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New Approach to Pell and Pell-Lucas Sequences

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Abstract. In this paper, we first define generalizations of Pell and Pell-Lucas sequences by the recurrence relations

$$\mathtt{p}_n = 2a\mathtt{p}_{n-1} + (b-a^2)\mathtt{p}_{n-2} \quad \text{and} \quad \mathtt{q}_n = 2a\mathtt{q}_{n-1} + (b-a^2)\mathtt{q}_{n-2}$$

with initial conditions $\mathtt{p}_0 = 0$, $\mathtt{p}_1 = 1$, and $\mathtt{q}_0 = 2$, $\mathtt{q}_1 = 2a$, respectively. We give generating functions and Binet's formulas for these sequences. Also, we obtain some identities of these sequences.

1. Introduction

Due partly to their innumerable applications in not only the field of science, but also of art and literature, there have been many studies on special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Fermat, Mersenne. In particular, there have been many studies on Fibonacci and Lucas sequences, and their generalizations. For a small sample of these studies, one can see [1, 2, 4, 6, 9]. Although to a lesser extent, Pell and Pell-Lucas sequences have also been well studied and generalized by many authors. Some examples of these studies can be found in [3, 5, 7, 8].

Fibonacci $\{F_n\}_{n=0}^{\infty}$ and Lucas $\{L_n\}_{n=0}^{\infty}$ sequences are defined by the recurrence relations $F_n = F_{n-1} + F_{n-2}$ with initial conditions $F_0 = 0$, $F_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_0 = 2$ and $L_1 = 1$, respectively. Similarly, Pell $\{P_n\}_{n=0}^{\infty}$ and Pell-Lucas $\{Q_n\}_{n=0}^{\infty}$ sequences are defined recursively by the relations $P_n = 2P_{n-1} + P_{n-2}$ with initial conditions $P_0 = 0$, $P_1 = 1$, and $Q_n = 2Q_{n-1} + Q_{n-2}$ with initial conditions $Q_0 = 2$, $Q_1 = 2$, respectively.

In [1], Bilgici gave generalizations for Fibonacci and Lucas sequences as follows:

$$f_0 = 0, f_1 = 1 \quad f_n = 2af_{n-1} + (b-a^2)f_{n-2} \quad (n \geq 2)$$

$$l_0 = 2, l_1 = 2a \quad l_n = 2al_{n-1} + (b-a^2)l_{n-2} \quad (n \geq 2)$$

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where a and b are any nonzero real numbers.

The main objective of this study is to give new generalizations for Pell and Pell-Lucas sequences, in much the same way that Bilgici did for Fibonacci and Lucas sequences in [1]. We shall then determine generating functions and Binet's formulas, and give well-known Catalan's, Cassini's and d'Ocagne's identities for these generalized sequences. In addition, some identities of these generalized sequences and some relations between these two generalized sequences are given.

2. Generalized Pell and Pell-Lucas Sequences

In this section, we define generalizations of Pell and Pell-Lucas sequences. Then, we give generating functions and Binet's formulas for these generalized sequences.

Definition 2.1. For any real nonzero numbers a and b, the *generalized Pell sequence* $\{\mathtt{p}_n\}_{n=0}^{\infty}$ and the *generalized Pell-Lucas sequence* $\{\mathtt{q}_n\}_{n=0}^{\infty}$ are defined recursively, for $n \geq 2$, by

$$\mathtt{p}_n = 2a\mathtt{p}_{n-1} + (b - a^2)\mathtt{p}_{n-2}, \tag{2.1}$$

$$\mathtt{q}_n = 2a\mathtt{q}_{n-1} + (b - a^2)\mathtt{q}_{n-2} \tag{2.2}$$

with initial conditions $\mathtt{p}_0 = 0$, $\mathtt{p}_1 = 1$, and $\mathtt{q}_0 = 2$, $\mathtt{q}_1 = 2a$, respectively.

