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CYLOTOMIC FUNCTION FIELDS OVER FINITE FIELDS WITH CLASS NUMBER THREE

Mehpare Bilhan, Dilek Buyruk, and Ferruh Özbudak

Abstract. We list all subfields of cyclotomic function fields over rational function fields with class number three. We also determine all the imaginary abelian extensions with relative class number three, explicitly.

1. Introduction

Let $k := \mathbb{F}_q(T)$ be the rational function field over the finite field $\mathbb{F}_q$ with q elements and let $\mathbb{A} := \mathbb{F}_q[T]$ be the ring of polynomials. The infinite prime divisor of k associated to $(1/T)$ is denoted by P_∞. For $N \in \mathbb{A}$, we construct the Carlitz module to the N^{th} *cyclotomic function field* K_N and its maximal real subfield K_N^+. Cyclotomic function fields were investigated by L. Carlitz in 1935 in [4]. In [5], Hayes developed the theory of cyclotomic function fields.

Throughout the paper, we assume K is a finite abelian extension of k such that $K \subseteq K_M$ for some $M \in \mathbb{A}$. The *conductor* $cond(K)$ of K is $N \in A$ such that K_N is the smallest cyclotomic function field containing K. Let K_N^+ be the maximal real subfield of K_N, that is, the maximal subfield of K on which P_∞ splits completely. Then $K^+ = K \cap K_N^+$ be the maximal real subfield of K. We say K is a *real extension* of k, if $K = K^+$, *imaginary* otherwise. If $K^+ = k$, K is a *totally imaginary extension* of k. Let the group of divisors of K and principle divisors of K be denoted by $Div^0(K)$ and $P(K)$, respectively. Then, the order of the quotient group $Div^0(K)/P(K)$ is called the *class number* of K. Let h_K and h_{K^+} denote the class number of K and K^+, respectively. Since K/K^+ is a finite separable extension, h_K is divisible by h_{K^+} and $h_K^- = h_K/h_{K^+}$ is called the *relative class number* of K.

In [8], the authors determined all cyclotomic function fields, their maximal real subfields with class number one. In [3], Bae and Kang determined all the cyclotomic function fields with relative class number one. In [7], Ahn and Jung determined all the abelian extensions of k which have the divisor class number one and all the imaginary abelian extensions with relative divisor class number one. In [6], they also determined all subfields of cyclotomic function fields with

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genus one. In [1], same authors determined all subfields of cyclotomic function fields with divisor class number two and they also gave the generators of such fields, explicitly.

In this paper, we determine all subfields of cyclotomic function fields with class number three and all the imaginary abelian extensions with relative divisor class number three. In Section 2, we recall some definitions and theorems required for the determination of subfields of cyclotomic function fields and we also gave some necessary conditions for a function field to have class number three. Then we conclude that if a subfield of a cyclotomic function field has class number three, then its genus is one or two. In Section 3, we classify subfields of cyclotomic function fields of genus one. In Section 4, we classify the subfields of genus two. In the last section, we determined imaginary subfields of cyclotomic function fields with relative class number three.

Throughout the paper, we fix the following notations:

- $\mathbb{F}_q$: a finite field with q elements.
- $k/\mathbb{F}_q$: the rational function field $\mathbb{F}_q(T)$ over $\mathbb{F}_q$.
- $\mathbb{A}$: the ring of polynomial $\mathbb{F}_q[T]$ of k.
- $K_N/\mathbb{F}_q$: Nth cyclotomic function field over $\mathbb{F}_q$ where $N \in \mathbb{A}$.
- K_N^+: the maximal real subfield of K_N.
- $K/\mathbb{F}_q$: a subfield K of a cyclotomic function field K_N over $\mathbb{F}_q$.
- K^+: the maximal real subfield of K.
- $Cond(K)$: the conductor of K.
- g_K: the genus of K.
- h_K, h_{K^+}: the class number of K and K^+, respectively.
- h_K^-: the relative class number of K.
- $h(O_K)$, $h(O_{K^+})$: the ideal class number of K and K^+, respectively.
- $h^-(O_K)$: the relative ideal class number of K.
- $D(F_1/F_2)$: the different of the function field extension of F_1 over F_2.
- $e(P, F_1/F_2)$: the ramification index of a place P of F_2 in the extension F_1/F_2.
- $f(P, F_1/F_2)$: the inertia degree of a place P of F_2 in the extension F_1/F_2.
- $Gal(F_1/F_2)$: the Galois group of F_1 over F_2.
- X_K: the character group of a function field K.
- N_j for $j \in \mathbb{N}^+$: the number of degree j places of K.

2. Preliminaries

Definition 2.1. Let F'/K be a finite separable extension of F/K. Then the divisor

$$D(F'/F) := \sum_{P \in \mathbb{P}_F} \sum_{P'|P} d(P'|P)P'$$

is called the different of F'/F. If $P'|P$ is tamely ramified, then $d(P'|P) = e(P'|P) - 1$. Otherwise

$$d(P'|P) = \sum_{n=0}^{\infty}(|G^0(P', F'/F)| - [G^0(P', F'/F) : G^n(P', F'/F)]),$$

where $G^n(P', F'/F)$ denotes the nth upper ramification group of P' in F'.

Theorem 2.2 (Hurwitz Genus formula)**.** *Let L/K be a finite abelian extension with the same constant field. Then we have*

$$2g_L - 2 = (2g_K - 2)[L : K] + \deg(D(L/K)),$$

where $D(L/K)$ is the different of L/K. Especially $g_L \geq g_K$.

Definition 2.3. The *ideal class number* $h(O_S)$ is defined as the class number of the Dedekind ring

$$O_S = \bigcap_{P \notin S} O_P,$$

where $O_P = \{z \in K, v_P(z) \geq 0\}$ is the local ring associated with the valuation v_P at P.

