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Necessary and Sufficient Condition for the Solutions of First-Order Neutral Differential Equations to be Oscillatory or Tend to Zero

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Abstract. In this work, we give necessary and sufficient conditions under which every solution of a class of first-order neutral differential equations of the form

$$\big(x(t)+p(t)x(\tau(t))\big)' + q(t)H\big(x(\sigma(t))\big) = 0$$

either oscillates or converges to zero as $t \to \infty$ for various ranges of the neutral coefficient p. Our main tools are the Knaster-Tarski fixed point theorem and the Banach's contraction mapping principle.

1. Introduction

Consider a class of first-order nonlinear neutral differential equations with variable delays of the form

$$\big(x(t)+p(t)x(\tau(t))\big)' + q(t)H\big(x(\sigma(t))\big) = 0, \tag{1.1}$$

where

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(A_1) $p \in C([0,\infty),\mathbb{R})$, $q,\tau,\sigma \in C(\mathbb{R}_+,\mathbb{R}_+)$ such that $\tau(t) < t$, $\sigma(t) < t$ with $\tau(t),\sigma(t) \to \infty$ as $t \to \infty$ and τ invertible when necessary, and

(A_2) $H \in C(\mathbb{R},\mathbb{R})$ is nondecreasing and satisfies $uH(u) > 0$ for all $u \neq 0$.

Santra [17] studied oscillatory behaviour of the solutions of neutral differential equations of the form

$$\big(x(t) + p(t)x(t-\tau)\big)' + q(t)H\big(x(t-\sigma)\big) = 0 \tag{1.2}$$

and

$$\big(x(t) + p(t)x(t-\tau)\big)' + q(t)H\big(x(t-\sigma)\big) = f(t) \tag{1.3}$$

for various ranges of the neutral coefficient p. The same work [17] also obtained sufficient conditions for the existence of bounded positive solutions of (1.3). Candan [5] obtained sufficient conditions for the existence of non-oscillatory solutions of first order neutral differential equations having both delay and advance terms (known as mixed equations) by using the Banach contraction mapping principle. Das and Misra [6] established necessary and sufficient conditions such that every solution of (1.3) either oscillates or tends to zero for constant $p(t) = p$, $-1 < p \leq 0$, $f \geq 0$ and H satisfying the generalized sublinear condition $\int_0^{\pm k} \frac{d\eta}{H(\eta)} < \infty$ for every positive constant k. Guo *et al.* [10] studied necessary and sufficient conditions for oscillations to occur in (1.2) with linear H and constant coefficients for $p \in (0,\infty)$.

The motivation of the present work comes from the above studies. In this work we make an attempt to establish the necessary and sufficient conditions such that every solution of (1.1) converges to zero as $t \to \infty$ for different ranges in the neutral coefficient p.

The increasing interest in the oscillation properties of solutions to functional differential equations (FDEs) during the last few decades has been stimulated by applications arising in engineering and natural sciences. The new classes of FDEs provide challenges in these application areas. Equations involving delay, and those involving advance and a combination of both arise in the models on loss-less transmission lines in high speed computers which are used to connect switching circuits. The construction of these models using delays is complemented by the mathematical investigation of nonlinear equations. Moreover, the delay differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. There has been many investigations into the oscillation and non-oscillation of first order nonlinear neutral delay differential equations (See for e.g. [1, 2, 3, 4, 5, 6, 7, 9, 13, 14, 15, 16, 17, 18, 19, 20]). However, the study of asymptotic behaviour of solutions of (1.1) has received much less attention for $|p(t)| < +\infty$.

Definition 1.1. We call a continuously differentiable function $x(t)$ a solution of (1.1), if there exists a t_0 such that $x(t)$ is defined for all $t \geq T^* = \min\{\tau(t_0), \sigma(t_0)\}$ and satisfies (1.1) for all $t \geq t_0$. In the sequel, it will always be assumed that the solution of (1.1) exists on some half line $[t_1, \infty)$ $t_1 \geq t_0$. A solution of (1.1) is said to be *oscillatory*, if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory*. An equation of the form (1.1) is called *oscillatory*, if all its solutions are oscillatory.

