KYUNGPOOK Math. J. 59(2019), 83-99
https://doi.org/10.5666/KMJ.2019.59.1.83
pISSN 1225-6951 eISSN 0454-8124

Weak and Strong Convergence of Hybrid Subgradient Method for Pseudomonotone Equilibrium Problems and Nonspreading-Type Mappings in Hilbert Spaces

Wanna Sriprad* and Somnuk Srisawat
Department of Mathematics and Computer Scicence, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi, Pathum Thani 12110, Thailand
e-mail : wanna_sriprad@rmutt.ac.th *and* somnuk_s@rmutt.ac.th

Abstract. In this paper, we introduce a hybrid subgradient method for finding an element common to both the solution set of a class of pseudomonotone equilibrium problems, and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space. We establish some weak and strong convergence theorems of the sequences generated by our iterative method under some suitable conditions. These convergence theorems are investigated without the Lipschitz condition for bifunctions. Our results complement many known recent results in the literature.

1. Introduction

Let H be a real Hilbert space in which the inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$ and we denote the set of fixed points of T by $F(T)$. Recall that a mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C,$$

and it is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x \in C, \text{ and } y \in F(T).$$

* Corresponding Author.
Received October 16, 2016; revised January 22, 2019; accepted January 28, 2019.
2010 Mathematics Subject Classification: 47H05, 47H09, 47H10.
Key words and phrases: pseudomonotone equilibrium problem, κ-strictly presudononspreading mapping, nonspreading mapping, hybrid subgradient method, fixed point.
This work was supported by Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT) Thailand.

A mapping $T : C \to C$ is said to be *a strict pseudocontraction* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \ \forall x, y \in C,$$

where I is the identity mapping on H. If $k = 0$, then T is nonexpansive on C.

In 2008, Kohsaka and Takahashi [15] defined a mapping T in a in Hilbert spaces H to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \text{ for all } x, y \in C.$$

Following the terminology of Browder-Petryshyn [10], Osilike and Isiogugu [17] called a mapping T of C into itself *κ-strictly pseudononspreading* if there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle + \kappa\|x - Tx - (y - Ty)\|^2, \text{ for all } x, y \in C.$$

Clearly, every nonspreading mapping is κ-strictly pseudononspreading but the converse is not true; see [17]. We note that the class of strict pseudocontraction mappings and the class of κ-strictly pseudononspreading mappings are independent.

In 2010, Kurokawa and Takahashi [16] obtained a weak mean ergodic theorem of Baillon's type [7] for nonspreading mappings in Hilbert spaces. Furthermore, using the idea of mean convergence in Hilbert spaces, they also proved a strong convergence theorem of Halpern's type [12] for this class of mappings. After that, in 2011, Osilike and Isiogugu [17] introduced the concept of κ-strictly pseudononspreading mappings and they proved a weak mean convergence theorem of Baillon's type similar to [16]. They further proved a strong convergence theorem using the idea of mean convergence. This theorem extended and improved the main theorems of [16] and gave an affirmative answer to an open problem posed by Kurokawa and Takahashi [16] for the case when the mapping T is averaged. In 2013 Kangtunyakarn [14] proposed a new technique, using the projection method, for κ-strictly pseudononspreading mappings. He obtained a strong convergence theorem for finding the common element of the set of solutions of a variational inequality, and the set of fixed points of κ-strictly pseudononspreading mappings in a real Hilbert space.

On the other hand, let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F, C)$. It is well known that there are several problems, such as complementarity problems, minimax problems, the Nash equilibrium problem in noncooperative games, fixed point problems, optimization problems, that can be written in the form of an EP. In other words, the EP is a unifying model for several problems arising in physics, engineering, science, optimization, economics, etc.; see [6, 8, 11] and the references therein.

In recent years the problem of finding an element common to the set of solutions of a equilibrium problems, and the set of fixed points of nonlinear mappings, has become a fascinating subject, and various methods have been developed by many authors for solving this problem (see [1, 4, 5, 20]). Most of all the existing algorithms for this problem are based on applying the proximal point method to the equilibrium problem $EP(F, C)$, and using a Mann's iteration to the fixed point problems of nonexpansive mappings. The convergence analysis has been considered when the bifunction F is monotone. This is because the proximal point method is not valid when the underlying operator F is pseudomonotone.

Recently, Anh [2] introduced a new hybrid extragradient iteration method for finding a element common to the set of fixed points of a nonexpansive mapping and the set of solutions of equilibrium problems for a pseudomonotone bifunctions. In this algorithm the equilibrium bifunction is not required to satisfy any monotonicity property, but it must satisfy a Lipschitz-type continuous bifunction i.e. there are two Lipschitz constants $c_1 > 0$ and $c_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \ \forall x,y,z \in C. \tag{1.2}$$

They obtained strongly convergent theorems for the sequences generated by these processes in a real Hilbert space.