Let $h(O_{K^+})$ be the ideal class number of K^+. Since K/K^+ is totally imaginary $h(O_{K^+})$ divides $h(O_K)$. Then $h^-(O_K) = h(O_K)/h(O_{K^+})$ is called the *relative ideal class number* of K.

It is known that K/K^+ is a finite separable extension, so the Zeta function of K^+ divides that of K. Hence h_{K^+} divides h_K and $h_K^- = h_K/h_{K^+}$ is called the *relative class number* of K.

By [11, Chapter 16], the analytic class number formula is

$$h_K^- = \prod_{\chi \text{ non-real}} (\sum_{A \in M_N} \chi(A)), \tag{2.1}$$

where $M_N = \{A \in \mathbb{F}_q[T] : A \text{ is monic, } \deg(A) < \deg(N) \text{ and } (A, N) = 1\}$ and χ is a non-real character in the character group X_K associated to the function field K.

Let K be a subfield of a cyclotomic function field and X_K be the character group of K. For a fixed monic irreducible polynomial $Q \in \mathbb{A}$, let $Y_K = \{\chi \in X_K : \chi(Q) \neq 0\}$ and $Z_K = \{\chi \in X_K : \chi(Q) = 1\}$. By [15, Chapter 3], we have $[Y_K : Z_K] = f(Q, K/k)$, the inertia degree of the associated place of Q in K/k and $\mid Z_K \mid$, the number of primes of K lying above the associated place of Q.

The following theorem was already proved in [14]. It presents only numerical necessary and sufficient conditions for the existence of algebraic function fields with class number three.

Theorem 2.4 ([14, Theorem 3.3.4])**.** *Let $K/\mathbb{F}_q$ be a function field of genus g. Then $h_K = 3$ if and only if one of the following conditions holds*:

(A) $g = 1$, $2 \leq q \leq 7$ *and* $N_1 = 3$.

(B) $g = 2$, $q = 2$ *and*
  (1) $N_1 = 0$, $N_2 = 5$ *or*
  (2) $N_1 = 1$, $N_2 = 4$ *or*
  (3) $N_1 = 2$, $N_2 = 2$.
(C) $g = 2$, $q = 3$ *and*
  (1) $N_1 = 0$, $N_2 = 6$ *or*
  (2) $N_1 = 1$, $N_2 = 5$ *or*
  (3) $N_1 = 2$, $N_2 = 3$.
(D) $g = 3$, $q = 2$ *and*
  (1) $N_1 = 0$, $N_3 = 3$, $N_2 \leq 13$ *or*
  (2) $N_1 = 1$, $N_2 + N_3 = 4$ *or*
  (3) $N_1 = 2$, $N_3 + 2N_2 = 3$.
(E) $g = 4$, $q = 2$ *and*
  (1) $N_1 = 0$, $2N_4 + N_2^2 - 3N_2 = 6$ *and*

$$\begin{cases} N_2 = 0, & N_3 \leq 11, & N_4 = 3 & or \\ N_2 = 1, & N_3 \leq 8, & N_4 = 4 & or \\ N_2 = 2, & N_3 \leq 6, & N_4 = 4 & or \\ N_2 = 3, & N_3 \leq 3, & N_4 = 3 & or \\ N_2 = 4, & N_3 = 0, & N_4 = 1, & \end{cases}$$

  or
  (2) $N_1 = 1$, $2N_4 + 2N_3 + N_2^2 - N_2 = 8$ *and*

$$\begin{cases} N_2 = 0, & N_3 \leq 3, & 1 \leq N_4 \leq 4 & or \\ N_2 = 1, & N_3 = 0, & N_4 = 4. & \end{cases}$$

(F) $g = 5$, $q = 2$ *and* $N_1 = 0, N_5 - 2N_3 + N_2N_3 = 3$, $N_5 \neq 0$.
(G) $g = 6$, $q = 2$ *and* $N_1 = 0, N_6 - 2N_4 + (N_3 + N_3^2)/2 = 3$, $N_2 = 0$, $N_5 \leq 6$.

From now on, we assume K is a subfield of a cyclotomic function field with class number 3. Let $S_\infty(K)$ denote the set of prime divisors of K lying above P_∞. Then $N_1 \geq |S_\infty(K)| = [K^+ : k] \geq 1$. For $q = 2$, $K = K^+$ and $N_1 \geq |S_\infty(K)| = [K : k] \geq 2$ in this case. Then by Theorem 2.4, we have the following cases:
(I) $g_K = 1$, ($q = 2, 3, 5, 7$, $N_1 = 3$),
(II) $g_K = 2$, ($q = 2$, $N_1 = 2, N_2 = 2$) or ($q = 3$, $N_1 = 1, N_2 = 5$ or $N_1 = 2, N_2 = 3$),
(III) $g_K = 3$, $q = 2$, $N_1 = 2, N_3 + 2N_2 = 3$.

For the last case, $2 = N_1 \geq |S_\infty(K)| = [K : k] \geq 2$, then the extension is quadratic and P_∞ split. That implies (T) and $(T + 1)$ are inert, thus $N_2 \geq 2$ and $N_3 \leq -1$, which is not possible. Hence we skip this case.

Lemma 2.5 ([6, Lemma 2.4]). *Let P be a monic irreducible polynomial in $\mathbb{F}_q[T]$. Let l be a natural number such that l divides $q-1$, $l > 1$. Let $d = \deg P$ and $d_0 = \gcd(l, d)$. Let n, $1 \leq n \leq l/d_0$ be such that $nd \equiv d_0 \pmod{l}$.*

Then the unique cyclic subextension K of K_P/k with degree l is given by $K = k(\sqrt[l]{(-1)^{d_0}P^n})$. *Furthermore, l divides d if and only if $K \subseteq K_P^+$.*

3. Genus one

In this section, we determine the subfield K with class number three when $g_K = 1$. By Theorem 2.4, we have $2 \leq q \leq 7$. First, we consider the real extensions.

