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On Regular Γ-semihyperrings and Idempotent Γ-semihyperrings

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Abstract. The Γ-semihyperring is a generalization of the concepts of a semiring, a semihyperring and a Γ-semiring. Here, the notions of (strongly) regular Γ-semihyperring, idempotent Γ-semihyperring; invertible set, invertible element in a Γ-semihyperring are introduced, and several examples given. It is proved that if all subsets of Γ-semihyperring are strongly regular then for every $\Delta \subseteq \Gamma$, there is a Δ-idempotent subset of R. Regularity conditions of Γ-semihyperrings in terms of ideals of Γ-semihyperrings are also characterized.

1. Introduction

In 1964, Nobusawa [15] introduced the notion of Γ-rings as a generalization of ternary rings. Barens [3] weakened the conditions in Nobusawa's definition of a Γ-ring. The notion of a Γ-semiring was introduced by Rao [17], generalizing both Γ-rings and semirings. In [13], Krishnamoorthy and Doss introduced the notion of commuting regular Γ-semiring.

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The concept of a hyperstructure was introduced in 1934 when Marty [14] defined hypergroups based on the notion of a hyperoperation during the 8^{th} Congress of Scandinavian Mathematicians. By analyzing their properties he applied the concepts to group theory. Corsini gave many applications of hyperstructure in several branches of both pure and applied sciences [4, 5].

In 1990, Vougiouklis [18] studied the notion of semihyperring in which both binary operations were hyperoperations; see [1]. The notion of a regular Γ-hyperring was introduced in [16]. Davvaz et al. [2, 12] introduced the notion of Γ-semihypergroup as a generalization of semihypergroup. Generalizations of semirings, semihyperrings and Γ-semirings to Γ-semihyperrings can be found in [7, 8, 9, 10, 11, 16].

This paper extends many classical notions of Γ-semirings [17] and Γ-hyperrings [16] to Γ-semihyperrings. Section 2 states some preliminary definitions which are useful to understand the main idea of the paper. The concept of a (strongly) regular Γ-semihyperring with some examples and results are introduced in Section 3. Section 4 introduces the notion of invertible sets (elements) in a Γ-semihyperring and an idempotent Γ-semihyperring that yield some important results.

2. Preliminaries

Here are some useful definitions; for others, the readers referred to [16].

Definition 2.1. Let H be a non-empty set and $\circ : H \times H \to \wp^*(H)$ be a hyperopertion, where $\wp^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid.* For any two non-empty subsets A and B of H and $x \in H$ we have

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, A \circ \{x\} = A \circ x \text{ and } \{x\} \circ A = x \circ A.$$

Definition 2.2. A hypergroupoid $(H, \circ)$ is called a *semihypergroup*, if for all $a, b, c \in H$ we have, $(a \circ b) \circ c = a \circ (b \circ c)$. In addition, if for every $a \in H, a \circ H = H = H \circ a$, then $(H, \circ)$ is called a *hypergroup* (for more details about hypergroups and semihypergroups see [4, 6]).

Definition 2.3. A *semihyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following properties:

(i) $(R, +)$ is a commutative semihypergroup;
i.e. $(x + y) + z = x + (y + z)$ and $x + y = y + x$, for all $x, y, z \in R$.

(ii) $(R, \cdot)$ is a semihypergroup.

(iii) The hyperoperation $\cdot$ is distributive with respect to the hyperoperation $+$,
i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$, $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in R$.

(iv) The element $0 \in R$ is an absorbing element;
i.e. $x \cdot 0 = 0 \cdot x = 0$, for all $x \in R$.

Definition 2.4. A semihyperring $(R, +, \cdot)$ is called *commutative* if and only if $a \cdot b = b \cdot a$ for all $a, b \in R$.

Definition 2.5. Let R be a commutative semihypergroup and Γ be a commutative group. Then, R is called a Γ-*semihyperring* if there is a map $R \times \Gamma \times R \to \wp^*(R)$ (the image (a, α, b) is denoted by $a\alpha b$ for all $a, b \in R$ and $\alpha \in \Gamma$) and $\wp^*(R)$ is the family of all non-empty subsets of R, satisfy the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$;

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$;

(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$;

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in R$ and for all $\alpha, \beta \in \Gamma$.