It is obvious that, in Eq. (2.1) and (2.2), if we take respectively

(1) $a = 1$, $b = 2$, we obtain classical Pell and Pell-Lucas sequences,

(2) $a = \frac{1}{2}$, $b = \frac{5}{4}$, we obtain classical Fibonacci and Lucas sequences,

(3) $a = \frac{1}{2}$, $b = \frac{9}{4}$, we obtain classical Jacobsthal and Jacobsthal-Lucas sequences,

(4) $a = \frac{3}{2}$, $b = \frac{1}{4}$, we obtain Mersenne and Fermat sequences.

The following theorem gives us generating functions for generalized Pell and Pell-Lucas sequences :

Theorem 2.2. *The generating functions of the generalized Pell sequence* $\{\mathtt{p}_n\}_{n=0}^{\infty}$ *and the generalized Pell-Lucas sequence* $\{\mathtt{q}_n\}_{n=0}^{\infty}$ *are given, respectively, by*

$$\mathbf{p}(x) = \frac{x}{1-2ax-(b-a^2)x^2} \quad \text{and} \quad \mathbf{q}(x) = \frac{2-2ax}{1-2ax-(b-a^2)x^2}.$$

Proof. The generating functions $\mathbf{p}(x)$ and $\mathbf{q}(x)$ can be written as $\mathbf{p}(x) = \sum_{n=0}^{\infty} \mathtt{p}_n x^n$ and $\mathbf{q}(x) = \sum_{n=0}^{\infty} \mathtt{q}_n x^n$. Then, we write

$$\begin{aligned}\mathbf{p}(x) &= \sum_{n=0}^{\infty} \mathtt{p}_n x^n = \mathtt{p}_0 + \mathtt{p}_1 x + \sum_{n=2}^{\infty} \mathtt{p}_n x^n \\ &= x + 2a \sum_{n=2}^{\infty} \mathtt{p}_{n-1} x^n + (b-a^2) \sum_{n=2}^{\infty} \mathtt{p}_{n-2} x^n \\ &= x + 2ax \sum_{n=0}^{\infty} \mathtt{p}_n x^n + (b-a^2)x^2 \sum_{n=0}^{\infty} \mathtt{p}_n x^n \\ &= x + 2ax\mathbf{p}(x) + (b-a^2)x^2\mathbf{p}(x).\end{aligned}$$

Thus, we obtain

$$(1 - 2ax - (b-a^2)x^2)\mathbf{p}(x) = x.$$

Hence, we have

$$\mathbf{p}(x) = \frac{x}{1 - 2ax - (b-a^2)x^2}.$$

Similarly, we have

$$\begin{aligned}\mathbf{q}(x) &= \sum_{n=0}^{\infty} \mathtt{q}_n x^n = \mathtt{q}_0 + \mathtt{q}_1 x + \sum_{n=2}^{\infty} \mathtt{q}_n x^n \\ &= 2 + 2ax + 2a \sum_{n=2}^{\infty} \mathtt{q}_{n-1} x^n + (b-a^2) \sum_{n=2}^{\infty} \mathtt{q}_{n-2} x^n \\ &= 2 - 2ax + 2ax \sum_{n=0}^{\infty} \mathtt{q}_n x^n + (b-a^2)x^2 \sum_{n=0}^{\infty} \mathtt{q}_n x^n \\ &= 2 - 2ax + 2ax\mathbf{q}(x) + (b-a^2)x^2\mathbf{q}(x).\end{aligned}$$

Thus, we obtain

$$\mathbf{q}(x) = \frac{2 - 2ax}{1 - 2ax - (b-a^2)x^2}. \qquad \square$$

We now give Binet's formulas for the generalized Pell and Pell-Lucas sequences by the following:

Theorem 2.3. *The nth terms of the generalized Pell and Pell-Lucas sequences are given by*

$$\mathtt{p}_n = \tfrac{\alpha^n - \beta^n}{\alpha - \beta} \quad \textit{and} \quad \mathtt{q}_n = \alpha^n + \beta^n$$

where $\alpha = a+\sqrt{b}$ *and* $\beta = a-\sqrt{b}$ *are the roots of the equation* $x^2-2ax-(b-a^2) = 0$.