Proposition 3.1. *Let K/k be a real extension of genus one with class number three, then $N_1 = 3 \geq [K : k] \geq 2$ and K satisfies one of the following cases*:

(i) $q = 2, 4$, *K is a quadratic extension of k with conductor P^4*, $\deg P = 1$,

(ii) $q = 4, 7$, $K = k(\sqrt[3]{P})$, $\deg P = 3$,

(iii) $q = 3, 5, 7$, $K = k(\sqrt{P_1P_2})$, $\deg P_1 = 1, \deg P_2 = 3$.

Proof. Proof is by [6, Theorem 3.3 and Theorem 3.4]. □

Theorem 3.2. *Let K/k be a real extension of genus 1 with class number 3. Then K is one of the following function fields up to isomorphism* ($T \to T + a$, $a \in \mathbb{F}_q^*$):

(1) $q = 2$, $K = k(y)$ *such that* $y^2 + y = 1/T^3$,

(2) $q = 4$, $K = k(y)$ *such that* $y^2 + wy = 1/T^3$ *where* $\langle w \rangle = \mathbb{F}_4^*$,

(3) $q = 4$, $K = k(y)$ *such that* $y^3 = T^3 + w$ *where* $\langle w \rangle = \mathbb{F}_4^*$,

(4) $q = 7$, $K = k(y)$ *such that* $y^3 = T^3 + 3$ *or* $y^3 = T^3 + 4$,

(5) $q = 3$, $K = k(y)$ *such that* $y^2 = T(T^3 + 2T^2 + T + 1)$ *or* $y^2 = T(T^3 + T^2 + T + 2)$,

(6) $q = 5$, $K = k(y)$ *such that* $y^2 = T(T^3 - T^2 - T - 1)$ *or* $y^2 = T(T^3 + 2T^2 + T + 3)$ *or* $y^2 = T(T^3 + 3T^2 + T + 2)$ *or* $y^2 = T(T^3 + T^2 - T + 1)$,

(7) $q = 7$, $K = k(y)$ *such that* $y^2 = T(T^3 + 2)$ *or* $y^2 = T(T^3 + 5)$,

Proof. Clearly, K satisfies one of the conditions of Proposition 3.1:

(i) $q = 2$ or 4 and K is a quadratic extension of k with conductor P^4, $\deg P = 1$.

• Let $q = 2$, then $|S_\infty(K)| = [K : k] = 2 = N_1 - 1$ and one of the finite places of k of degree one is ramified. Up to isomorphism, let (T) be ramified. Since K is an elliptic function field, up to isomorphism, $K = \mathbb{F}_2(x, y)$ with $y^2 + y = x^3$, where P_∞ is ramified and (x) splits in $K/\mathbb{F}_2(x)$. Using the substitution, $x \to 1/T$, we get $K = k(y)$ with $y^2 + y = 1/T^3$ where P_∞ splits and (T) is ramified in K/k.

By Hurwitz Genus formula, we check our result:

$$0 = -2[K : k] + \deg(Diff(K/k)).$$

That is, $d((T), K/k) = 4$ and $v_T(u) = -3$, where $u = 1/T^3$. Since $u \neq w^2 - w$ for any $w \in \mathbb{F}_2(T)$, by Artin-Schreier extension, $y^2 + y = u$ where (T) is totally ramified in K/k.

• Let $q = 4$, then $|S_\infty(K)| = [K : k] = 2 = N_1 - 1$ and one of the finite places of k of degree one is ramified. Up to isomorphism, let (T) be ramified.

K is an elliptic function field, then up to isomorphism, $K = \mathbb{F}_2(x, y)$ with $y^2+wy+x^3 = 0$ where $\langle w\rangle = \mathbb{F}_4^*$. then P_∞ is ramified and (x) splits in $K/\mathbb{F}_2(x)$. Using the substitution, $x \to 1/T$, we get $K = k(y)$ with $y^2 + wy = 1/T^3$ where P_∞ splits and (T) is ramified in K/k.

(ii) $q = 4, 7$, $K = k(\sqrt[3]{P})$, $\deg P = 3$.

• Assume $q = 4$. Let X_{K_P}, X_K be the character groups of K_P and K, respectively. $X_{K_P} \cong (\mathbb{A}/P)^*$ is a cyclic group of order 63. Let χ be a generator, then $X_K = \langle\chi^a\rangle$ for some integer a and the order of χ^a is $[K : k] = 3$. Hence we may assume $X_K = \langle\chi^{21}\rangle$. As $|S_\infty(K)| = [K : k] = 3$, none of the finite places of k of degree one splits in K/k, that is, $[Y_K : Z_K] = f((Q), K/k) > 1$. Then, we have

$$\chi^{21}(T + a) \neq 1$$

for all $a \in \mathbb{F}_4$. Up to isomorphism $T \to T+\alpha$, $\alpha \in \mathbb{F}_4^*$, we have 5 possibilities for P. Among them, for $P = T^3+w$ where $\langle w\rangle = \mathbb{F}_4^*$, we get the solution. That is, let $P = T^3+w$, then $X_{K_P} = \langle\chi\rangle$ and $T+1$ is a primitive element of $(\mathbb{F}_4(T)/P)^*$. We have $\chi^{21}(T) = \chi(T^{21}) = \chi(w) = \exp(2\pi i/3)$, $\chi^{21}(T + 1) = \exp(4\pi i/3)$, $\chi^{21}(T + w) = \exp(4\pi i/3)$, $\chi^{21}(T + w^2) = \exp(4\pi i/3)$. Hence all finite places of k of degree one are inert and

$$K = k(\sqrt[3]{T^3 + w}) \text{ where } \langle w\rangle = \mathbb{F}_4^*. \tag{3.1}$$

• Assume $q = 7$. Let X_{K_P}, X_K be the character groups of K_P and K, respectively. $X_{K_P} \cong (\mathbb{A}/P)^*$ is a cyclic group of order $7^3 - 1$. Let χ be a generator, then $X_K = \langle\chi^a\rangle$ for some integer a and the order of χ^a is $[K : k] = 3$. Hence we may assume $X_K = \langle\chi^{114}\rangle$. As $|S_\infty(K)| = [K : k] = 3$, none of the finite places of k of degree one splits in K/k, that is, $[Y_K : Z_K] = f((Q), K/k) > 1$. Then, we have

$$\chi^{114}(T + a) \neq 1$$

for all $a \in \mathbb{F}_7$. Up to isomorphism $T \to T + \alpha, \alpha \in \mathbb{F}_7^*$, we have 16 possibilities for P. Among them, for $P = T^3 + 3$ and $T^3 + 4$, the result follows.