In the above definition, if R is a semigroup, then R is called a *multiplicative* Γ-*semihyperring.*

Definition 2.6. A Γ-semihyperring R is called *commutative* if $a\alpha b = b\alpha a$, for all $a, b \in R$ and $\alpha \in \Gamma$.

Definition 2.7. A Γ-semihyperring R is said to be *with zero*, if there exists $0 \in R$ such that $a \in a + 0$, $0 \in 0\alpha a$ and $0 \in a\alpha 0$, for all $a \in R$ and $\alpha \in \Gamma$.

Let A and B be two non-empty subsets of a Γ-semihyperring R and $x \in R$. Then,

$$A + B = \{x | x \in a + b, a \in A, b \in B\}$$
$$A\Gamma B = \{x | x \in a\alpha b, a \in A, b \in B, \alpha \in \Gamma\}.$$

Definition 2.8. A non-empty subset R_1 of a Γ-semihyperring R is called a Γ-*sub semihyperring* if it is closed with respect to the multiplication and addition, i.e. $R_1 + R_1 \subseteq R_1$ and $R_1 \Gamma R_1 \subseteq R_1$.

Definition 2.9. A *right (left) ideal* I of a Γ-semihyperring R is an additive sub semihypergroup of $(R, +)$ such that $I\Gamma R \subseteq I$ $(R\Gamma I \subseteq I)$. If I is both right and left ideal of R, then we say that I is *two sided ideal* or simply an *ideal* of R.

To see some examples on Γ-semihyperrings; the notions of Noetherian, Artinian, simple Γ-semihyperringa; and regular relations on Γ-semihyperrings, refer to [16].

3. Regular Γ-semihyperring

In this section, notions of regular sets, regular elements in a Γ-semihyperring are introduced and thereby (strongly) regular Γ-semihyperring is defined. Few examples in this context and characterization of regularity conditions of Γ-semihyperrings in terms of ideals of Γ-semihyperring are discussed.

Definition 3.1. A subset A of a Γ-semihyperring R is said to be *regular (strongly regular)* if there exist $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq R$ such that $A \subseteq A\Gamma_1 B\Gamma_2 A$ ($A = A\Gamma_1 B\Gamma_2 A$).

A singleton set $\{a\}$ of a Γ-semihyperring is *regular* if there exist $\Gamma_1, \Gamma_2 \subseteq \Gamma$ and $B \subseteq R$ such that

$$\{a\} = a \in a\Gamma_1 B\Gamma_2 a = \{x \in R \mid x \in a\alpha b\beta a, \alpha \in \Gamma_1, \beta \in \Gamma_2, b \in B\}.$$

That is, a singleton set $\{a\}$ of Γ-semihyperring is regular if there exist $\alpha, \beta \in \Gamma, b \in R$ such that $a \in a\alpha b\beta a$. Similarly, a singleton set $\{a\}$ of a Γ-semihyperring is strongly regular if there exist $\alpha, \beta \in \Gamma, b \in R$ such that $\{a\} = a = a\alpha b\beta a$. Simply an element $a \in R$ is said to be *regular (strongly regular)* instead of a singleton set $\{a\}$.

Definition 3.2. A Γ-semihyperring R is said to be *regular (strongly regular)*, if every element of R is regular (strongly regular).

Definition 3.3. An element $e \in R$ is said to be a Γ-*identity* of a Γ-semihyperring R, if $a\alpha e = e\alpha a = a$, for all $a \in R, \alpha \in \Gamma$.

Definition 3.4. A pair (A, B) of subsets of Γ-semihyperring R is said to be (Γ_1, Γ_2) *strongly regular (regular)* for some $\Gamma_1, \Gamma_2 \subseteq \Gamma$, if $A = A\Gamma_1 B\Gamma_2 A$ ($A \subseteq A\Gamma_1 B\Gamma_2 A$) and $B = B\Gamma_2 A\Gamma_1 B$ ($B \subseteq B\Gamma_2 A\Gamma_1 B$).

