in Banach spaces*, Comput. Math. Appl., **54**(2007), 24–30.