Let $P = T^3 + 3$, then $X_{K_P} = \langle\chi\rangle$ and $T + 1$ is a primitive element of $(\mathbb{F}_7(T)/P)^*$. We have $\chi^{114}(T) = \chi(T^{114}) = \chi(T^6) = \chi(2) = \exp(4\pi i/3)$, $\chi^{114}(T + 1) = \exp(2\pi i/3)$, $\chi^{114}(T + 2) = \exp(2\pi i/3)$, $\chi^{114}(T + 3) = \chi((T + 3)^{114}) = \chi(2) = \exp(4\pi i/3)$, $\chi^{114}(T + 4) = \chi(4) = \exp(2\pi i/3)$, $\chi^{114}(T + 5) = \chi(2) = \exp(4\pi i/3) = \chi^{114}(T + 6)$. Hence all finite places of k of degree one are inert and

$$K = k(\sqrt[3]{T^3 + 3}). \tag{3.2}$$

Similarly, for $P = T^3 + 4$, $\chi(T + a) \neq 1$ for all $a \in \mathbb{F}_7$ and

$$K = k(\sqrt[3]{T^3 + 4}). \tag{3.3}$$

(iii) $q = 3, 5, 7$, $K = k(\sqrt{P_1P_2})$, $\deg P_1 = 1, \deg P_2 = 3$.

• Let $q = 3$. $|S_\infty(K)| = [K : k] = 2 = N_1 - 1$ and one of the finite places of k of degree one, say P_1, is ramified and all other places of degree one are inert, except P_∞. Assume $P_1 = (T)$ and let the associated polynomial of P_2 be $T^3 + aT^2 + bT + c$ where $a, b, c \in \mathbb{F}_3$. Using [9, Lemma 2.6 and Proposition 2.7],

$$K = k(y) \text{ with } y^2 = T(T^3 + aT^2 + bT + c). \tag{3.4}$$

Let $P_1P_2(T)$ denote the product of the associated polynomials of P_1 and P_2, respectively. $P_1P_2(1) = P_1P_2(2) = 2 \in \mathbb{F}_3^* \backslash \mathbb{F}_3^{*2}$ and P_2 is irreducible. By [9, Lemma 2.6], we have $b = 1$, $a = 2$, $c = 1$ or $b = 1$, $a = 1$, $c = 2$. That is,

$$y^2 = T(T^3 + 2T^2 + T + 1) \text{ or } y^2 = T(T^3 + T^2 + T + 2). \tag{3.5}$$

• Let $q = 5$. Similarly, $K = k(y)$ where $y^2 = P_1P_2$. Assume $P_1 = (T)$ and $P_2 = T^3 + aT^2 + bT + c$ where $a, b, c \in \mathbb{F}_5$. Also $P_1P_2(\alpha) = 2$ or 3 for $\alpha \in \mathbb{F}_5$. Checking all possibilities we have $K = k(y)$ satisfying one of the following equations:

$$y^2 = T(T^3 - T^2 - T - 1), \tag{3.6}$$

$$y^2 = T(T^3 + 2T^2 + T + 3), \tag{3.7}$$

$$y^2 = T(T^3 + 3T^2 + T + 2), \tag{3.8}$$

$$y^2 = T(T^3 + T^2 - T + 1). \tag{3.9}$$

• Let $q = 7$. $K = k(y)$ where $y^2 = P_1P_2$. Assume $P_1 = (T)$ and $P_2 = T^3 + aT^2 + bT + c$ where $a, b, c \in \mathbb{F}_7$. Also $P_1P_2(\alpha) = 3, 5$ or 6 for $\alpha \in \mathbb{F}_7$. Checking all possibilities for P_2, we have $K = k(y)$ where

$$y^2 = T(T^3 + 2) \text{ or } y^2 = T(T^3 + 5). \tag{3.10}$$

□

Remark 3.3 ([6, Lemma 4.1]). Let K/k be an imaginary extension of k with $g_K = 1$. Then $g_{K^+} = 0$.

Proposition 3.4. *Let K/k be a totally imaginary extension of genus* 1 *with class number* 3. *Then K satisfies one of the following cases*:
(i) $q = 4, 7$, $K = k(\sqrt[3]{P_1P_2})$, $\deg P_i = 1$,
(ii) $q = 7$, $K = k(\sqrt{-P_1}, \sqrt[3]{-P_2})$, $\deg P_i = 1$,
(iii) $q = 3, 5, 7$, $K = k(\sqrt{-P})$, $\deg P = 3$.

Proof. Proof follows from [6, Theorem 4.2]. □

Theorem 3.5. *Let K/k be a totally imaginary extension of genus* 1 *with class number* 3. *Then, up to isomorphism,* $(x \to x + a,\ a \in \mathbb{F}_q^*)$ *K is one of the following function fields*:
(1) $q = 4$, $K = k(y)$ *such that* $y^3 = T(T + w)$ *where* $\langle w \rangle = \mathbb{F}_4^*$,
(2) $q = 7$, $K = k(y)$ *such that* $y^3 = T(T + 3)$ *or* $y^3 = T(T + 4)$,
(3) $q = 7$, $K = k(y, z)$ *such that* $y^2 + T = 0$ *and* $z^3 + T + 4 = 0$,
(4) $q = 3$, $K = k(y)$ *such that* $y^2 + T^3 + 2T^2 + 1 = 0$,
(5) $q = 5$, $K = k(y)$ *such that* $y^2 + T^3 + 4T + 2 = 0$ *or* $y^2 + T^3 + 4T + 3 = 0$,

(6) $q = 7$, $K = k(y)$ *such that* $y^2 + T^3 + 3 = 0$.