Anh and Muu [3] reiterated that the Lipschitz-type condition (1.2) is not in general satisfied, and if it is, that finding the constants c_1 and c_2 is not easy. They further observed that solving strongly convex programs is also difficult except in special cases when C has a simple structure. They introduced and studied a new algorithm, which is called a hybrid subgradient algorithm for finding a common point in the set of fixed points of nonexpansive mappings and the solution set of a class of pseudomonotone equilibrium problems in a real Hilbert space. The proposed algorithm is a combination of the well-known Mann's iterative scheme for fixed point and the projection method for equilibrium problems. Furthermore, the proposed algorithm uses only one projection and does not require any Lipschitz condition for the bifunctions. To be more precise, they proposed the following iterative method:

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \gamma_n w_n), \ \gamma_n = \frac{\beta_n}{\max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tu_n, \ \text{for each } n = 1,2,3,..., \end{cases} \tag{1.3}$$

where $\partial_\epsilon F(x, \cdot)(x)$ stands for ϵ-subdifferential of the convex function $F(x, \cdot)$ at x and $\{\epsilon_n\}$, $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\}$, and $\{\alpha_n\}$ were chosen appropriately. Under certain conditions, they prove that $\{x_n\}$ converges strongly to a common point in the set of a class of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mapping. Using the idea of Anh and Muu [3], Thailert et al. [21] proposed a new algorithm for finding a common point in the solution set of a class of pseudomonotone equilibrium problems and the set of common fixed points of a family of strict pseudocontraction mappings in a real Hilbert space. Then Thailert et al. [22] introduced new general iterative methods for finding a common element in the solution set of pseudomonotone equilibrium problems and the set of fixed points of nonexpansive mappings which is a solution of a certain optimization problem related to a strongly positive linear operator. Under suitable control conditions, They proved the strong convergence theorems of such iterative schemes in a real Hilbert space.

In this paper, motivated by Anh and Muu [3], Kangtunyakarn [14], and other research going on in this direction, we proposed a hybrid subgradient method for the pseudomonotone equilibrium problem and the finite family of κ-strictly pseudononspreading mapping in a real Hilbert space. The weak and strong convergence of the proposed methods is investigated under certain assumptions. Our results improve and extend many recent results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. It is well-known that for all $x, y, z \in H$ and α, β, $\gamma \in [0, 1]$, with $\alpha+\beta+\gamma = 1$ there holds

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{2.1}$$

and

$$\| \alpha x + \beta y + \gamma z \|^2 = \alpha \| x \|^2 + \beta \| y \|^2 + \gamma\|z\|^2 - \alpha\beta \| x - y \|^2 - \beta\gamma\|y - z\|^2. \tag{2.2}$$

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\| x - P_C x \| \leq \| x - y \|$ for all $y \in C$. Such a P_C is called the metric projection from H into C. We know that P_C is nonexpansive. It is also known that, $P_C x \in C$ and

$$\langle x - P_C x, P_C x - z\rangle \geq 0, \quad \text{for all } x \in H \text{ and } z \in C. \tag{2.3}$$

It is easy to see that (2.3) equivalent to

$$\|x - z\|^2 \geq \|x - P_C x\|^2 + \|z - P_C x\|^2, \quad \text{for all } x \in H \text{ and } z \in C. \tag{2.4}$$

Lemma 2.1.([19]) *Let H be a real Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u,$$

where P_C is the metric projection of H onto C.

Recall that a bifunction $F : C \times C \to \mathbb{R}$ is said to be

(i) *η-strongly monotone* if there exists a number $\eta > 0$ such that

$$F(x,y) + F(y,x) \leq -\eta\|x-y\|^2, \text{ for all } x,y \in C,$$

(ii) *monotone* on C if

$$F(x,y) + F(y,x) \leq 0, \text{ for all } x,y \in C,$$

(iii) *pseudomonotone* on C with respect to $x \in C$ if

$$F(x,y) \geq 0 \text{ implies } F(y,x) \leq 0, \text{ for all } y \in C.$$

It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), for every $x \in C$. Moreover, F is said to be *pseudomonotone* on C with respect to $A \subseteq C$, if it is pseudomonotone on C with respect to every $x \in A$. When $A \equiv C$, F is called pseudomonotone on C.

The following example, taken from [18], shows that a bifunction may not be pseudomonotone on C, but yet is pseudomonotone on C with respect to the solution set of the equilibrium problem defined by F and C:

$$F(x,y) := 2y|x|(y-x) + xy|y-x|, \text{ for all } x,y \in \mathbb{R},\ C := [-1,1].$$

Clearly, $EP(F) = \{0\}$. Since $F(y,0) = 0$ for every $y \in C$, this bifunction is pseudomonotone on C with respect to the solution $x^* = 0$, However, F is not pseudomonotone on C. In fact, both $F(-0.5, 0.5) = 0.25 > 0$ and $F(0.5, -0.5) = 0.25 > 0$.