Proof. Using the partial fraction decomposition, $\mathbf{p}(x)$ and $\mathbf{q}(x)$ can be expressed as

$$\mathbf{p}(x) = -\tfrac{1}{2\sqrt{b}}\tfrac{1}{\alpha x-1} + \tfrac{1}{2\sqrt{b}}\tfrac{1}{\beta x-1} \quad \text{and} \quad \mathbf{q}(x) = -\tfrac{1}{\alpha x-1} - \tfrac{1}{\beta x-1}.$$

However, note that, $\alpha+\beta = 2a$, $\alpha-\beta = 2\sqrt{b}$ and $\alpha\beta = a^2-b$.
Then, we have

$$\begin{aligned}\mathbf{p}(x) = \sum_{n=0}^{\infty}\mathtt{p}_n x^n &= \frac{1}{2\sqrt{b}}\frac{1}{1-\alpha x} - \frac{1}{2\sqrt{b}}\frac{1}{1-\beta x}\\ &= \frac{1}{2\sqrt{b}}\sum_{n=0}^{\infty}(\alpha^n-\beta^n)x^n\\ &= \sum_{n=0}^{\infty}\frac{\alpha^n-\beta^n}{\alpha-\beta}x^n.\end{aligned}$$

Also

$$\begin{aligned}\mathbf{q}(x) = \sum_{n=0}^{\infty}\mathtt{q}_n x^n &= \frac{1}{1-\alpha x} + \frac{1}{1-\beta x}\\ &= \sum_{n=0}^{\infty}(\alpha^n+\beta^n)x^n.\end{aligned}$$

Inspecting the above expressions, we get the following:

$$\mathtt{p}_n = \tfrac{\alpha^n-\beta^n}{\alpha-\beta} \quad \text{and} \quad \mathtt{q}_n = \alpha^n+\beta^n.$$ □

The following corollary is also consequence of Theorem 2.3:

Corollary 2.4. *Let* n *be any integer. Then,*

$$\mathtt{p}_{-n} = -(a^2-b)^{-n}\mathtt{p}_n \quad \textit{and} \quad \mathtt{q}_{-n} = (a^2-b)^{-n}\mathtt{q}_n.$$

3. Some Identities on Generalized Pell and Pell-Lucas Sequences

In this section, we give well-known identities Catalan's, Cassini's and d'Ocagne's for the generalized Pell and Pell-Lucas sequences. Also, we investigate some nameless identities of these generalized sequences, and give some relations between these sequences.

Theorem 3.1.(Catalan's Identity) *For any integers* n *and* r, *we have*

$$\begin{aligned}\mathtt{p}_{n+r}\mathtt{p}_{n-r} - \mathtt{p}_n{}^2 &= -(a^2-b)^{n-r}\mathtt{p}_r{}^2,\\ \mathtt{q}_{n+r}\mathtt{q}_{n-r} - \mathtt{q}_n{}^2 &= (a^2-b)^{n-r}4b\mathtt{p}_r{}^2.\end{aligned}$$

Proof. Using Binet's formula of the generalized Pell sequence, we write

$$\begin{aligned} \mathtt{p}_{n+r}\mathtt{p}_{n-r} - {\mathtt{p}_n}^2 &= \frac{\alpha^{n+r}-\beta^{n+r}}{\alpha-\beta}\frac{\alpha^{n-r}-\beta^{n-r}}{\alpha-\beta} - (\frac{\alpha^n-\beta^n}{\alpha-\beta})^2 \\ &= -\frac{1}{(\alpha-\beta)^2}(\alpha\beta)^{n-r}(\alpha^{2r}+\beta^{2r}-2\alpha^r\beta^r) \\ &= -(\alpha\beta)^{n-r}(\frac{\alpha^r-\beta^r}{\alpha-\beta})^2. \end{aligned}$$

Thus, we obtain

$$\mathtt{p}_{n+r}\mathtt{p}_{n-r} - {\mathtt{p}_n}^2 = -(a^2-b)^{n-r}{\mathtt{p}_r}^2.$$

Using Binet's formula of the generalized Pell-Lucas sequence, the second identity can be proved in a similar manner. □

Taking $r = 1$ in Theorem 3.1, we obtain the following:

Corollary 3.2.(Cassini's Identity) *For every integer n, we have*

$$\begin{aligned} \mathtt{p}_{n+1}\mathtt{p}_{n-1} - {\mathtt{p}_n}^2 &= -(a^2-b)^{n-1}, \\ \mathtt{q}_{n+1}\mathtt{q}_{n-1} - {\mathtt{q}_n}^2 &= 4b(a^2-b)^{n-1}. \end{aligned}$$

Theorem 3.3.(d'Ocagne's Identity) *Let m and n be any integers. Then,*

$$\begin{aligned} \mathtt{p}_m\mathtt{p}_{n+1} - \mathtt{p}_n\mathtt{p}_{m+1} &= (a^2-b)^n\mathtt{p}_{m-n}, \\ \mathtt{q}_m\mathtt{q}_{n+1} - \mathtt{q}_n\mathtt{q}_{m+1} &= -2(b)^{\frac{1}{2}}(a^2-b)^n(\alpha^{m-n}-\beta^{m-n}). \end{aligned}$$

Proof. Using Binet's formula of the generalized Pell sequence, we write

$$\begin{aligned} \mathtt{p}_m\mathtt{p}_{n+1} - \mathtt{p}_n\mathtt{p}_{m+1} &= \frac{\alpha^m-\beta^m}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - \frac{\alpha^n-\beta^n}{\alpha-\beta}\frac{\alpha^{m+1}-\beta^{m+1}}{\alpha-\beta} \\ &= \frac{-\alpha^m\beta^{n+1}-\beta^m\alpha^{n+1}+\alpha^n\beta^{m+1}+\beta^n\alpha^{m+1}}{(\alpha-\beta)^2} \\ &= \frac{\alpha^m\beta^n-\alpha^n\beta^m}{\alpha-\beta} \\ &= (\alpha\beta)^n\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}. \end{aligned}$$

Thus, we obtain

$$\mathtt{p}_m\mathtt{p}_{n+1} - \mathtt{p}_n\mathtt{p}_{m+1} = (a^2-b)^n\mathtt{p}_{m-n}.$$

The second statement of the theorem can be proved similarly. □

Theorem 3.4. *Let n be any integer. Then,*

$$\begin{aligned} 4b\mathtt{p}_n &= (b-a^2)\mathtt{q}_{n-1} + \mathtt{q}_{n+1}, \\ \mathtt{q}_n &= (b-a^2)\mathtt{p}_{n-1} + \mathtt{p}_{n+1}. \end{aligned}$$

Proof. Using Binet's formula of the generalized Pell-Lucas sequence, we write

$$\begin{aligned} (b-a^2)\mathtt{q}_{n-1} + \mathtt{q}_{n+1} &= -\alpha\beta(\alpha^{n-1}+\beta^{n-1}) + (\alpha^{n+1}+\beta^{n+1}) \\ &= \alpha^n(\alpha-\beta) + \beta^n(\beta-\alpha) \\ &= (\alpha-\beta)^2\frac{\alpha^n-\beta^n}{\alpha-\beta} \\ &= 4b\mathtt{p}_n. \end{aligned}$$

This proves the first identity. The second identity can be proved in a similar manner. □

Theorem 3.5. *Let m and n be any integers. Then,*

$$\begin{aligned} \mathtt{p}_{m+n} &= \frac{1}{2}(\mathtt{p}_m\mathtt{q}_n + \mathtt{p}_n\mathtt{q}_m), \\ \mathtt{q}_{m+n} &= 2b\mathtt{p}_m\mathtt{p}_n + \frac{1}{2}\mathtt{q}_m\mathtt{q}_n. \end{aligned}$$

Proof. Using Binet's formulas of the generalized Pell and Pell-Lucas sequences, the theorem can be proved easily. □

Taking $n=-n$ in Theorem 3.5 and using Corollary 2.4, we obtain the following:

Corollary 3.6. *For any integers m and n, we have*

$$\begin{aligned} \mathtt{p}_{m-n} &= \frac{1}{2}(a^2-b)^{-n}(\mathtt{p}_m\mathtt{q}_n - \mathtt{p}_n\mathtt{q}_m), \\ \mathtt{q}_{m-n} &= \frac{1}{2}(a^2-b)^{-n}(\mathtt{q}_m\mathtt{q}_n - 4b\mathtt{p}_m\mathtt{p}_n). \end{aligned}$$