Proof. Clearly, K satisfies one of the conditions of Proposition 3.4:

(i) $q = 4$ or 7, $K = k(\sqrt[3]{P_1P_2})$, $\deg P_i = 1$.

• Let $q = 4$. $|S_\infty(K)| = 1$ and P_1 and P_2 are totally ramified. Then all of the other finite places of k of degree one are inert in K/k. Assume $P_1 = (T)$ and $P_2 = (T + a)$ for $a \in \mathbb{F}_4^*$. Let $X_{K_{P_i}}, X_K$ be the character groups of K_{P_i} and K, respectively. $X_{K_{P_i}} \cong (\mathbb{A}/P_i)^*$ is a cyclic group of order 3. Let χ_i be the generator of $X_{K_{P_i}}$, then $X_K = \langle \chi_1\chi_2 \rangle$ where $\chi_i(w) = \exp(2\pi i/3)$ such that w is a generator of $\mathbb{F}_4^*$.

If $a = 1$, then $\chi_1\chi_2(T + w) = 1$ and $N_1 \geq 6$.

If $a = w$, then $\chi_1\chi_2(T+1) = \chi_2(w+1) = \exp(4\pi i/3)$ and $\chi_1\chi 2(T+w+1) = \exp(4\pi i/3)$, then the places of K associated to the polynomial $T+1$ and $T+w^2$ are inert. That is,

$$K = k(y) \text{ where } y^3 = T(T + w). \tag{3.11}$$

• Let $q = 7$. Assume $P_1 = (T)$ and $P_2 = (T + a)$ for $a \in \mathbb{F}_7^*$. Let $X_{K_{P_i}}, X_K$ be the character groups of K_{P_i} and K, respectively. $X_{K_{P_i}} \cong (\mathbb{A}/P_i)^*$ is a cyclic group of order 6. Let χ_i be the generator of $X_{K_{P_i}}$, then $X_K = \langle \chi_1^2\chi_2^2 \rangle$ where $\chi_i(3) = \exp(2\pi i/6)$.

If $a = 1$, then $\chi_1^2\chi_2^2(T + 3) = 1$ and $N_1 \geq 6$. If $a = 2$, then $\chi_1^2\chi_2^2(T + 1) = 1$ and $N_1 \geq 6$. If $a = 5$, then $\chi_1^2\chi_2^2(T + 6) = 1$ and $N_1 \geq 6$. If $a = 6$, then $\chi_1^2\chi_2^2(T + 2) = 1$ and $N_1 \geq 6$.

Let $a = 3$, then $\chi_1^2\chi_2^2(T + 1) = \exp(10\pi i/3)$, $\chi_1^2\chi_2^2(T + 2) = \exp(4\pi i/3)$, $\chi_1^2\chi_2^2(T+4) = \exp(2\pi i/3)$, $\chi_1^2\chi_2^2(T+5) = \exp(2\pi i/3)$, $\chi_1^2\chi_2^2(T+6) = \exp(2\pi i/3)$ and all of the places of degree one, except (T) and $(T + 3)$ are inert. That is,

$$K = k(y) \text{ where } y^3 = T(T + 3). \tag{3.12}$$

Similarly, result follows for $a = 4$ and

$$K = k(y) \text{ where } y^3 = T(T + 4). \tag{3.13}$$

(ii) $q = 7$, $K = k(\sqrt{-P_1}, \sqrt[3]{-P_2})$, $\deg P_i = 1$.

Let $P_1 = (T)$ and $P_2 = (T + a)$ for $a \in \mathbb{F}_7^*$. Using the notation of part (i), $X_K = \langle \chi_1^3, \chi_2^2 \rangle$. Since $N_1 = 3$, P_1 is inert in $k(\sqrt[3]{-P_2})/k$, P_2 splits in $k(\sqrt{-P_1})/k$ and all the other finite places of k of degree one do not split in K/k. Result follows only for $a = 4$. In this case, $\chi_2^2(T + 1) = \exp(2\pi i/3)$, $\chi_2^2(T + 2) = \exp(10\pi i/3)$, $\chi_1^3(T + 3) = -1$, $\chi_1^3(T + 5) = -1$, $\chi_2^2(T + 6) = \exp(4\pi i/3)$. Also $\chi_1^3(T + 4) = 1$ and $\chi_2^2(T) = \exp(\pi i/3)$. Hence,

$$K = k(y, z) \text{ where } y^2 + T = 0 \text{ and } z^3 + T + 4 = 0. \tag{3.14}$$

(iii) $q = 3, 5, 7$, $K = k(\sqrt{-P})$, $\deg P = 3$.

• Let $q = 3$. Since extension is totally imaginary, one of the finite places of k of degree one splits and the others are inert in K/k. Up to isomorphism ($T \to T + a$, $a \in F_3^*$) there exist four possibilities for P. These are $T^3 + 2T + 1$, $T^3 + 2T + 2$, $T^3 + T^2 + 2$ and $T^3 + 2T^2 + 1$. Among them, result follows for

$P = T^3 + 2T^2 + 1$. In this case, $(\mathbb{F}_3(T)/P)^* = \langle T \rangle$. Let $X_{K_P} = \langle \chi \rangle$, we have $X_K = \langle \chi^{13} \rangle$ and $\chi^{13}(T+2) = 1$ and $\chi^{13}(T) = \chi^{13}(T+1) = -1$. Hence $N_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 2T^2 + 1 = 0. \tag{3.15}$$