A pair of elements (a, b) of a Γ-semihyperring R is said to be (α, β) *strongly regular (regular)* for some $\alpha, \beta \in \Gamma$, if $a = a\alpha b\beta a (a \in a\alpha b\beta a)$ and $b = b\beta a\alpha b (b \in b\beta a\alpha b)$.

Example 3.5. If $R = \{a, b, c, d\}$ then R is a commutative semihypergroup with the following hyperoperations:

$+$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,d\}$
c	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,d\}$	$\{b,d\}$	$\{c,d\}$	$\{d\}$

$\cdot$	a	b	c	d
a	$\{a\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
b	$\{a,b\}$	$\{b\}$	$\{b,c\}$	$\{b,c,d\}$
c	$\{a,b,c\}$	$\{b,c\}$	$\{c\}$	$\{c,d\}$
d	$\{a,b,c,d\}$	$\{b,c,d\}$	$\{c,d\}$	$\{d\}$

Then R is a Γ-semihyperring with the operation $x\alpha y \to x \cdot y$ for $x, y \in R$ and $\alpha \in \Gamma$, where Γ is any commutative group. Also every subset of R is strongly regular since $A\Gamma_1 A\Gamma_2 A = A$, for any $\Gamma_1, \Gamma_2 \subseteq \Gamma$. Since all singleton sets of R are strongly regular, it follows that Γ-semihyperring R is strongly regular.

Example 3.6. Consider the following sets:

$$R = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} | x, y, z, w \in \mathbb{R} \right\}$$
$$\Gamma = \{z | z \in \mathbb{Z}\}$$
$$A_\alpha = \left\{ \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} | a, b \in \mathbb{R}, \alpha \in \Gamma \right\}.$$

Then R is a Γ-semihyperring under the matrix addition and the hyperoperation $M\alpha N \to MA_\alpha N$ for all $M, N \in R$ and $\alpha \in \Gamma$.

Now every invertible matrix $M \in R$ is regular, since $M \in M\alpha M^{-1}\alpha M$ where $\alpha = 1 \in \Gamma$ and $M^{-1} \in R$ is an inverse of M. Also, if we consider K as a collection of all invertible matrices in R then K is a regular subset of R, since $K \subseteq K\alpha K\alpha K$, where $\alpha = 1 \in \Gamma$.

Example 3.7.([16]) Let $(R, +, \cdot)$ be a semihyperring such that $x \cdot y = x \cdot y + x \cdot y$ and Γ be a commutative group. Define $x\alpha y \to x \cdot y$, for every $x, y \in R$ and $\alpha \in \Gamma$. Then R is a Γ-semihyperring.

Example 3.8. Let X be a non-empty finite set and τ be a topology on X. Define the addition hyperoperation and the multiplication on τ as $A, B \in \tau,\ A + B = A \cup B,\ A \cdot B = A \cap B$. Then $(\tau, +, \cdot)$ is a semihyperring with the absorbing element, the additive identity ϕ, and multiplicative identity X.

Example 3.9. Continuing with Example 3.7 and Example 3.8 observe that τ is a Γ-semihyperring if we define $x\alpha y \to x \cdot y$, for every $x, y \in \tau$ and $\alpha \in \Gamma$, where Γ is a commutative group. Further it is strongly regular since $A \in \tau$, then $A\alpha A\beta A = A$, for any $\alpha, \beta \in \Gamma$. Here X is a Γ-identity, since $X\alpha A = A = A\alpha X$, for all $A \in \tau$ and $\alpha \in \Gamma$.

Example 3.10. Let $R = Q^+, \Gamma = \{z | z \in \mathbb{Z}\}$ and $A_\alpha = \alpha\mathbb{Z}^+$. If we define $x\alpha y \to xA_\alpha y, \alpha \in \Gamma$ and $x, y \in R$, then R is a Γ-semihyperring under the ordinary addition and multiplication.