2. Main Results

Lemma 2.1.([11]) *Let $p, x, z \in C([0,\infty), \mathbb{R})$ be such that $z(t) = x(t) + p(t)x(\tau(t))$, $t \geq \tau(t) > 0$ and$x(t) > 0$ for all $t \geq t_1 > \tau$, $\liminf_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = L$ exists. Let $p(t)$ satisfy one of the following conditions:*

(i) $0 \leq p_1 \leq p(t) \leq p_2 < 1$,

(ii) $1 < p_3 \leq p(t) \leq p_4 < \infty$,

(iii) $-\infty < -p_5 \leq p(t) \leq 0$,

where $r_i > 0$, $1 \leq i \leq 5$. Then $L = 0$.

Remark 2.1. If, in the above lemma, $x(t) < 0$ for $t \geq \tau(t) > 0$, $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z(t) = L \in \mathbb{R}$, exists, then $L = 0$.

Theorem 2.1. *Assume that (A_1) and (A_2) hold and $0 \leq p_1 \leq p(t) \leq p_2 < 1$ for all $t \in \mathbb{R}_+$. Let H be Lipschitz continuous on all intervals of the form $[\alpha, \beta]$ with $0 < \alpha < \beta < \infty$. Then every solution of (1.1) either oscillates or converges to zero as $t \to \infty$ if and only if*

(A_3) $\int_0^\infty q(\eta)\, d\eta = \infty$.

Proof. Suppose that (A_3) holds. Let $x(t)$ be a solution of (1.1) on $[t_0, \infty]$ where $t_0 \geq 0$. If $x(t)$ is oscillatory, then there is nothing to prove. We assume that $x(t)$ is non-oscillatory, that is, $x(t) > 0$ or $x(t) < 0$ for all $t \geq t_0$. We shall show that $x(t) \to \infty$ as $t \to \infty$. We set

$$z(t) = x(t) + p(t)x\big(\tau(t)\big), \; t \geq t_0. \tag{2.1}$$

From (1.1) it follows that

$$z'(t) = -q(t)H\big(x(\sigma(t))\big) < 0, \tag{2.2}$$

holds and hence $z(t)$ is a decreasing function for $t \geq t_1 > t_0 + \rho$. Since $z(t) > 0$ for $t \geq t_1$, the limit $\lim_{t\to\infty} z(t)$ must exist. Consequently, $z(t) > x(t)$ implies that $x(t)$ is bounded. Our objective is to show that $\lim_{t\to\infty} x(t) = 0$. For this, we need to show that $\liminf_{t\to\infty} x(t) = 0$. If $\liminf_{t\to\infty} x(t) \neq 0$, then there exists $t_2 > t_1$ and $\beta > 0$ such that $x\big(\sigma(t)\big) \geq \beta > 0$ for all $t \geq t_2$. This implies

$$\int_{t_2}^t q(\eta)H\big(x(\sigma(\eta))\big)\, d\eta \geq H(\beta)\left[\int_{t_2}^t q(\eta)\, d\eta\right] \\ \to +\infty, \quad as \quad t \to \infty,$$

due to (A_3). On the other hand, we integrate (2.2) from t_2 to $t(> t_2)$ to obtain

$$\int_{t_2}^{t} q(\eta)H\big(x(\sigma(\eta))\big)\,d\eta = -\big[z(\eta)\big]_{t_2}^{t} < \infty, \quad as \quad t \to \infty,$$

which results in the desired contradiction. Therefore, $\liminf_{t\to\infty} x(t) = 0$. Consequently, $\lim_{t\to\infty} z(t) = 0$ due to Lemma 2.1. As a result,

$$0 = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty}\big(x(t) + p(t)x(\tau(t))\big) \geq \limsup_{t\to\infty} x(t),$$

which implies that $\limsup_{t\to\infty} x(t) = 0$, that is, $\lim_{t\to\infty} x(t) = 0$.