• Let $q = 5$. Since $|S_\infty(K)| = 1$, one of the finite places of k of degree one splits and others are inert in K/k. Let $X_{K_P} = \langle \chi \rangle$ where $|X_{K_P}| = 5^3 - 1$. $X_K = \langle \chi^a \rangle$ for some integer a and $|X_K| = [K : k]$ which is equal to 2. That is, order of χ^a is 2. So we may assume $X_K = \langle \chi^{62} \rangle$. Up to isomorphism, $(T \to T + a,\ a \in F_5^*)$ we have 8 possibilities for P. These are $T^3 + T + 1$, $T^3 + T + 4$, $T^3 + 3T + 2$, $T^3 + 3T + 3$, $T^3 + 2T + 1$, $T^3 + 2T + 4$, $T^3 + 4T + 2$ and $T^3 + 4T + 3$. Among them, we have solutions for $P = T^3 + 4T + 2$ and $P = T^3 + 4T + 3$.

Let $P = T^3 + 4T + 2$. $\chi^{62}(T) = \chi^{62}(T+1) = \chi^{62}(T+3) = \chi^{62}(T+4) = -1$ and $\chi^{62}(T+2) = 1$. That is, $N_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 4T + 2 = 0. \tag{3.16}$$

Let $P = T^3 + 4T + 3$. $\chi^{62}(T) = \chi^{62}(T+1) = \chi^{62}(T+2) = \chi^{62}(T+4) = -1$ and $\chi^{62}(T+3) = 1$. That is, $N_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 4T + 3 = 0. \tag{3.17}$$

• Let $q = 7$. Since $|S_\infty(K)| = 1$, one of the finite places of k of degree one splits and others are inert in K/k. Let $X_{K_P} = \langle \chi \rangle$ where $|X_{K_P}| = 7^3 - 1$. $X_K = \langle \chi^a \rangle$ for some $a \in \mathbb{Z}$ and $|X_K| = [K : k]$ which is equal to 2. That is, order of χ^a is 2. Then we may assume $X_K = \langle \chi^{171} \rangle$. Up to isomorphism $(T \to T + a,\ a \in F_7^*)$ we have 16 possibilities for P. Among them, the result follows for only $T^3 + 3$. That is, let $P = T^3 + 3$, then $(\mathbb{F}_7[T]/P)^* = \langle T + 1 \rangle$ and we have $\chi^{171}(T+1) = \chi^{171}(T+2) = \chi^{171}(T+3) = \chi^{171}(T+4) = \chi^{171}(T+5) = \chi^{171}(T+6) = -1$ and $\chi^{171}(T) = \chi(T^{171}) = \chi(1) = 1$, that is $N_1 = 3$ and $K = k(y)$ where

$$y^2 + T^3 + 3 = 0. \tag{3.18}$$ □

Proposition 3.6. *Let K/k be an imaginary (not totally imaginary) extension of genus 1 with class number 3. Then K satisfies one of the following cases:*

(i) *$q = 4$ or 7, $K = k(\sqrt[3]{-P_1}, \sqrt[3]{-P_2})$, $\deg P_i = 1$,*

(ii) *$q = 7$, $K = k(\sqrt{-P_1}\sqrt[6]{-P_2})$, $\deg P_i = 1$,*

(iii) *$q = 4$, $K = k(\sqrt[3]{-P}, u)$ where $k(u)$ is a quadratic subfield of $K^+_{P^2}$, $\deg P = 1$.*

Proof. We have $K \neq K^+ \neq k$ and $|S_\infty(K)| \geq 2$. By [6, Theorem 4.4 and Theorem 4.6], proof is clear. □

Theorem 3.7. *Let K/k be an imaginary (not totally imaginary) extension of genus one with class number three. Then, up to isomorphism, $(x \to x + a,\ a \in \mathbb{F}_q^*)$ K is one of the following function fields:*

(1) $q = 4$, $K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+w)})$ *where* $\langle w \rangle = \mathbb{F}_4^*$,
(2) $q = 7$, $K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+3)})$,
(3) $q = 7$, $K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+4)})$,
(4) $q = 7$, $K = k(\sqrt{-(T+2)}\sqrt[6]{-T})$,
(5) $q = 7$, $K = k(\sqrt{-(T+5)}\sqrt[6]{-T})$,
(6) $q = 4$, $K = k(\sqrt[3]{-T}, u)$ *where* $u^2 + u + w/T = 0$ *for* $\langle w \rangle = \mathbb{F}_4^*$.

Proof. K satisfies one of the conditions of Proposition 3.6:

(i) Let $q = 4$ or 7 and $K = k(\sqrt[3]{-P_1}, \sqrt[3]{-P_2})$ where $\deg P_i = 1$.

• Let $q = 4$. Assume $P_1 = (T)$ and $P_2 = (T+a)$ for some $a \in \mathbb{F}_4^*$. Since $|S_\infty(K)| = 3 = N_1$, inertia degree $f(P, K/k)$ of a finite place P of k of degree one is greater than 1. Let $X_{K_{P_i}} = \langle \chi_i \rangle$, then $o(\chi_i) = 3$ and $X_K = \langle \chi_1, \chi_2 \rangle$. Define χ_i such that $\chi_i(w) = \exp(2\pi i/3)$ for $i = 1, 2$ and $\langle w \rangle = \mathbb{F}_4^*$. For $a = 1$, $\chi_1(T+1) = 1$, then $f((T+1), K/k) = 1$ and $e((T+1), K/k) = 3$. Hence $h_K = N_1 \geq 6$. For $a = w$, $\chi_2(T+1) = \exp(4\pi i/3) \neq 1$, $\chi_1(T+w^2) = \exp(4\pi i/3) \neq 1$, $\chi_1(T+w) = \exp(2\pi i/3) \neq 1$, $\chi_2(T) = \exp(2\pi i/3) \neq 1$. Hence $N_1 = 3$ and

$$K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+w)}) \text{ where } \langle w \rangle = \mathbb{F}_4^*. \tag{3.19}$$

• Let $q = 7$. Assume $P_1 = (T)$ and $P_2 = (T+a)$ for some $a \in \mathbb{F}_7^*$. Let $X_{K_{P_i}}$ denote the character group of K_{P_i} for $i = 1, 2$. Let $X_{K_{P_i}} = \langle \chi_i \rangle$, then $o(\chi_i) = 6$ and $X_K = \langle \chi_1^2, \chi_2^2 \rangle$. Since $(\mathbb{F}_7(T)/P_i)^* \cong \mathbb{F}_7^*$, we define χ_i such that $\chi_i(3) = \exp(2\pi i/6)$ for $i = 1, 2$. As $|S_\infty(K)| = 3 = N_1$, none of the places of K of degree one splits. For $a = 3$ and 4, result follows.