For solving the equilibrium problem (1.1), let us assume that Δ is an open convex set containing C and the bifunction $F : \Delta \times \Delta \to \mathbb{R}$ satisfies the following assumptions:

(A1) $F(x,x) = 0$ for all $x \in C$ and $F(x,\cdot)$ is convex and lower semicontinuous on C;

(A2) for each $y \in C$, $F(\cdot, y)$ is weakly upper semicontinuous on the open set Δ;

(A3) F is pseudomonotone on C with respect to $EP(F,C)$ and satisfies the strict paramonotonicity property, i.e., $F(y,x) = 0$ for $x \in EP(F,C)$ and $y \in C$ implies $y \in EP(F,C)$;

(A4) if $\{x_n\} \subseteq C$ is bounded and $\epsilon_n \to 0$ as $n \to \infty$, then the sequence $\{w_n\}$ with $w_n \in \partial_n F(x_n, \cdot)x_n$ is bounded, where $\partial_\epsilon F(x,\cdot)x$ stands for the ϵ-subdifferential of the convex function $F(x,\cdot)$ at x.

The following idea of the ϵ-subdierential of convex functions can be found in the work of Bronsted and Rockafellar [9] but the theory of ϵ-subdierential calculus was given by Hiriart-Urruty [13].

Definition 2.2. Consider a proper convex function $\phi : C \to \overline{\mathbb{R}}$. For a given $\epsilon > 0$, the ϵ-subdierential of ϕ at $x_0 \in Dom\phi$ is given by

$$\partial_\epsilon \phi(x_0) = \{x \in C : \phi(y) - \phi(x_0) \geq \langle x, y - x_0\rangle - \epsilon,\ \forall y \in C\}.$$

Remark 2.3. It is known that if the function ϕ is proper lower semicontinuous convex, then for every $x \in Dom\phi$, the ϵ-subdierential $\partial_\epsilon \phi(x)$ is a nonempty closed convex set (see [13]).

Next, throughout this paper, weak and strong convergence of a sequence $\{x_n\}$ in H to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. In order to prove our main results, we need the following lemmas.

Lemma 2.4.([17]) *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a κ-strictly pseudonospreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Remark 2.5. If $T : C \to C$ is a κ-strictly pseudononspreading mapping with $F(T) \neq \emptyset$, then from Lemma 2.8 in [14] and Lemma 2.1, we have $F(T) = VI(C, (I - T)) = F(P_C(I - \lambda(I - T)))$, for all $\lambda > 0$.

Lemma 2.6. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. For every $i = 1, 2, ..., N$, let $T_i : C \to C$ be a finite family of κ_i-strictly pseudononspreading mappings with $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{a_1, a_2, ..., a_n\} \subset (0, 1)$ with $\Sigma_{i=1}^{N} a_i = 1$, let $\bar{\kappa} = max\{\kappa_1, \kappa_2, ..., \kappa_N\}$ and let $\lambda \in (0, 1 - \bar{\kappa})$. Then*

(i) $\bigcap_{i=1}^{N} F(T_i) = F(\Sigma_{i=1}^{N} a_i P_C(I - \lambda(I - T_i)))$.

(ii) $\|\Sigma_{i=1}^{N} a_i P_C(I - \lambda(I - T_i))x - y\|^2 \leq \|x - y\|^2$, *for all $x \in C$ and $y \in \bigcap_{i=1}^{N} F(T_i)$, i.e. $\Sigma_{i=1}^{N} a_i P_C(I - \lambda(I - T_i))$ is quasi-nonexpansive.*

Proof. (i) It easy to see that $\bigcap_{i=1}^{N} F(T_i) \subseteq F(\Sigma_{i=1}^{N} a_i P_C(I - \lambda(I - T_i)))$. Let $x \in F(\Sigma_{i=1}^{N} a_i P_C(I - \lambda(I - T_i)))$ and let $x^* \in \bigcap_{i=1}^{N} F(T_i) \subseteq F(\Sigma_{i=1}^{N} a_i P_C(I - \lambda(I - T_i)))$. Note that for every $i = 1, 2, 3, ..., N$ we have

$$\begin{aligned} \|P_C(I - \lambda(I - T_i))x - x^*\|^2 &\leq \|x - x^* - \lambda(I - T_i)\|^2 \\ &= \|x - x^*\|^2 - 2\lambda\langle x - x^*, (I - T_i)x\rangle \\ &\quad + \lambda^2\|(I - T_i)x\|^2. \end{aligned} \tag{2.5}$$

Put $A_i = I - T_i$, for all $i = 1, 2, ..., N$, we have $T_i = I - A_i$ and

$$\begin{aligned} \|T_i x - T_i x^*\|^2 &= \|(I - A_i)x - (I - A_i)x^*\|^2 \\ &= \|(x - x^*) - A_i x\|^2 \\ &= \|x - x^*\|^2 - 2\langle x - x^*, A_i x\rangle + \|A_i x\|^2 \\ &\leq \|x - x^*\|^2 + \kappa_i\|(I - T_i)x - (I - T_i)x^*\|^2 + 2\langle x - T_i x, x^* - T_i x^*\rangle \\ &= \|x - x^*\|^2 + \kappa_i\|(I - T_i)x\|^2, \end{aligned} \tag{2.6}$$

which implies that

$$(1 - \kappa_i)\|(I - T_i)x\|^2 \leq 2\langle x - x^*, A_i x\rangle, \quad \text{for all } i = 1, 2, 3, ..., N$$