The consequence of the above is the following:

Corollary 3.7. *Let n be any integer. Then,*

$$\begin{aligned} \mathtt{p}_{n-1} &= \frac{1}{b-a^2}(\frac{1}{2}\mathtt{q}_n - a\mathtt{p}_n), \\ \mathtt{q}_{n-1} &= \frac{1}{b-a^2}(2b\mathtt{p}_n - a\mathtt{q}_n). \end{aligned}$$

From the second identity in Corollary 3.6, we obtain the following:

Corollary 3.8. *For every integer n, we have*

$$\mathtt{q}_n{}^2 - 4b\mathtt{p}_n{}^2 = 4(a^2-b)^n.$$

Setting $m = n$ in Theorem 3.5, it is easy to see the following:

Corollary 3.9. *Let n be any integer. Then,*

$$\begin{aligned} \mathtt{p}_{2n} &= \mathtt{p}_n\mathtt{q}_n, \\ \mathtt{q}_{2n} &= 2b\mathtt{p}_n^2 + \frac{1}{2}\mathtt{q}_n^2. \end{aligned}$$

The following corollary follows from Corollary 3.8 and 3.9:

Corollary 3.10. *For any integer n, we have*

$$\mathtt{q}_{2n} = \mathtt{q}_n^2 - 2(a^2-b)^n.$$

If we take $m = 1$ in Theorem 3.5, we have the following:

Corollary 3.11. *For every integer n, we have*

$$\begin{aligned} \mathtt{p}_{n+1} &= a\mathtt{p}_n + \frac{1}{2}\mathtt{q}_n, \\ \mathtt{q}_{n+1} &= a\mathtt{q}_n + 2b\mathtt{p}_n. \end{aligned}$$

Theorem 3.12. *Let m and n be any integers. Then,*

$$\begin{aligned} \mathtt{p}_{m+n} &= \mathtt{p}_m\mathtt{p}_{n+1} - (a^2-b)\mathtt{p}_{m-1}\mathtt{p}_n, \\ \mathtt{p}_{m+n} &= \mathtt{p}_m\mathtt{q}_n - (a^2-b)^n\mathtt{p}_{m-n}, \\ \mathtt{p}_{m-n} &= (a^2-b)^{1-n}[\mathtt{p}_{m-1}\mathtt{p}_n - \mathtt{p}_m\mathtt{p}_{n-1}]. \end{aligned}$$

Proof. Using Binet's formula of the generalized Pell sequence, we write

$$\begin{aligned} \mathtt{p}_m\mathtt{p}_{n+1} - (a^2-b)\mathtt{p}_{m-1}\mathtt{p}_n &= \frac{\alpha^m-\beta^m}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} - \alpha\beta\frac{\alpha^{m-1}-\beta^{m-1}}{\alpha-\beta}\frac{\alpha^n-\beta^n}{\alpha-\beta} \\ &= \frac{1}{(\alpha-\beta)^2}[\alpha^{m+n}(\alpha-\beta) + \beta^{m+n}(\beta-\alpha)] \\ &= \frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta} \\ &= \mathtt{p}_{m+n}. \end{aligned}$$

This completes the proof of the first identity in theorem. Setting $n = -n$ in the first identity, the last identity can be obtained. The second identity can be proved in a similar manner. □

Theorem 3.13. *Let m and n be any integers. Then,*

$$\begin{aligned} \mathtt{p}_m\mathtt{p}_n &= \frac{1}{4b}[\mathtt{q}_{m+n} - (a^2-b)^n\mathtt{q}_{m-n}], \\ \mathtt{q}_m\mathtt{q}_n &= \mathtt{q}_{m+n} + (a^2-b)^n\mathtt{q}_{m-n}, \\ \mathtt{p}_m\mathtt{q}_n &= \mathtt{p}_{m+n} + (a^2-b)^n\mathtt{p}_{m-n}. \end{aligned}$$