If $x(t) < 0$ for $t \geq t_0$, then we may set $y(t) = -x(t)$ for $t \geq t_0$ in (1.1) and we find

$$\big(y(t) + p(t)y(\tau(t))\big)' + q(t)G\big(y(\sigma(t))\big) = 0$$

where $G(u) = -H(-u)$. Clearly, G also satisfies (A_2). Thus, we can apply the same arguments as above to prove that $\lim_{t\to\infty} x(t) = 0$.

Next, we show that (A_3) is necessary, that is, we need to show that the equation (1.1) admits a nonoscillatory solution which does not tend to zero as $t \to \infty$ if the limit exists. Suppose that (A_3) does not hold such that the integral in (A_3) is finite for some $1 - p_2 > 0$. Then there exists a $t_1 > 0$ such that

$$\int_{t_1}^{\infty} q(\eta)\,d\eta < \frac{1-p_2}{10K},$$

where $K = \max\{K_1, H(1)\}$ and K_1 is the Lipschitz constant of H on $\big[\frac{2(1-p_2)}{5}, 1\big]$. For $t_2 > t_1$ we set $Y = BC([t_2,\infty),\mathbb{R})$, the space of real valued bounded continuous functions on $[t_2,\infty)$. Clearly, Y is a Banach space with respect to the sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Let us define the set

$$S = \left\{u \in Y : \ \frac{2(1-p_2)}{5} \leq u(t) \leq 1, \ t \geq t_2\right\}.$$

This set S is a closed and convex subspace of Y. Let $\Phi : S \to S$ be defined by

$$(\Phi x)(t) = \begin{cases} (\Phi x)(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -p(t)x(\tau(t)) + \frac{2+3p_2}{5} + \int_t^{\infty} q(\eta)H\big(x(\sigma(\eta))\big)\,d\eta, & t \geq t_2+\rho. \end{cases}$$

For an arbitrary $x \in S$

$$(\Phi x)(t) \leq \frac{2+3p_2}{5} + H(1)\left[\int_t^{\infty} q(\eta)\,d\eta\right] < \frac{2+3p_2}{5} + \frac{1-p_2}{10} = \frac{1+p_2}{2} < 1$$

and

$$(\Phi x)(t) \geq -p(t)x(\tau(t)) + \frac{2+3p_2}{5} \geq -p_2 + \frac{2+3p_2}{5} = \frac{2(1-p_2)}{5}.$$

The above two inequalities imply that $\Phi x \in S$, too. Thus, Φ maps S back into itself. For arbitrary $y_1, y_2 \in S$ we have

$$\begin{aligned}|(\Phi y_1)(t) - (\Phi y_2)(t)| &\leq |p(t)||y_1\big(\tau(t)\big) - y_2\big(\tau(t)\big)| \\ &\quad + K_1 \int_t^\infty q(\eta)|y_1\big(\sigma(\eta)\big) - y_2\big(\sigma(\eta)\big)|\, d\eta.\end{aligned}$$

Hence,

$$\begin{aligned}|(\Phi y_1)(t) - (\Phi y_2)(t)| &\leq p_2||y_1 - y_2|| + K_1||y_1 - y_2|| \left[\int_t^\infty q(\eta)\, d\eta\right] \\ &< \left(p_2 + \frac{1-p_2}{10}\right)||y_1 - y_2||,\end{aligned}$$

which implies that

$$||\Phi y_1 - \Phi y_1|| \leq \mu||y_1 - y_2||,$$

such that Φ is a contraction mapping, because $\mu = p_2 + \frac{1-p_2}{10} = \frac{1+9p_2}{10} < 1$. Since S is complete and Φ is a contraction on S, Banach's contraction mapping principle implies that Φ has a unique fixed point on S. Hence $\Phi x = x$, which implies that

$$x(t) = \begin{cases} x(t_2+\rho), & t \in [t_2, t_2+\rho] \\ -p(t)x(\tau(t)) + \frac{2+3p_2}{5} + \int_t^\infty q(\eta)H\big(x(\sigma(\eta))\big)\, d\eta, & t \geq t_2+\rho \end{cases}$$

is a non-oscillatory solution of (1.1) and stays between the bounds of S, $\big[\frac{2(1-p_2)}{5}, 1\big]$. Thus, x cannot tend to zero for $t \to \infty$. Therefore, (A_3) is the necessary condition. This completes the proof of the theorem. □