From (2.5) and (2.6), we have

$$\begin{aligned}\|P_C(I-\lambda(I-T_i))x-x^*\|^2 &\leq \|x-x^*\|^2-2\lambda\langle x-x^*,(I-T_i)x\rangle\\ &\quad+\lambda^2\|(I-T_i)x\|^2\\ &\leq \|x-x^*\|^2-\lambda(1-\kappa_i)\|(I-T_i)x\|^2\\ &\quad+\lambda^2\|(I-T_i)x\|^2\\ &= \|x-x^*\|^2-\lambda[(1-\kappa_i)-\lambda]\|(I-T_i)x\|^2\\ &\leq \|x-x^*\|^2, \qquad (2.7)\end{aligned}$$

for all $i=1,2,3,\ldots,N$.
From the definition of x and (2.7), we have

$$\begin{aligned}\|x-x^*\|^2 &= \|\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i))x-x^*\|^2\\ &= a_1\|P_C(I-\lambda(I-T_1))x-x^*\|^2+a_2\|P_C(I-\lambda(I-T_2))x-x^*\|^2+\cdots\\ &\quad+a_N\|P_C(I-\lambda(I-T_N))x-x^*\|^2-a_1a_2\|P_C(I-\lambda(I-T_1))x\\ &\quad-P_C(I-\lambda(I-T_2))x\|^2-a_2a_3\|P_C(I-\lambda(I-T_2))x-\\ &\quad P_C(I-\lambda(I-T_3))x\|^2-\cdots-a_{N-1}a_N\|P_C(I-\lambda(I-T_{N-1}))x-\\ &\quad P_C(I-\lambda(I-T_N))x\|^2\\ &\leq \|x-x^*\|^2-a_1a_2\|P_C(I-\lambda(I-T_1))x-P_C(I-\lambda(I-T_2))x\|^2\\ &\quad-a_2a_3\|P_C(I-\lambda(I-T_2))x-P_C(I-\lambda(I-T_3))x\|^2-\cdots\\ &\quad-a_{N-1}a_N\|P_C(I-\lambda(I-T_{N-1}))x-P_C(I-\lambda(I-T_N))x\|^2.\end{aligned}$$

This implies that

$$P_C(I-\lambda(I-T_1))x=P_C(I-\lambda(I-T_2))x=\cdots=P_C(I-\lambda(I-T_N))x$$

Since $x\in F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))$, we get that $x=P_C(I-\lambda(I-T_i))x$, for all $i=1,2,3,...,N$ From Remark 2.5, we have $x\in F(T_i)$, for all $i=1,2,3,...,N$. That is $x\in\bigcap_{i=1}^N F(T_i)$. Hence $F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))\subseteq\bigcap_{i=1}^N F(T_i)$.
(ii) Let $x\in C$ and $y\in\bigcap_{i=1}^N F(T_i)=F(\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i)))$
As the same argument as in (i), we can show that

$$\|P_C(I-\lambda(I-T_i))x-y\|^2\leq\|x-y\|^2, \qquad (2.8)$$

for all $i=1,2,3,...,N$. Thus

$$\begin{aligned}\|\Sigma_{i=1}^N a_iP_C(I-\lambda(I-T_i))x-y\|^2 &\leq a_1\|P_C(I-\lambda(I-T_1))x-y\|^2\\ &\quad+a_2\|P_C(I-\lambda(I-T_2))x-y\|^2+\cdots\\ &\quad+a_N\|P_C(I-\lambda(I-T_N))x-y\|^2\\ &\leq \Sigma_{i=1}^N a_i\|x-y\|^2=\|x-y\|^2. \qquad (2.9)\end{aligned}$$

□

Lemma 2.7.([23]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \ \ n \geq 1,$$

where $\sum_{n=0}^{\infty} b_n < \infty$. Then the sequence $\{a_n\}$ is convergent.

3. Weak Convergence Theorem

In this section, we prove weak convergence theorem for finding a common element in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{\kappa_1, \kappa_2, ..., \kappa_N\} \subset [0, 1)$ and $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i \leq \min\{1 - \kappa_1, 1 - \kappa_2, ..., 1 - \kappa_N\}$ *and* $\Sigma_{n=1}^{\infty} \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$;

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$;

(iii) $\sum_{n=0}^{\infty} \delta_n = \infty$, $\sum_{n=0}^{\infty} \delta_n^2 < \infty$, *and* $\sum_{n=0}^{\infty} \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

Proof. First, we will show that $\{x_n\}$ is bounded. Let $p \in \Omega$. Then we have

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\langle x_n - u_n, p - u_n\rangle \\ &\leq \|x_n - p\|^2 + 2\langle x_n - u_n, p - u_n\rangle. \end{aligned} \tag{3.2}$$

Since $u_n = P_C(x_n - \rho_n w_n)$ and $p \in C$, we get that

$$\langle x_n - u_n, p - u_n\rangle \leq \rho_n\langle w_n, p - u_n\rangle. \tag{3.3}$$