Proof. Using Binet's formulas of the generalized Pell and Pell-Lucas sequences, we write

$$\begin{aligned} \mathtt{p}_m\mathtt{q}_n &= \frac{\alpha^m-\beta^m}{\alpha-\beta}(\alpha^n+\beta^n) \\ &= \frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta} + \frac{\alpha^m\beta^n-\beta^m\alpha^n}{\alpha-\beta} \\ &= \frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta} + (\alpha\beta)^n\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta} \\ &= \mathtt{p}_{m+n} + (a^2-b)^n\mathtt{p}_{m-n}. \end{aligned}$$

This completes the proof of the last identity. The others can be proved similarly. □

Theorem 3.14. *For any integers m and n, we have*

$$\mathtt{q}_n = \mathtt{q}_m\mathtt{p}_{n-m+1} + (b-a^2)\mathtt{q}_{m-1}\mathtt{p}_{n-m}.$$

The theorem can be proved easily using Binet's formulas of the generalized Pell and Pell-Lucas sequences.

Theorem 3.15. *Let m and n be any integers. Then,*

$$\begin{aligned} \mathtt{p}_{2m}\mathtt{p}_{2n} &= \frac{1}{4b}[\mathtt{q}^2_{m+n} - (a^2-b)^{2n}\mathtt{q}^2_{m-n}], \\ \mathtt{p}_{2m}\mathtt{p}_{2n} &= \mathtt{p}^2_{m+n} - (a^2-b)^{2n}\mathtt{p}^2_{m-n}, \\ \mathtt{q}_{2m}\mathtt{q}_{2n} &= 4b\mathtt{p}^2_{m+n} + (a^2-b)^{2n}\mathtt{q}^2_{m-n}, \\ \mathtt{q}_{2m}\mathtt{q}_{2n} &= \mathtt{q}^2_{m+n} + 4b(a^2-b)^{2n}\mathtt{p}^2_{m-n}. \end{aligned}$$

Proof. Using Binet's formula of the generalized Pell-Lucas sequence, we write

$$\begin{aligned} \frac{1}{4b}[\mathtt{q}^2_{m+n} - (a^2-b)^{2n}\mathtt{q}^2_{m-n}] &= \frac{1}{4b}[(\alpha^{m+n}+\beta^{m+n})^2 - (\alpha\beta)^{2n}(\alpha^{m-n}+\beta^{m-n})^2] \\ &= \frac{\alpha^{2m+2n}+\beta^{2m+2n}-\alpha^{2m}\beta^{2n}-\beta^{2m}\alpha^{2n}}{4b}. \end{aligned}$$

On the other hand, from Binet's formula of the generalized Pell sequence, we get

$$\begin{aligned} \mathtt{p}_{2m}\mathtt{p}_{2n} &= \frac{\alpha^{2m}-\beta^{2m}}{\alpha-\beta}\frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} \\ &= \frac{\alpha^{2m+2n}+\beta^{2m+2n}-\alpha^{2m}\beta^{2n}-\beta^{2m}\alpha^{2n}}{4b}. \end{aligned}$$

Hence, we obtain

$$\mathtt{p}_{2m}\mathtt{p}_{2n} = \frac{1}{4b}[\mathtt{q}_{m+n}^2 - (a^2-b)^{2n}\mathtt{q}_{m-n}^2].$$

Similarly,

$$\begin{aligned}\mathtt{p}_{m+n}^2 - (a^2-b)^{2n}\mathtt{p}_{m-n}^2 &= (\frac{\alpha^{m+n}-\beta^{m+n}}{\alpha-\beta})^2 - (\alpha\beta)^{2n}(\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta})^2\\ &= \frac{\alpha^{2m+2n}+\beta^{2m+2n}-\alpha^{2m}\beta^{2n}-\beta^{2m}\alpha^{2n}}{4b}\\ &= \mathtt{p}_{2m}\mathtt{p}_{2n}.\end{aligned}$$

The others can be proved in a similar manner. □

Theorem 3.16. *Let n be any integer. Then,*

$$\begin{aligned}\mathtt{p}_n\mathtt{p}_{n+1} &= \frac{1}{4b}[\mathtt{q}_{2n+1} - 2a(a^2-b)^n],\\ \mathtt{q}_n\mathtt{q}_{n+1} &= \mathtt{q}_{2n+1} + 2a(a^2-b)^n,\\ \mathtt{p}_{n-1}\mathtt{p}_{n+1} &= \mathtt{p}_n^2 - (a^2-b)^{n-1}.\end{aligned}$$