Let $a = 3$. $\chi_2^2(T+1) = \exp(10\pi i/3) \neq 1$, $\chi_1^2(T+2) = \exp(8\pi i/6) \neq 1$, $\chi_1^2(T+3) = \exp(4\pi i/6) \neq 1$, $\chi_1^2(T+4) = \exp(8\pi i/3) \neq 1$, $\chi_2^2(T+5) = \exp(4\pi i/3) \neq 1$, $\chi_2^2(T+6) = \exp(4\pi i/6) \neq 1$. Hence $N_1 = 3$ and

$$K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+3)}). \tag{3.20}$$

Let $a = 4$. $\chi_2^2(T+1) = \exp(8\pi i/3) \neq 1$, $\chi_1^2(T+2) = \exp(8\pi i/6) \neq 1$, $\chi_1^2(T+3) = \exp(4\pi i/6) \neq 1$, $\chi_1^2(T+4) = \exp(8\pi i/3) \neq 1$, $\chi_1^2(T+5) = \exp(10\pi i/3) \neq 1$, $\chi_2^2(T+6) = \exp(4\pi i/3) \neq 1$. That is, $N_1 = 3$ and

$$K = k(\sqrt[3]{-T}, \sqrt[3]{-(T+4)}). \tag{3.21}$$

(ii) $q = 7$, $K = k(\sqrt{-P_1}\sqrt[6]{-P_2})$, $\deg P_i = 1$.

Assume $P_1 = (T+a)$ and $P_2 = (T)$ for some $a \in \mathbb{F}_7^*$. By the proof of Theorem 4.4 of [6], $|S_\infty(K)| = 2$. Since P_2 is totally ramified, $f((T+b), K/k) \neq 1$ for all $b \in \mathbb{F}_7^*$. Let $X_{K_{P_i}}$ denote the character group of K_{P_i} for $i = 1, 2$. Let $X_{K_{P_i}} = \langle \chi_i \rangle$, then $o(\chi_i) = 6$ and $X_K = \langle \chi_1^3 \chi_2 \rangle$. Since $(\mathbb{F}_7(T)/P_i)^* \cong \mathbb{F}_7^*$, we define χ_i such that $\chi_i(3) = \exp(2\pi i/6)$ for $i = 1, 2$. We have solutions for $a = 2$ and $a = 5$.

Let $a = 2$. $\chi_1^3\chi_2(T+1) = -1$, $\chi_1^3\chi_2(T+3) = \exp(\pi i/3)$, $\chi_1^3\chi_2(T+4) = \exp(4\pi i/3)$, $\chi_1^3\chi_2(T+5) = (-1)\exp(5\pi i/3)$, $\chi_1^3\chi_2(T+6) = (-1)$, and $\chi_2(T+$

$2) = \exp(2\pi i/3)$. Hence $N_1 = 3$ and

$$K = k(\sqrt{-(T+2)}\sqrt[6]{-T}). \tag{3.22}$$

Let $a = 5$. $\chi_1^3\chi_2(T+1) = -1$, $\chi_1^3\chi_2(T+2) = \exp(2\pi i/3)$, $\chi_1^3\chi_2(T+3) = (-1)\exp(\pi i/3)$, $\chi_1^3\chi_2(T+4) = (-1)\exp(4\pi i/3)$, $\chi_1^3\chi_2(T+6) = (-1)$, and $\chi_2(T+5) = \exp(5\pi i/3)$. Hence none of them splits and

$$K = k(\sqrt{-(T+5)}\sqrt[6]{-T}). \tag{3.23}$$

(iii) $q = 4$, $K = k(\sqrt[3]{-P}, u)$ where $k(u)$ is a quadratic subfield of $K^+_{P^2}$.

K is contained in $K_P K^+_{P^2}$. Up to isomorphism, assume $P = (T)$. Then P is totally ramified and $N_1 \geq 3$. The character group $X_{K_{P^2}} \simeq (\mathbb{A}/T^2)^* = \langle T+w\rangle \times \langle T+1\rangle$ is isomorphic to $\mathbb{Z}_6 \times \mathbb{Z}_2$. Let X_K denote the character group of K. It is a subgroup of $\langle\chi_1\rangle \times \langle\chi_2\rangle \times \langle\chi_3\rangle$ where $\langle\chi_1\rangle = X_P$ is of order 3 and $\langle\chi_2\rangle \times \langle\chi_3\rangle$ is the character group associated to K_{P^2} where $\langle\chi_2\rangle \simeq \langle T+w\rangle$ and $\langle\chi_3\rangle \simeq \langle T+1\rangle$. Then $X_K = \langle\chi_1(\chi_2^3)^a(\chi_3)^b\rangle$ is of order six where $0 \leq a, b \leq 1$. Since $(T+1)$ is inert in the extension K/k, $\chi_1(\chi_2^3)^a(\chi_3)^b(T+1) = -1$, then $b = 1$. $(T+w)$ and $(T+w^2)$ are also inert, but $\chi_1(\chi_2^3)^a(\chi_3)(T+w) = \exp(2\pi i/3)\exp(3a(2\pi i/6)1 \neq 1$ and $\chi_1(\chi_2^3)^a(\chi_3)(T+w^2) = \exp(22\pi i/3)\exp(15a(2\pi i/6)(-1) \neq 1$ for $0 \leq a \leq 1$. We have $g_{K^+} = 0$ by [6, Lemma 4.1], then $K^+ = k(u)$ is a function field with class number one where P_∞ splits and (T) is ramified. Then

$$K = k(\sqrt[3]{-T}, u) \text{ where } u^2 + u + w/T = 0. \tag{3.24}$$

□

4. Genus two

In this section, we determine the subfields K of the cyclotomic function fields with class number three when $g_K = 2$. By Theorem 2.4, we have $q = 2$ or 3.