Substuting (3.3) into (3.2), we have

$$\begin{aligned}
\|u_n - p\|^2 &\leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - u_n\rangle \\
&= \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\rho_n\langle w_n, x_n - u_n\rangle \\
&\leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\rho_n\|w_n\|\|x_n - u_n\| \\
&\leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\delta_n\|x_n - u_n\|. \qquad (3.4)
\end{aligned}$$

By using $u_n = P_C(x_n - \rho_n w_n)$ and $x_n \in C$ again, we get

$$\begin{aligned}
\|x_n - u_n\|^2 &= \langle x_n - u_n, x_n - u_n\rangle \\
&\leq \rho_n\langle w_n, x_n - u_n\rangle \\
&\leq \rho_n\|w_n\|\|x_n - u_n\| \\
&\leq \delta_n\|x_n - u_n\|, \qquad (3.5)
\end{aligned}$$

which implies that

$$\|x_n - u_n\| \leq \delta_n. \qquad (3.6)$$

By condition (iii), we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \qquad (3.7)$$

Combining (3.4) and (3.6), we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + 2\rho_n\langle w_n, p - x_n\rangle + 2\delta_n^2. \qquad (3.8)$$

Since $w_n \in \partial_{\epsilon_n}F(x_n, \cdot)x_n$, $p \in C$ and $F(x, x) = 0$ for each $x \in C$, we obtain that

$$\begin{aligned}
\langle w_n, p - x_n\rangle &\leq F(x_n, p) - F(x_n, x_n) + \epsilon_n \\
&= F(x_n, p) + \epsilon_n. \qquad (3.9)
\end{aligned}$$

Thus, it follows from (3.8) and (3.9) that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + 2\rho_n F(x_n, p) + 2\rho_n\epsilon_n + 2\delta_n^2. \qquad (3.10)$$

Form Lemma 2.6 (ii), we have

$$\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - p\|^2 \leq \|x_n - p\|^2. \qquad (3.11)$$

From (3.1), (3.10) and (3.11), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n x_n + \beta_n\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \beta_n\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - p\|^2 \\
&\quad + \gamma_n\|u_n - p\|^2 - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\Big(\|x_n - p\|^2 + 2\rho_n F(x_n, p) \\
&\quad + 2\rho_n\epsilon_n + 2\delta_n^2\Big) - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\
&= \|x_n - p\|^2 + 2\gamma_n\rho_n F(x_n, p) + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\
&\quad - \alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2. \qquad (3.12)
\end{aligned}$$

Since $p \in EP(F, C)$ and F is pseudomonotone on F with respect to p, we get that $F(x_n, p) \leq 0$ for all $n \in \mathbb{N}$. Then from (3.12) it follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\quad -\alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\ &\leq \|x_n - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2. \end{aligned} \tag{3.13}$$

Let $\eta_n = 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2$ for all $n \geq 0$. From condition (ii) and (iii), we get that

$$\Sigma_{n=0}^{\infty}\eta_n = \Sigma_{n=0}^{\infty}(2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2) \leq 2b\Sigma_{n=0}^{\infty}\rho_n\epsilon_n + 2b\Sigma_{n=0}^{\infty}\delta_n^2 < +\infty$$

Now applying Lemma 2.7 to (3.13), we obtain that the $\lim_{n\to\infty} \|x_n - p\|$ exists, i.e. $\lim_{n\to\infty} \|x_n - p\| = \bar{a}$ for some $\bar{a} \in C$. Thus $\{x_n\}$ is bounded. Also, it easy to verify that $\{u_n\}$ and $\{\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\}$ are also bounded.

Next, we will show that $\limsup_{n\to\infty} F(x_n, p) = 0$ for any $p \in \Omega$. Since F is pseudomonotone on C and $F(p, x_n) \geq 0$, we have $-F(x_n, p) \geq 0$. From (3.12) and condition (ii), we have

$$\begin{aligned} 2\gamma_n\rho_n[-F(x_n, p)] &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad +2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2b\rho_n\epsilon_n + 2b\delta_n^2. \end{aligned} \tag{3.14}$$

Summing up (3.14) for every n, we obtain

$$\begin{aligned} 0 &\leq 2\sum_{n=0}^{\infty}\gamma_n\rho_n[-F(x_n, p)] \\ &\leq \|x_0 - p\|^2 + 2b\sum_{n=0}^{\infty}\rho_n\epsilon_n + 2b\sum_{n=0}^{\infty}\delta_n^2 < +\infty. \end{aligned} \tag{3.15}$$

By the assumption (A_4), we can find a real number w such that $\|w_n\| \leq w$ for every n. Setting $\mathit{\Gamma} := \max\{\sigma, w\}$, where σ is a real number such that $0 < \sigma_n < \sigma$ for every n, it follows from (ii) that

$$0 \leq \frac{2a}{\mathit{\Gamma}}\sum_{n=0}^{\infty}\delta_n[-F(x_n, p)] \tag{3.16}$$

$$\leq 2\sum_{n=0}^{\infty}\gamma_n\rho_n[-F(x_n, p)] < +\infty, \tag{3.17}$$

which implies that

$$0 \leq \sum_{n=0}^{\infty}\delta_n[-F(x_n, p)] < +\infty. \tag{3.18}$$

Combining with $-F(x_n, p) \geq 0$ and $\sum_{n=0}^{\infty} \delta_n = \infty$, we can deduced that $\limsup_{n\to\infty} F(x_n, p) = 0$ as desired.