Theorem 2.2. *Assume that (A_1) and (A_2) hold and $1 < p_3 \leq p(t) \leq p_4 < \infty$ such that $p_3^2 > p_4$ for $t \in \mathbb{R}_+$. Let H is Lipschitz continuous on intervals of the form $[\alpha, \beta]$ where $0 < \alpha < \beta < \infty$. Then every solution of (1.1) either oscillates or converges to zero as $t \to \infty$ if and only if (A_3) holds.*

Proof. The sufficient part follows from the proof of Theorem 2.1. For the necessary part, we assume that (A_3) does not hold. Then it is possible to find a $t_1 > 0$ such that

$$\int_{t_1}^\infty q(\eta)\, d\eta < \frac{p_3 - 1}{2K},$$

where $K = \max\{K_1, K_2\}$ and K_1 is the Lipschitz constant of H on $[a,b]$ and $K_2 = H(b)$. Thus,

$$a = \frac{2\lambda({p_3}^2 - p_4) - p_4(p_3 - 1)}{2{p_3}^2 p_4}$$

$$b = \frac{p_3 - 1 + 2\lambda}{2p_3}, \qquad \lambda > \frac{p_4(p_3 - 1)}{2({p_3}^2 - p_4)} > 0.$$

Let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of real valued bounded continuous functions on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Define

$$S = \{u \in Y : \ a \leq u(t) \leq b, \ t \geq t_2\}.$$

It is easy to verify that S is a closed convex subspace of Y. Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(t_2 + \rho), & t \in [t_2, t_2 + \rho] \\ -\frac{x\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)} + \frac{\lambda}{p\left(\tau^{-1}(t)\right)} + \frac{1}{p\left(\tau^{-1}(t)\right)} \left[\int_{\tau^{-1}(t)}^{\infty} q(\eta) H\big(x(\sigma(\eta))\big)\, d\eta\right], & t \geq t_2 + \rho. \end{cases}$$

For an arbitrary $x \in S$

$$(\Phi x)(t) \leq \frac{H(b)}{p\left(\tau^{-1}(t)\right)} \left[\int_{\tau^{-1}(t)}^{\infty} q(\eta)\, d\eta\right] + \frac{\lambda}{p\left(\tau^{-1}(t)\right)} \leq \frac{1}{p_3}\left[\frac{p_3 - 1}{2} + \lambda\right] = b$$

and

$$\begin{aligned}(\Phi x)(t) \geq -\frac{x\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)} + \frac{\lambda}{p\left(\tau^{-1}(t)\right)} > -\frac{b}{p_3} + \frac{\lambda}{p_4} &= -\frac{p_3 - 1 + 2\lambda}{2p_3^2} + \frac{\lambda}{p_4} \\ &= \frac{2\lambda({p_3}^2 - p_4) - p_4(p_3 - 1)}{2{p_3}^2 p_4} = a.\end{aligned}$$

This implies that $\Phi x \in S$, such that Φ maps S back into itself. For arbitrary $y_1, y_2 \in S$

$$\begin{aligned}|(\Phi y_1)(t) - (\Phi y_2)(t)| \leq{}& \frac{1}{|p\left(\tau^{-1}(t)\right)|} |y_1\left(\tau^{-1}(t)\right) - y_2\left(\tau^{-1}(t)\right)| \\ &+ \frac{K}{|p\left(\tau^{-1}(t)\right)|}\left[\int_{\tau^{-1}(t)}^{\infty} q(\eta) |y_1(\sigma(\eta)) - y_2(\sigma(\eta))|\, d\eta\right].\end{aligned}$$

Hence,

$$\begin{aligned}|(\Phi y_1)(t) - (\Phi y_1)(t)| &\leq \frac{1}{p_3}||y_1 - y_2|| + \frac{K}{p_3}||y_1 - y_2|| \left[\int_{\tau^{-1}(t)}^{\infty} q(\eta)\, d\eta\right] \\ &< \left(\frac{1}{p_3} + \frac{p_3 - 1}{2p_3}\right) ||y_1 - y_2||,\end{aligned}$$

which implies that

$$||\Phi y_1 - \Phi y_2|| \leq \mu ||y_1 - y_2||.$$

Consequently Φ is a contraction, because $\mu = \left(\frac{1}{p_3} + \frac{p_3-1}{2p_3}\right) < 1$. Hence, by Banach's contraction mapping principle Φ has a unique fixed point in S, which is a nonoscillatory solution of (1.1) on $[a, b]$.