Clearly R is a regular Γ-semihyperring, since for any $x \in R, x \in x\alpha\frac{1}{x}\alpha x$, where $\alpha = 1 \in \Gamma$, that is, every element of R is regular and $(x, \frac{1}{x})$ is a regular pair of elements of Γ-semihyperring R.

Theorem 3.11. *Let R be a Γ-semihyperring. If I_1 and I_2 are ideals of R then $I_1 \cap I_2$ is an ideal of R too.*

Theorem 3.12. *Let R be a Γ-semihyperring with an identity. Then R is regular if and only if for any left ideal A and right ideal B of R, $A \cap B = B\Gamma A$.*

Proof. Since A is a left ideal of R, it follows that $B\Gamma A \subseteq R\Gamma A \subseteq A$. Further, since B is a right ideal of R, it follows that $B\Gamma A \subseteq B\Gamma R \subseteq B$. Hence,

$$B\Gamma A \subseteq A \cap B. \tag{3.1}$$

Let R be regular and $a \in A \cap B$. Then R is regular and there exist $b \in R$ and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha b\beta a \subseteq a\alpha A \subseteq B\Gamma A$. Hence we get,

$$A \cap B \subseteq B\Gamma A. \tag{3.2}$$

From Equation (3.1) and (3.2), we have $A \cap B = B\Gamma A$.

In order to prove the converse, assume that $A \cap B \subseteq B\Gamma A$ in R and $a \in R$. Now, $A = R\Gamma a$ is a left ideal and $B = a\Gamma R$ is a right ideal in R. Since R is a Γ-semihyperring with an identity, both ideals A and B contain a. Hence $a \in A \cap B = B\Gamma A = (a\Gamma R)\Gamma(R\Gamma a) = a\Gamma(R\Gamma R)\Gamma a \subseteq a\Gamma R\Gamma a$. Thus there exist $\alpha, \beta \in \Gamma$ and $b \in R$ such that $a \in a\alpha b\beta a$, i.e., $a \in R$ is a regular element. Since a is arbitrary, it follows that R is a regular Γ-semihyperring. □

Corollary 3.13. *Let R be a commutative Γ-semihyperring with an identity. Then R is regular if and only if $A = A\Gamma A$ for each ideal A of R.*

Proof. Suppose that R is regular and A is an ideal of R. Then by Theorem 3.12, $A\Gamma A = A \cap A = A$. Conversely, let R be a Γ-semihyperring satisfying the given condition and $a \in R$. Then $A = a\Gamma R$ is an ideal of R. Since R is a Γ-semihyperring with an identity element, it follows that $a \in A = A\Gamma A = (a\Gamma R)\Gamma(a\Gamma R) = a\Gamma(R\Gamma R)\Gamma a \subseteq a\Gamma R\Gamma a$. Hence an arbitrary element $a \in R$ is regular whence R is a regular Γ-semihyperring. □

Theorem 3.12. *Let I be an ideal of a regular Γ-semihyperring R. Then I is regular and any ideal J of I is an ideal of R.*

Proof. Suppose that I is an ideal of a regular Γ-semihyperring R and $a \in I \subset R$. Since R is regular, it follows that there exist $\alpha, \beta \in \Gamma$ and $b \in R$ such that $a \in a\alpha b\beta a$. Also let $C = b\beta a\alpha b \subseteq I$. Then $a \in a\alpha C\beta a = a\alpha b\beta a\alpha b\beta a$. Hence the ideal I is regular.

We now prove that if $a \in J \subset I$ and $r \in R$, then both $a\alpha r$ and $r\alpha a$ are subsets of J, where $\alpha \in \Gamma$. Let $a\alpha r \subseteq I$. Then each $k \in a\alpha r \subseteq I$ is a regular element in I. Hence there exist $k_1 \in I$ and $\alpha_1, \alpha_2 \in \Gamma$ such that $k \in k\alpha_1 k_1 \alpha_2 k \subseteq k\alpha_1 I \subseteq a\alpha r\alpha_1 I \subseteq a\alpha I \subseteq J$ since J is an ideal of I. Thus $k \in a\alpha r$ gives $k \in J$. Hence $a\alpha r \subseteq J$. Similar steps leads to $r\alpha a \subseteq J$. Thus an ideal J of I is an ideal of R. □

4. Inverse Sets in Γ-semihyperring

In this section notions of an idempotent Γ-semihyperring and inverse sets (elements) in Γ-semihyperring are introduced and demonstrated with examples. Also presented their characterizations and obtained some conditions of existence of strongly regular pairs and idempotent subsets in Γ-semihyperrings.