Next, we will show that $\omega_\omega(x_n) \subset \Omega$, where $\omega_\omega(x_n) = \{x \in H : x_{n_i} \rightharpoonup x$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}\}$. In deed since $\{x_n\}$ is bounded and H is reflexive, $\omega_\omega(x_n)$ is nonempty. Let $\bar{x} \in \omega_\omega(x_n)$. Then there exists subsequence $\{x_{n_i}\}$ of $\{x_n\}$. converging weakly to $\bar{x}$, that is $x_{n_i} \rightharpoonup \bar{x}$ as $i \to \infty$. By the convexity, C is weakly closed and hence $\bar{x} \in C$. Since $F(\cdot, p)$ is weakly upper semicontinuous for $p \in \Omega$, we obtain

$$\begin{aligned} F(\bar{x}, p) &\geq \limsup_{i\to\infty} F(x_n, p) \\ &= \lim_{i\to\infty} F(x_{n_i}, p) \\ &= \limsup_{n\to\infty} F(x_n, p) \\ &= 0. \end{aligned} \tag{3.19}$$

Since F is pseudomontone with respect to p and $F(p, \bar{x}) \geq 0$, we obtain $F(\bar{x}, p) \leq 0$. Thus $F(\bar{x}, p) = 0$. Furthermore, by assumption (A_3), we get that $\bar{x} \in EP(F, C)$. On the other hand, from (3.13) and conditions (ii)–(iii), we have

$$\begin{aligned} &\alpha_n\beta_n\|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\|^2 \\ &\qquad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\gamma_n\rho_n\epsilon_n + 2\gamma_n\delta_n^2 \\ &\qquad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2b\rho_n\epsilon_n + 2b\delta_n^2 \end{aligned} \tag{3.20}$$

taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} \|x_n - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n\| = 0. \tag{3.21}$$

Now, we will show that $\bar{x} \in \bigcap_{i=1}^N F(T_i)$. Assume that $\bar{x} \notin \bigcap_{i=1}^N F(T_i)$. By Lemma 2.6, we have $\bar{x} \notin F(\Sigma_{i=1}^N a_i P_C(I - \lambda_n(I - T_i)))$. From the Opial's condition, (3.21) and condition (i), we can write

$$\begin{aligned} \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| &< \liminf_{i\to\infty} \|x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))\bar{x}\| \\ &\leq \liminf_{i\to\infty} \Big(\|x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_{n_i}\| \\ &\quad + \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_{n_i} - \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))\bar{x}\| \Big) \\ &\leq \liminf_{i\to\infty} \Big(\|x_{n_i} - \bar{x}\| + \Sigma_{i=1}^N a_i \lambda_n^i \|(I - T_i)x_{n_i} - (I - T_i)\bar{x}\| \Big) \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\|. \end{aligned}$$

This is a contradiction. Then $\bar{x} \in \bigcap_{i=1}^N F(T_i)$. Thus $\bar{x} \in EP(F, C) \cap F(T) = \Omega$ and so $\omega_\omega(x_n) \subset \Omega$.

Finally, we prove that $\{x_n\}$ converge weakly to an element of Ω. It's sufficient to show that $\omega_\omega(x_n)$ is a single point set. Taking $z_1, z_2 \in \omega_\omega(x_n)$ arbitrarily, and let $\{x_{n_k}\}$ and $\{x_{n_m}\}$ be subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z_1$ and $x_{n_m} \rightharpoonup z_2$ respectively. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega$ and $z_1, z_2 \in \Omega$, we get that $\lim_{n\to\infty} \|x_n - z_1\|$ and $\lim_{n\to\infty} \|x_n - z_2\|$ exist. Now, assume that $z_1 \neq z_2$, then by the Opial's condition,

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{k\to\infty} \|x_{n_k} - z_1\| \\
&< \lim_{k\to\infty} \|x_{n_k} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{m\to\infty} \|x_{n_m} - z_2\| \\
&< \lim_{m\to\infty} \|x_{n_m} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|, \qquad (3.22)
\end{aligned}
$$

which is a contradiction. Thus $z_1 = z_2$. This show that $\omega_\omega(x_n)$ is single point set. i.e. $x_n \rightharpoonup \bar{x}$. This completes the proof. □

If we set $\kappa_i = 0$ for all $i = 1, 2, ..., N$ then we get the following Corollary.

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$
(3.23) \qquad \begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases}
$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i < 1$ *and* $\Sigma_{n=1}^\infty \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$*;*

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$*;*

(ii) $\sum_{n=0}^\infty \delta_n = \infty$, $\sum_{n=0}^\infty \delta_n^2 < \infty$, *and* $\sum_{n=0}^\infty \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

4. Strong Convergence Theorem

In this section, to obtain strong convergence result, we add the control condition $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$, and then we get the strong convergence theorem for finding a common element in the solution set of a class of pseudomonotone equilibrium problems and the set of fixed points of a finite family of κ-strictly presudononspreading mappings in a real Hilbert space.