Thus the proof of the theorem is complete. □

Theorem 2.3. *Assume that* (A_1) *and* (A_2) *hold and* $-1 < -p_5 \leq p(t) \leq 0$ *for* $t \in \mathbb{R}_+$ *and* $p_5 > 0$. *Then every solution of* (1.1) *either oscillates or converges to zero as* $t \to \infty$ *if and only if* (A_3) *holds.*

Proof. Proceeding as in the proof of Theorem 2.1, we have obtained (2.2). Hence, $z(t)$ is monotonic on $[t_2, \infty)$, $t_2 > t_1$. Let $z(t) > 0$ for $t \geq t_2$, the limit $\lim_{t\to\infty} z(t)$ must exist. Let $z(t) < 0$ for $t \geq t_2$. We claim that $x(t)$ is bounded. If not, there exists $\{\eta_n\}$ such that $\tau(\eta_n) \leq \tau_n$ and $\eta_n \to \infty$ as $n \to \infty$, $x(\eta_n) \to \infty$ as $n \to \infty$ and

$$x(\eta_n) = \max\{x(s) : t_2 \leq s \leq \eta_n\}.$$

Therefore,

$$z(\eta_n) = x(\eta_n) + p(\eta_n)x\big(\tau(\eta_n)\big) \geq (1 - p_5)x(\eta_n) \to +\infty, \quad as \quad n \to \infty,$$

a contradiction to the face $z(t) > 0$. So, our claim holds. Consequently, $z(t) \leq x(t)$ implies that $\lim_{t\to\infty} z(t)$ exists. Hence for any $z(t)$, $x(t)$ is bounded. Using the same type of argument as in the proof of Theorem 2.1, it is easy to show that $\liminf_{t\to\infty} x(t) = 0$ and by Lemma 2.1, we have $\lim_{t\to\infty} z(t) = 0$. Indeed,

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) = \limsup_{t\to\infty}\big(x(t) + p(t)x(\tau(t))\big) &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty}\big(-p_5 x(\tau(t))\big) \\ &= (1 - p_5)\limsup_{t\to\infty} x(t). \end{aligned}$$

Hence, $\limsup_{t\to\infty} x(t) = 0$. The rest of the proof follows from Theorem 2.1.

Next, we suppose that (A_3) does not hold such that the integral in (A_3) is finite for some $1 - p_5 > 0$. Then there exist $t_1 > 0$ such that

$$\int_{t_1}^{\infty} q(\eta)\, d\eta < \frac{1 - p_5}{5H(1)}, \quad t \geq t_1.$$

For $t_2 > t_1$, we let $Y = BC([t_2, \infty), \mathbb{R})$ be the space of all real valued bounded continuous functions defined on $[t_2, \infty)$. Clearly, Y is a Banach space with respect to sup norm defined by

$$||y|| = \sup\{|y(t)| : t \geq t_2\}.$$

Let $K = \{y \in Y : y(t) \geq 0,\ t \geq t_2\}$. Then, Y is a partially ordered Banach space (p.30, [11]). For $u, v \in Y$, we define $u \leq v$ if and only if $u - v \in K$. Let

$$S = \left\{ X \in Y : \ \frac{1-p_5}{5} \leq x(t) \leq 1,\ t \geq t_2 \right\}.$$

If $x_0(t) = \frac{1-p_5}{5}$, then $x_0 \in S$ and $x_0 = $ g.l.b S. Further, if $\phi \subset S^* \subset S$, then

$$S^* = \left\{ x \in Y : \ l_1 \leq x(t) \leq l_2,\ \frac{1-p_5}{5} \leq l_1,\ l_2 \leq 1 \right\}.$$