Proof. Using Binet's formula of the generalized Pell sequence, we write

$$\begin{aligned}\mathtt{p}_{n-1}\mathtt{p}_{n+1} &= \frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\\ &= \frac{\alpha^{2n}+\beta^{2n}-(\alpha\beta)^{n-1}(\alpha^2+\beta^2)}{(\alpha-\beta)^2}\\ &= (\frac{\alpha^n-\beta^n}{\alpha-\beta})^2 - (\alpha\beta)^{n-1}\\ &= \mathtt{p}_n^2 - (a^2-b)^{n-1}.\end{aligned}$$

This completes the last identity. The others can be proved similarly. □

Theorem 3.17. *For every integer n, we have*

$$\mathtt{q}_n\mathtt{q}_{n+2} - 4b\mathtt{p}_{n-1}\mathtt{p}_{n+3} = (a^2-b)^{n-1}4a^2(a^2+3b).$$

Proof. Using Binet's formulas of the generalized Pell and Pell-Lucas sequences, we write

$$\begin{aligned}\mathtt{q}_n\mathtt{q}_{n+2} - 4b\mathtt{p}_{n-1}\mathtt{p}_{n+3} &= (\alpha^n+\beta^n)(\alpha^{n+2}+\beta^{n+2})\\ &\quad - (\alpha-\beta)^2\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\\ &= (\alpha\beta)^{n+1}(\frac{\beta}{\alpha}+\frac{\alpha}{\beta}+\frac{\beta^2}{\alpha^2}+\frac{\alpha^2}{\beta^2})\\ &= (\alpha\beta)^{n-1}(\alpha+\beta)^2(\alpha^2-\alpha\beta+\beta^2).\end{aligned}$$

Considering $\alpha = a + \sqrt{b}$ and $\beta = a - \sqrt{b}$, we obtain desired result. □

Theorem 3.18. *Let n be any integer. Then,*

$$\begin{aligned} \mathtt{p}_{4n+1} - (a^2 - b)^{2n} &= \mathtt{q}_{2n+1}\mathtt{p}_{2n}, \\ \mathtt{p}_{4n+3} - (a^2 - b)^{2n+1} &= \mathtt{q}_{2n+1}\mathtt{p}_{2n+2}. \end{aligned}$$

Proof. Using Binet's formulas of the generalized Pell and Pell-Lucas sequences, we write

$$\begin{aligned} \mathtt{q}_{2n+1}\mathtt{p}_{2n} &= (\alpha^{2n+1} + \beta^{2n+1})\frac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} \\ &= \frac{\alpha^{4n+1} - \beta^{4n+1}}{\alpha - \beta} + \frac{\beta^{2n+1}\alpha^{2n} - \alpha^{2n+1}\beta^{2n}}{\alpha - \beta} \\ &= \mathtt{p}_{4n+1} - (\alpha\beta)^{2n} \\ &= \mathtt{p}_{4n+1} - (a^2 - b)^{2n}. \end{aligned}$$

This completes the proof of the first identity. The proof of the second identity can be done in a similar manner. □

Theorem 3.19. *For any integer n, we have*

$$\mathtt{q}_n^2 - \frac{1}{a^2 - b}\mathtt{q}_{n+1}^2 = \mathtt{q}_{2n} - \frac{1}{a^2 - b}\mathtt{q}_{2n+2}.$$

Proof. The theorem can be proved easily using Binet's formula of the generalized Pell-Lucas sequence. □

Theorem 3.20. *Let m and n be any integers. Then,*

$$\begin{aligned} \mathtt{p}_{mn} &= \mathtt{q}_m\mathtt{p}_{m(n-1)} - (a^2 - b)^m\mathtt{p}_{m(n-2)}, \\ \mathtt{q}_{mn} &= \mathtt{q}_m\mathtt{q}_{m(n-1)} - (a^2 - b)^m\mathtt{q}_{m(n-2)}. \end{aligned}$$