Theorem 4.1. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{\kappa_1, \kappa_2, ..., \kappa_N\} \subset [0, 1)$ and $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudononspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F, C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0, 1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0, \infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i \leq \min\{1 - \kappa_1, 1 - \kappa_2, ..., 1 - \kappa_N\}$ *and* $\Sigma_{n=1}^\infty \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$;

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$ *and* $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$;

(iii) $\sum_{n=0}^\infty \delta_n = \infty$, $\sum_{n=0}^\infty \delta_n^2 < \infty$, *and* $\sum_{n=0}^\infty \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Omega$.

Proof. By a similar argument to the proof of Theorem 3.1 and (2.4), we have

$$\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - P_\Omega(x_n)\|^2 \leq \|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2$$

and

$$\|u_n - P_\Omega(x_n)\|^2 \leq \|u_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2. \tag{4.2}$$

It follows from (4.2) and condition (ii) that

$$
\begin{aligned}
&\|x_{n+1} - P_\Omega(x_{n+1})\|^2 \\
&\quad \leq \|\alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n - P_\Omega(x_n)\|^2 \\
&\quad \leq \alpha_n\|x_n - P_\Omega(x_n)\|^2 + \beta_n\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - P_\Omega(x_n))\|^2 \\
&\qquad +\gamma_n\|u_n - P_\Omega(x_n)\|^2 \\
&\quad \leq \alpha_n\|x_n - P_\Omega(x_n)\|^2 + \beta_n\Big(\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - x_n\|^2 \\
&\qquad -\|x_n - P_\Omega(x_n)\|^2\Big) + \gamma_n\Big(\|u_n - x_n\|^2 - \|x_n - P_\Omega(x_n)\|^2\Big) \\
&\quad = (\alpha_n - (\beta_n + \gamma_n))\|x_n - P_\Omega(x_n)\|^2 + \beta_n\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - x_n\|^2 \\
&\qquad +\gamma_n\|u_n - x_n\|^2. \\
&\quad \leq (2\alpha_n - 1)\|x_n - P_\Omega(x_n)\|^2 + b\|\Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n - x_n\|^2 \\
&\qquad +b\|u_n - x_n\|^2.
\end{aligned}
$$

Combining (3.7), (3.21), conditions (ii)–(iii), and the boundedness of the sequence $\{x_n - P_\Omega(x_n)\}$, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - P_\Omega(x_{n+1})\| = 0 \tag{4.3}$$

Since Ω is convex, for all $m > n$, we have $\frac{1}{2}(P_\Omega(x_m) + P_\Omega(x_n)) \in \Omega$, and therefore

$$
\begin{aligned}
\|P_\Omega(x_m) - P_\Omega(x_n)\|^2 &= 2\|x_m - P_\Omega(x_m)\|^2 + 2\|x_m - P_\Omega(x_n)\|^2 \\
&\quad -4\|x_m - \frac{1}{2}(P_\Omega(x_m) + P_\Omega(x_n))\|^2 \\
&\leq 2\|x_m - P_\Omega(x_m)\|^2 + 2\|x_m - P_\Omega(x_n)\|^2 \\
&\quad -4\|x_m - P_\Omega(x_m)\|^2 \\
&= 2\|x_m - P_\Omega(x_n)\|^2 - 2\|x_m - P_\Omega(x_m)\|^2. \qquad (4.4)
\end{aligned}
$$

Using (3.13) with $p = P_\Omega(x_n)$, we have

$$
\begin{aligned}
\|x_m - P_\Omega(x_n)\|^2 &\leq \|x_{m-1} - P_\Omega(x_n)\|^2 + \eta_{m-1} \\
&\leq \|x_{m-2} - P_\Omega(x_n)\|^2 + \eta_{m-1} + \eta_{m-2} \\
&\leq \ldots \\
&\leq \|x_n - P_\Omega(x_n)\|^2 + \sum_{j=n}^{m-1}\eta_j, \qquad (4.5)
\end{aligned}
$$

where $\eta_j = 2\gamma_j\rho_j\epsilon_j + 2\gamma_j\delta_j^2$. It follows from (4.4) and (4.5) that

$$\|P_\Omega(x_m) - P_\Omega(x_n)\|^2 \leq 2\|x_n - P_\Omega(x_n)\|^2 + 2\sum_{j=n}^{m-1}\eta_j - 2\|x_m - P_\Omega(x_m)\|^2. \tag{4.6}$$

Together with (4.3) and $\sum_{j=0}^{\infty} \eta_j < +\infty$, this implies that $\{P_\Omega(x_n)\}$ is a Cauchy sequence, Hence $\{P_\Omega(x_n)\}$ strongly converges to some point $x^* \in \Omega$. Moreover, we obtain

$$x^* = \lim_{i\to\infty} P_\Omega(x_{n_i}) = P_\Omega(\bar{x}) = \bar{x}, \tag{4.7}$$

which implies that $P_\Omega(x_i) \to x^* = \bar{x} \in \Omega$. Then from (4.3) and (4.7), we can conclude that $x_n \to \bar{x}$. This completes the proof. □

If we set $\kappa_i = 0$ for all $i = 1, 2, ..., N$ then we get the following Corollary.