Let $v_0(t) = l_2'$, $t \geq t_3$, where $l_2' = \sup\{l_2 : \frac{1-p_5}{5} \leq l_2 \leq 1\}$. Then $v_0 \in S$ and $v_0 =$ l.u.b S^*. For $t_3 = t_2 + \rho$, define $\Phi : S \to S$ by

$$(\Phi x)(t) = \begin{cases} (\Phi x)(t_3), & t \in [t_2, t_3] \\ -p(t)x(\tau(t)) + \frac{1-p_5}{5} + \int_t^\infty q(\eta) H\big(x(\sigma(\eta))\big)\, d\eta, & t \geq t_3. \end{cases}$$

For every $x \in S$, $(\Phi x)(t) \geq \frac{1-p_5}{5}$ and

$$\begin{aligned} (\Phi x)(t) &\leq p_5 + \frac{1-p_5}{5} + H(1) \left[\int_t^\infty q(\eta)\, d\eta \right] \\ &< \frac{2+3p_5}{5} < 1. \end{aligned}$$

The above two inequalities imply that $\Phi x \in S$, too. Thus Phi maps S back into itself. For arbitary $x_1, x_2 \in S$, it is easy to verify that $x_1 \leq x_2$ implies that $\Phi x_1 \leq \Phi x_2$. Hence by Knaster-Tarski fixed point theorem (Theorem 1.7.3, [11]), Φ has a unique fixed point such that $\lim_{t\to\infty} x(t) \neq 0$. This completes the proof of the theorem. □

Theorem 2.4. *Assume that* (A_1) *and* (A_2) *hold and* $-\infty < -p_6 \leq p(t) \leq -p_7 < -1$ *for* $t \in \mathbb{R}_+$ *and* $p_6, p_7 > 0$. *Let* H *be Lipschitz continuous on intervals of the form* $[\alpha, \beta]$, $0 < \alpha < \beta < \infty$. *Then every bounded solutions of* (1.1) *either oscillates or converges to zero as* $t \to \infty$ *if and only if* (A_3) *holds.*

Proof. The proof of the theorem follows from the proof of Theorem 2.2. For necessary part, we need to mention the followings:

$$\int_{t_1}^\infty q(\eta)\, d\eta < \frac{p_7 - 1}{2K},$$

where $K = \max\{K_1, K_2\}$ and K_1 is the Lipschitz constant of H on $[a, b]$, $K_2 = H(b)$ such that

$$a = \frac{2\lambda p_7 - p_6(p_7 - 1)}{2p_6 p_7}, \quad b = \frac{\lambda}{p_7 - 1}$$

for

$$\lambda > \frac{p_6(p_7 - 1)}{2p_7} > 0,$$

and

$$(\Phi x)(t) = \begin{cases} (\Phi x)(t_2+\rho), \quad t \in [t_2, t_2+\rho] \\ -\frac{x\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)} - \frac{\lambda}{p\left(\tau^{-1}(t)\right)} + \frac{1}{p\left(\tau^{-1}(t)\right)} \left[\int_{\tau^{-1}(t)}^{\infty} q(\eta) H\big(x(\sigma(\eta))\big)\, d\eta\right], t \ge t_2+\rho. \end{cases}$$

This completes the proof of the theorem. □

Remark 2.2. In the above theorems, H could be linear, sublinear or superlinear.

Remark 2.3. Lemma 2.1 does not include $p(t) \equiv 1$ for all t (see for e.g. [11]). The present analysis does not allow the case $p(t) \equiv -1$ for all t. Hence in this work, a necessary and sufficient condition is established excluding $p(t) = \pm 1$ for all t. It seems that a different approach is necessary to study the case $p(t) = \pm 1$.

3. Example

Consider the differential equations

$$\big(x(t) + e^{-\pi} x(\tau(t))\big)' + 2e^{2t-6\pi}\big(x(\sigma(t))\big)^3 = 0, \tag{3.1}$$

where $0 < p(t) = e^{-\pi} < 1$, $\tau(t) = t - \pi$, $\sigma(t) = t - 2\pi$ and $H(x) = x^3$. Clearly, all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1 every solutions of (3.1) converges to zero as $t \to \infty$. Indeed, $x(t) = e^{-t}$ is such a solution of (3.1).