Proof. Using Binet's formulas of the generalized Pell and Pell-Lucas sequences, we write

$$\begin{aligned} \mathtt{q}_m\mathtt{p}_{m(n-1)} - (a^2 - b)^m\mathtt{p}_{m(n-2)} &= (\alpha^m + \beta^m)\frac{\alpha^{mn-m} - \beta^{mn-m}}{\alpha - \beta} \\ &\quad - (\alpha\beta)^m\frac{\alpha^{mn-2m} - \beta^{mn-2m}}{\alpha - \beta} \\ &= \frac{\alpha^{mn} - \beta^{mn}}{\alpha - \beta} \\ &= \mathtt{p}_{mn}. \end{aligned}$$

This completes the proof of the first identity. The second identity is just as easy. □

Theorem 3.21. *Let n be any integer. Then,*

$$\begin{aligned} \mathtt{p}_{2n+1} &= (b-a^2)\mathtt{p}_n^2 + \mathtt{p}_{n+1}^2, \\ \mathtt{p}_{2n+1} &= \frac{1}{b-a^2}[\mathtt{p}_{n+1}\mathtt{q}_{n+2} - 2a\mathtt{p}_{n+2}\mathtt{q}_n + (a^2-b)^n(5a^2-b)]. \end{aligned}$$

Proof. Using Binet's formula of the generalized Pell sequence, we write

$$\begin{aligned} (b-a^2)\mathtt{p}_n^2 + \mathtt{p}_{n+1}^2 &= -\alpha\beta(\frac{\alpha^n-\beta^n}{\alpha-\beta})^2 + (\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta})^2 \\ &= \frac{1}{(\alpha-\beta)^2}[\alpha^{2n+1}(\alpha-\beta) - \beta^{2n+1}(\alpha-\beta)] \\ &= \frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta} \\ &= \mathtt{p}_{2n+1}. \end{aligned}$$

Also from Binet's formulas of the generalized Pell and Pell-Lucas sequences, we get

$$\begin{aligned} &\frac{1}{b-a^2}[\mathtt{p}_{n+1}\mathtt{q}_{n+2} - 2a\mathtt{p}_{n+2}\mathtt{q}_n + (a^2-b)^n(5a^2-b)] \\ &= \frac{1}{-\alpha\beta}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}(\alpha^{n+2}+\beta^{n+2}) + \frac{\alpha+\beta}{\alpha\beta}\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}(\alpha^n+\beta^n) \\ &\quad + \frac{(a^2-b)^n}{b-a^2}(5a^2-b) \\ &= \frac{1}{\alpha-\beta}(\alpha^{2n+1}-\beta^{2n+1}) + \frac{(\alpha\beta)^n(\alpha-\beta)}{\alpha-\beta}[1+\frac{(\alpha+\beta)^2}{\alpha\beta}] \\ &\quad + \frac{(a^2-b)^n}{b-a^2}(5a^2-b) \\ &= \frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta} \\ &= \mathtt{p}_{2n+1}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 3.22. *Let m and n be any integers. Then,*

$$\begin{aligned} \mathtt{p}_{3n} &= \mathtt{p}_n\mathtt{q}_{2n} + (a^2-b)^n\mathtt{p}_n, \\ \mathtt{p}_{3n} &= \mathtt{p}_{2n}\mathtt{q}_n - (a^2-b)^n\mathtt{p}_n, \\ \mathtt{q}_{3n} &= \mathtt{q}_n\mathtt{q}_{2n} - (a^2-b)^n\mathtt{q}_n, \\ \mathtt{p}_{2m+n} &= \mathtt{p}_m\mathtt{q}_{m+n} + (a^2-b)^m\mathtt{p}_n, \\ \mathtt{q}_{2m+n} &= \mathtt{q}_m\mathtt{q}_{m+n} - (a^2-b)^m\mathtt{q}_n. \end{aligned}$$

The theorem can be proved easily using Binet's formulas of the generalized Pell and Pell-Lucas sequences.

We would also like to point out that, if we take $a = 1$, $b = 2$ in the most of these theorems and corollaries, we obtain known identities of classical Pell and Pell-Lucas sequences.

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