Corollary 4.2. *Let C be a closed convex subset of a real Hilbert space H and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into itself such that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(F,C) \neq \emptyset$. Let $x_0 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ w_n \in \partial_{\epsilon_n} F(x_n, \cdot)x_n, \\ u_n = P_C(x_n - \rho_n w_n), \ \rho_n = \frac{\delta_n}{max\{\sigma_n, \|w_n\|\}}, \\ x_{n+1} = \alpha_n x_n + \beta_n \Sigma_{i=1}^N a_i P_C(I - \lambda_n^i(I - T_i))x_n + \gamma_n u_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{4.8}$$

where $a, b, c, d, \lambda \in \mathbb{R}$, $a_i \in (0,1)$, for all $i = 1, 2, ..., N$ with $\Sigma_{i=1}^N a_i = 1$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \subset [0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\delta_n\}$, $\{\epsilon_n\}$,$\{\lambda_n^i\} \subset (0,\infty)$ satisfying the following conditions:

(i) $0 < \lambda \leq \lambda_n^i < 1$ *and* $\Sigma_{n=1}^\infty \lambda_n^i < \infty$ *for all* $i = 1, 2, ..., N$;

(ii) $0 < a < \alpha_n, \beta_n, \gamma_n < b < 1$ *and* $\lim_{n\to\infty} \alpha_n = \frac{1}{2}$;

(iii) $\sum_{n=0}^\infty \delta_n = \infty$, $\sum_{n=0}^\infty \delta_n^2 < \infty$, *and* $\sum_{n=0}^\infty \delta_n \epsilon_n < \infty$.

Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Omega$.

Acknowledgements. The authors would like to thank the faculty of science and technology, Rajamangala University of Technology Thanyaburi (RMUTT), Thailand for the financial support. Moreover, the authors would like to thank the referees for their valuable suggestions and comments which helped to improve the quality and readability of the paper.

References

[1] P. N. Anh, *Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities*, J. Optim. Theory Appl., **154**(2012), 303–320.

[2] P. N. Anh, *A hybrid extragradient method extended to fixed point problems and equilibrium problems*, Optimization, **62(2)**(2013), 271–283.

[3] P. N. Anh and L. D. Muu, *A hybrid subgradient algorithm for nonexpansive mappings and equilibrium problems*, Optim. Lett., **8**(2014), 727–738.

[4] P. N. Anh and D. X. Son, *A new method for a finite family of pseudocontractions and equilibrium problems*, J. Appl. Math. Inform., **29**(2011), 1179–1191.

[5] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal., **67**(2007), 2350–2360.

[6] A. Auslender, M. Teboulle and S. Ben-Tiba, *A logarithmic quadratic proximal method for variational inequalities*,Comput. Optim. Appl., **12**(1999), 31–40.

[7] J. Baillon, *Un theoreme de type ergodique pour les contractions nonlineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B, **280**(1975), A1511–A1514.

[8] E. Blum and W. Oettli ,*From optimization and variational inequalities to equilibrium problems*, Math. Student, **63**(1994), 123–145.

[9] A. Brøndsted and R. T. Rockafellar, *On the subdifferentiability of convex functions*, Proc. Am. Math. Soc., **16**(1965), 605–611.

[10] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20**(1967), 197–228.

[11] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6**(2005), 117–136.

[12] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73**(1967), 957–961.

[13] J.-B. Hiriart-Urruty, *ε-Subdifferential calculus*, Convex Analysis and Optimization, Res. Notes in Math., **57**(1982), 43–92.

[14] A. Kangtunyakarn, *The methods for variational inequality problems and fixed point of κ-strictly pseudononspreading mapping*, Fixed Point Theory Appl., **2013:171**(2013), 15 pp.

[15] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel), **91**(2008), 166–177.

[16] Y. Kurokawa and W. Takahashi, *Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces*, Nonlinear Anal., **73(6)**(2010), 1562–1568.

[17] M. O. Osilike and F. O. Isiogugu, *Weak and strong convergence theorems for nonspreading-type mappings in Hilbert spaces*, Nonlinear Anal., **74(5)**(2011), 1814–1822.

[18] P. Santos and S. Scheimberg, *An inexact subgradient algorithm for equilibrium problems*, Comput. Appl. Math., **30**(2011), 91–107.

[19] W. Takahashi, *Introduction to nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[20] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331(1)**(2007), 506–515.

[21] E. Thailert, R. Wangkeeree and C. Khantree, *A Hybrid subgradient algorithm for finding a common solution of an equilibrium problem and a family of strict pseudo-contraction mappings,* J. Appl. Math., (2014), Art. ID 142671, 8 pp.

[22] E. Thailert, R. Wangkeeree and P. Preechasilp, *A new general iterative methods for solving the equilibrium problems, variational inequality problems and fixed point problems of nonexpansive mappings,* Thai J. Math., **14(1)**(2016), 53–67.

[23] H.-K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279–291.