Definition 4.1. A subset B of a Γ-semihyperring R is said to be a (Γ_1, Γ_2) *inverse* of A if there exist $\Gamma_1, \Gamma_2 \subseteq \Gamma$ such that $A = A\Gamma_1 B\Gamma_2 A$ and $B = B\Gamma_2 A\Gamma_1 B$ denoted by $B \in V_{\Gamma_1}^{\Gamma_2}(A)$.

An element b of a Γ-semihyperring R is said to be a (α, β) *inverse* of $a \in R$ if there exist $\alpha, \beta \in \Gamma$ such that $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$ denoted by $b \in V_{\alpha}^{\beta}(a)$.

If $B \in V_{\Gamma_1}^{\Gamma_2}(A)$, then the subsets A and B of R are strongly regular subsets of the Γ-semihyperring R.

Lemma 4.2. *Let A be a strongly regular subset of* Γ*-semihyperring R. Then there exists a strongly regular subset B of R such that $B \in V_{\Gamma_1}^{\Gamma_2}(A)$.*

Proof. Suppose that A is a strongly regular subset of a Γ-semihyperring R. Then there exist $C \subseteq R$ and $\Gamma_1, \Gamma_2 \subseteq \Gamma$ such that $A = A\Gamma_1 C\Gamma_2 A$. Let $B = C\Gamma_2 A\Gamma_1 C$. Then

$$\begin{aligned} A\Gamma_1 B\Gamma_2 A &= A\Gamma_1(C\Gamma_2 A\Gamma_1 C)\Gamma_2 A \\ &= (A\Gamma_1 C\Gamma_2 A)\Gamma_1 C\Gamma_2 A \\ &= A\Gamma_1 C\Gamma_2 A \\ &= A \end{aligned}$$

and

$$\begin{aligned} B\Gamma_2 A\Gamma_1 B &= (C\Gamma_2 A\Gamma_1 C)\Gamma_2 A\Gamma_1(C\Gamma_2 A\Gamma_1 C) \\ &= C\Gamma_2(A\Gamma_1 C\Gamma_2 A)\Gamma_1 C\Gamma_2 A\Gamma_1 C \\ &= C\Gamma_2(A\Gamma_1 C\Gamma_2 A)\Gamma_1 C \\ &= C\Gamma_2 A\Gamma_1 C \\ &= B. \end{aligned}$$

Thus $B \in V_{\Gamma_1}^{\Gamma_2}(A)$. □

Definition 4.3. A subset E of a Γ-semihyperring R is an *idempotent set* if there exits $\Gamma_1 \subseteq \Gamma$ such that $E\Gamma_1 E = E$. It is referred as E is Γ_1*-idempotent*.

An element $e \in R$ is said to be *idempotent* if there exits $\alpha \in \Gamma$ such that $e\alpha e = e$. Then e is said to be α*-idempotent*.

Definition 4.4. A Γ-semihyperring R is said to be an *idempotent* Γ*-semihyperring* if every element of R is idempotent.

Example 4.5.([16]) Let $R = \{a, b, c, d\}, \Gamma = \mathbb{Z}_2$ and $\alpha = \bar{0}, \beta = \bar{1}$. Then R is a Γ-semihyperring with the following hyperoperations

$+$	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
b	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
c	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$	$\{a,b\}$
d	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{a,b\}$

β	a	b	c	d
a	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$
d	$\{a,b\}$	$\{a,b\}$	$\{c,d\}$	$\{c,d\}$

For any $x, y \in R$ we define $x\alpha y = \{a,b\}$. Then clearly $\{a,b\}$ is α-idempotent, β idempotent as well as $\Gamma_1 = \{\alpha,\beta\}$-idempotent and $\{c,d\}$ is β-idempotent.