References

[1] F. N. Ahmed, R. R. Ahmad, U. K. S. Din and M. S. M. Noorani, *Oscillation criteria of first order neutral delay differential equations with variable coefficients,* Abst. Appl. Anal., (2013), Art. ID 489804, 5 pp.

[2] F. N. Ahmed, R. R. Ahmad, U. K. S. Din and M. S. M. Noorani, *Oscillations for nonlinear neutral delay differential equations with variable coefficients,* Abst. Appl. Anal., (2014), Art. ID 179195, 6 pp.

[3] F. N. Ahmed, R. R. Ahmad, U. K. S. Din and M. S. M. Noorani, *Oscillation criteria for nonlinear functional differential equations of neutral type,* J. Ineq. Appl., (2015), 2015:97, 11 pp.

[4] L. Berezansky and E. Braverman, *Oscillation criteria for a linear neutral differential equation,* J. Math. Anal. Appl., **286**(2003), 601–617.

[5] T. Candan, *Existence of non-oscillatory solutions to first-order neutral differential equations,* Elect. J. Diff. Equ., **2016(39)**(2016), 11 pp.

[6] P. Das and N. Misra, *A necessary and sufficient condition for the solutions of a functional-differential equation to be oscillatory or tend to zero,* J. Math. Anal. Appl., **205**(1997), 78–87.

[7] E. M. Elabbasy, T. S. Hassan and S. H. Saker, *Oscillation criteria for first-order nonlinear neutral delay differential equations,* Elect. J. Diff. Equ., **2005(134)**(2005), 18 pp.

[8] L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations,* Marcel Dekker, Inc., New York, 1995.

[9] J. R. Graef, R. Savithri and E. Thandapani, *Oscillation of First Order Neutral Delay Differential Equations,* Elect. J. Qual. Theo. Diff. Equ., Proc. 7th Coll. QTDE, (12):(2004), 1–11.

[10] S. Guo, W. Ma and B. G. S. A. Pradeep, *Necessary and sufficient conditions for oscillation of neutral delay differential equations,* Elect. J. Diff. Equ., **2014(138)**(2014), 12 pp.

[11] I. Gyori and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications,* Oxford University press., 1991.

[12] J. K. Hale, *Theory of Functional Differential Equations,* Spinger, New York, 1977.

[13] I. Kubiaczyk, S. H. Saker and J. Morchalo, *New oscillation criteria for first order nonlinear neutral delay differential equations,* Appl. Math. Comput., **142**(2003), 225–242.

[14] Z. Liu, S. M. Kang and J. S. Ume, *Existence of bounded nonoscillatory solutions of first-order nonlinear neutral delay differential equations,* Comput. Math. Appl., **59**(2010), 3535–3547.

[15] S. Pinelas and S. S. Santra, *Necessary and sufficient condition for oscillation of nonlinear neutral first order differential equations with several delays,* J. Fixed Point Theory Appl., **20(1)**(2018), Article Id. 27, 13 pp.

[16] S. S. Santra, *Oscillation criteria for nonlinear neutral differential equations of first order with several delays,* Mathematica, **57(80)(1-2)**(2015), 75–89.

[17] S. S. Santra, *Existence of positive solution and new oscillation criteria for nonlinear first order neutral delay differential equations,* Differ. Equ. Appl., **8(1)**(2016), 33–51.

[18] S. S. Santra, *Necessary and sufficient condition for oscillation of nonlinear neutral first order differential equations with several delays,* Mathematica, **58(81)(1-2)**(2016), 85–94.

[19] X. H. Tang and X. Lin, *Necessary and sufficient conditions for oscillation of first order nonlinear neutral differential equations,* J. Math. Anal. Appl., **321**(2006), 553–568.

[20] W. P. Zhang, W. Feng, J. R. Yan and J. S. Song, *Existence of Nonoscillatory Solutions of First-Order Linear Neutral Delay Differential Equations,* Comput. Math. Appl., **49**(2005), 1021–1027.