Observe that Examples 3.5 and 3.9 are the illustrations of an idempotent Γ-semihyperring. It is obvious that if E is Γ_1-idempotent then E is strongly regular subset of R and $E \in V_{\Gamma_1}^{\Gamma_1}(E)$, i.e., E is the inverse of itself. Also, idempotent elements (subsets) of Γ-semihyperring R are strongly regular.

Theorem 4.6. *Let R be a Γ-semihyperring. Then $a \in R$ is strongly regular element of R if there is an idempotent element $e \in R$ such that $a = e\alpha x$ and $e = a\beta y$, for some $x, y \in R$ and $\alpha, \beta \in \Gamma$.*

Proof. Suppose that R is a Γ-semihyperring with condition that for $a \in R$ there is an idempotent element $e \in R$ such that $a = e\alpha x$ and $e = a\beta y$, for some $x, y \in R$ and $\alpha, \beta \in \Gamma$. Since e is an idempotent of R, it follows that there exists $\gamma \in \Gamma$ such that $e = e\gamma e$ which implies that $a = e\alpha x = (e\gamma e)\alpha x = (a\beta y)\gamma e\alpha x = a\beta y\gamma a$. Consequently, a is strongly regular element of the Γ-semihyperring.

Similarly, one can prove that when R is a Γ-semihyperring, then $a \in R$ is strongly regular element of R, if there is an idempotent element $e \in R$ such that $a = x\alpha e$ and $e = y\beta a$, for some $x, y \in R$ and $\alpha, \beta \in \Gamma$. □

Lemma 4.7. *Let R be a Γ-semihyperring. Let (A, A') be a (Γ_1, Γ_2) strongly regular pair and (B, B') be a (Γ_3, Γ_4) strongly regular pair. Then $A'\Gamma_2 A\Gamma_1 B\Gamma_3 B'$ is a Γ_4-idempotent and $B\Gamma_3 B'\Gamma_4 A'\Gamma_2 A$ is a Γ_1-idempotent if and only if $(A\Gamma_1 B, B'\Gamma_4 A')$ is a (Γ_3, Γ_2) strongly regular pair.*

Proof. Suppose that $A'\Gamma_2 A\Gamma_1 B\Gamma_3 B'$ is Γ_4-idempotent and $B\Gamma_3 B'\Gamma_4 A'\Gamma_2 A$ is Γ_1-idempotent. Then, we have

$$\begin{aligned}(A\Gamma_1 B)\Gamma_3(B'\Gamma_4 A')\Gamma_2(A\Gamma_1 B) &= A\Gamma_1 A'\Gamma_2 A\Gamma_1 B\Gamma_3 B'\Gamma_4 A'\Gamma_2 A\Gamma_1 B\Gamma_3 B'\Gamma_4 B\\ &= A\Gamma_1 A'\Gamma_2 A\Gamma_1 B\Gamma_3 B'\Gamma_4 B\\ &= A\Gamma_1 B.\\ (B'\Gamma_4 A')\Gamma_2(A\Gamma_1 B)\Gamma_3(B'\Gamma_4 A') &= B'\Gamma_4 B\Gamma_3 B'\Gamma_4 A'\Gamma_2 A\Gamma 1 B\Gamma_3 B'\Gamma_4 A'\Gamma_2 A\Gamma_1 A'\\ &= B'\Gamma_4 B\Gamma_3 B'\Gamma_4 A'\Gamma_2 A\Gamma_1 A'\\ &= B'\Gamma_4 A'.\end{aligned}$$

Hence $(A\Gamma_1 B, B'\Gamma_4 A')$ is a (Γ_3, Γ_2) strongly regular pair.

For the converse, we have

$$(A^{'}\Gamma_2 A\Gamma_1 B\Gamma_3 B^{'})\Gamma_4(A^{'}\Gamma_2 A\Gamma_1 B\Gamma_3 B^{'}) = A^{'}\Gamma_2(A\Gamma_1 B\Gamma_3 B^{'}\Gamma_4 A^{'}\Gamma_2 A\Gamma_1 B)\Gamma_3 B^{'}$$
$$= A^{'}\Gamma_2 A\Gamma_1 B\Gamma_3 B^{'}.$$

That is $A^{'}\Gamma_2 A\Gamma_1 B\Gamma_3 B^{'}$ is Γ_4-idempotent. Similarly, it can be proved that $B\Gamma_3 B^{'}\Gamma_4 A^{'}\Gamma_2 A$ is Γ_1-idempotent. □

Lemma 4.8. *Let R be a commutative Γ-semihyperring. Let $E = E\Gamma_1 E$ and $F = F\Gamma_2 F$ be two idempotent subsets of R. Then there exists an idempotent G such that $(E\Gamma_1 F, G)$ is a (Γ_2, Γ_1) strongly regular pair.*

Proof. Since R is a commutative Γ-semihyperring, it follows that $(E\Gamma_1 F)\Gamma_2(E\Gamma_1 F) = E\Gamma_1 F$, i.e., $E\Gamma_1 F$ is idempotent and so strongly regular set. Hence there exist $\Gamma_3, \Gamma_4 \subseteq \Gamma$ and $K \subseteq R$ such that K is an (Γ_3, Γ_4) inverse of $E\Gamma_1 F$. That is $E\Gamma_1 F\Gamma_3 K\Gamma_4 E\Gamma_1 F = E\Gamma_1 F$ and $K\Gamma_4 E\Gamma_1 F\Gamma_3 K = K$. Let $G = F\Gamma_3 K\Gamma_4 E$, then $G\Gamma_1 G = F\Gamma_3 K\Gamma_4 E\Gamma_1 F\Gamma_3 K\Gamma_4 E = F\Gamma_3 K\Gamma_4 E = G$. Therefore G is an idempotent.

$$\begin{aligned}(E\Gamma_1 F)\Gamma_2 G\Gamma_1(E\Gamma_1 F) &= E\Gamma_1 F\Gamma_2 F\Gamma_3 K\Gamma_4 E\Gamma_1 E\Gamma_1 F \\ &= E\Gamma_1 F\Gamma_3 K\Gamma_4 E\Gamma_1 F \\ &= E\Gamma_1 F.\end{aligned}$$

Also, we have

$$\begin{aligned}G\Gamma_1(E\Gamma_1 F)\Gamma_2 G &= F\Gamma_3 K\Gamma_4 E\Gamma_1 E\Gamma_1 F\Gamma_2 F\Gamma_3 K\Gamma_4 E \\ &= F\Gamma_3 K\Gamma_4 E\Gamma_1 F\Gamma_3 K\Gamma_4 E \\ &= F\Gamma_3 K\Gamma_4 E \\ &= G.\end{aligned}$$

Thus, $(E\Gamma_1 F, G)$ is a (Γ_2, Γ_1) strongly regular pair. □

Lemma 4.8 shows that, if a commutative Γ-semihyperring R containing Δ_1-idempotent, Δ_2-idempotent subsets then it has (Δ_1, Δ_2) strongly regular pair. Further, using Lemma 4.2, we got the following results.

Lemma 4.9. *Let R be a Γ-semihyperring in which all subsets are strongly regular. Then for every $\Delta \subseteq \Gamma$, there is a Δ-idempotent subset of R.*

Proof. Suppose that A is any subset of a Γ-semihyperring R. Then $A\Delta A$ is strongly regular subset of R, for each $\Delta \subseteq \Gamma$. By Lemma 4.2 there exist $B \subseteq R$ and $\Gamma_1, \Gamma_2 \subseteq \Gamma$ such that $B \in V_{\Gamma_1}^{\Gamma_2}(A\Delta A)$ that is $(A\Delta A)\Gamma_1 B\Gamma_2(A\Delta A) = A\Delta A$ and $B = B\Gamma_2(A\Delta A)\Gamma_1 B$. Let $E = A\Gamma_1 B\Gamma_2 A$. Then

$$\begin{aligned}E\Delta E &= A\Gamma_1 B\Gamma_2 A\Delta A\Gamma_1 B\Gamma_2 A \\ &= A\Gamma_1(B\Gamma_2 A\Delta A\Gamma_1 B)\Gamma_2 A \\ &= A\Gamma_1 B\Gamma_2 A \\ &= E.\end{aligned}$$

Therefore E is a Δ-idempotent subset of R. □

Let R be a Γ-semihyperring and R^* be a collection of all non-empty subsets of R. Consider $R^*_\Delta = \{A\Delta B | A, B \in R^* \text{ and } \Delta \subseteq \Gamma\}$. Clearly, $R^\star_\Delta$ is a semigroup since $A, B \in R^\star$ implies that $A\Delta B \in R^\star$ and $(A\Delta B)\Delta C = A\Delta(B\Delta C)$. By Lemma 4.9, it can be proved that every element of R^*_Δ has right and left identity as well as right and left inverse with the given conditions.

Theorem 4.10. *Let R be a Γ-semihyperring and all subsets of R be strongly regular. Then $R^\star_\Delta$ has an identity and every element of $R^\star_\Delta$ has a right (left) inverse if $E\Delta F = F$ and $E\Delta_1 F = E$ for any Δ-idempotent subset E and Δ_1-idempotent subset F of R.*

Proof. Suppose that $E\Delta F = F$ and $E\Delta_1 F = E$ for any Δ-idempotent E and Δ_1-idempotent F. Then by Lemma 4.9, for $\Delta \subseteq \Gamma$ there exists an idempotent $E \subseteq R$ such that $E = E\Delta E$. Since all subsets of R are strongly regular, it follows that for any $G \subseteq R$ there exist Γ_1, Γ_2 subsets of Γ and $B \subseteq R$ such that $G = G\Gamma_1 B\Gamma_2 G$. Now, $(G\Gamma_1 B)\Gamma_2(G\Gamma_1 B) = (G\Gamma_1 B\Gamma_2 G)\Gamma_1 B = G\Gamma_1 B$ and $(B\Gamma_2 G)\Gamma_1(B\Gamma_2 G) = B\Gamma_2(G\Gamma_1 B\Gamma_2 G) = B\Gamma_2 G$. Thus $G\Gamma_1 B$ is Γ_2-idempotent and $B\Gamma_2 G$ is a Γ_1- idempotent. Then by given condition, we have $(B\Gamma_2 G)\Delta E = B\Gamma_2 G$. Then $G\Delta E = (G\Gamma_1 B\Gamma_2 G)\Delta E = G\Gamma_1(B\Gamma_2 G\Delta E) = G\Gamma_1 B\Gamma_2 G = G$. Hence E is the right identity in $R^\star_\Delta$. Similarly we can show E is the left identity in $R^\star_\Delta$. That is E is an identity in $R^\star_\Delta$. Again, since $G\Delta G \subseteq R$, it follows that there exist subsets Δ_1, Δ_2 of Γ and $D \subseteq R$ such that $(G\Delta G)\Delta_1 D\Delta_2(G\Delta G) = G\Delta G$, then $G\Delta G\Delta_1 D$ is an Δ_2-idempotent. Hence by the given condition; $(G\Delta G\Delta_1 D)\Delta_2 E = E$ we get $G\Delta(G\Delta_1 D\Delta_2 E) = E$. Hence for any $G \subseteq R$, there is the right inverse of G in $R^\star_\Delta$. Similarly it can be proved $D\Delta_2 G\Delta G$ is the left inverse of G in $R^\star_\Delta$. □

Conclusion. In this paper we have studied the nature of sets and elements of Γ-semihyperring with variety of examples and results. The relation between strongly regular elements and idempotent elements of a Γ-semihyperring has been developed. The notion of ideals of a Γ-semihyperring could characterize the regularity conditions of Γ-semihyperring.

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