4.1. $q = 2$

Theorem 4.1. *Let $q = 2$ and K be an extension of k of genus 2 with class number 3. Then, up to isomorphism, $K = k(y)$ where $y^2 + y = 1/(T^3 + T + 1)$ and $L(t) = 4t^4 - 2t^3 + t^2 - t + 1$.*

Proof. For $q = 2$, K is a real extension of k and $N_1 = 2 = N_2$ by Theorem 2.4. That implies $[K : k] = 2 = |S_\infty(K)|$. Thus extension is quadratic and any finite place of k of degree one is inert in K/k. Since $N_2 = 2$ and (T) and $(T+1)$ are inert, $(T^2 + T + 1)$ is also inert. Then P does not divide $N := cond(K)$, when $\deg P \leq 2$. Assume $N = \prod_{i=1}^r P_i^{m_i}$, then $\deg P_i \geq 3$. By Hurwitz's Genus Formula for K/k, $\deg(D(K/k)) = 6$. Since P_i are wildly ramified, $6 \geq 2(\sum_{i=1}^r \deg P_i)$. Equality holds if and only if $m_i = 2$ for all i. Hence $N = P^2$ where $\deg P = 3$.

Up to isomorphism $T \to T+1$, we assume $P = T^3 + T + 1$. Using [9, Proposition 2.8 and Proposition 2.9], $K = k(y)$, where $y^2 + (T^3 + T + 1)y = (T^3 + T + 1)g(T)$ where $0 \neq g(T) \in \mathbb{F}_2[T]$ is of degree less than 4 and $g(0) = g(1) = 1$. Also let α be a root of $T^2 + T + 1$. Since $(T^2 + T + 1)$ is inert, by [9, Lemma 2.8], $g(\alpha)/\alpha^2 + g(\alpha^2)/\alpha = 1$. That implies $g(T) = 1$. Hence $K = k(y)$ where $y^2 + y = 1/(T^3 + T + 1)$. □

4.2. $q = 3$

Theorem 4.2. *Let $q = 3$ and K be a real extension of k of genus 2 with class number three. Then, up to isomorphism, $K = k(y)$ where $y^2 = T^6 + T^4 + T^3 + T^2 + 2T + 2$ or $y^2 = T^6 + T^4 + 2T^3 + T^2 + T + 2$ and for each case $L(t) = 9t^4 - 6t^3 + t^2 - 2t + 1$.*

Proof. Let K/k be a real extension, then $[K : K^+] = [K : k]$ divides $q - 1 = 2$. Hence, $[K : k]$ is quadratic and $|S_\infty(K)| = 2$. Thus we are in the case $N_1 = 2, N_2 = 3$ by Theorem 2.4. Since all finite places of k of degree one are inert, all finite places of k of degree two are also inert. That is, any place P of k of degree less than or equal to two does not divide the conductor N of K. Since the extension degree is prime to q, we may assume $N = \prod_{i=1}^{r} P_i$ where $P_i \in \mathbb{P}_k$. By Hurwitz's Genus formula for K/k, $\sum_{i=1}^{r} \deg P_i = 6$ where $\deg P_i \geq 3$. Thus $N = P_1 P_2$ where $\deg P_i = 3$ or $N = P$ where $\deg P = 6$. Then by Lemma 2.5, $K = k(\sqrt{P_1 P_2}) \subset k(\sqrt{-P_1}, \sqrt{-P_2})$ or $K = k(\sqrt{P})$. By [9, Theorem 2.5 and Lemma 2.6], $y^2 = N$ such that N is not a square modulo Q for a place Q of k of degree one or two. Considering each case, the result follows for only $N = T^6 + T^4 + T^3 + T^2 + 2T + 2$ and $N = T^6 + T^4 + 2T^3 + T^2 + T + 2$. □

Theorem 4.3. *Let $q = 3$ and K be an imaginary extension of k of genus two with class number three. Then, up to isomorphism, $K = k(y)$ where $y^2 + T^5 + T^3 + T + 1 = 0$ and $L(t) = 9t^4 - 9t^3 + 5t^2 - 3t + 1$.*

Proof. Let K/k be an imaginary extension, then $2 \leq [K : K^+]$ divides $q-1 = 2$. That is, $[K : K^+] = 2$. Then by Theorem 2.4, we have

(i) $N_1 = 1$, $N_2 = 5$ or

(ii) $N_1 = 2$, $N_2 = 3$

(i) Let $N_1 = 1$ and $N_2 = 5$. Then $|S_\infty(K)| = 1$ and $K^+ = k$. That is K/k is quadratic, P_∞ is ramified and all finite places of k of degree one are inert in the extension. Then we have two possibilities: either two of the places of k of degree two are ramified and the third one is inert or one of them splits and the others are inert. Extension degree is prime to q and we assume $N = \prod_{i=1}^{r} P_i$ where $P_i \in \mathbb{P}_k$. By Hurwitz's Genus formula for K/k, $\sum_{i=1}^{r} \deg P_i = 5$ where $\deg P_i \geq 2$. Thus $N = P$ where $\deg P = 5$. By Lemma 2.5, $K = k(\sqrt{-P})$ and by [9, Theorem 2.5 and Lemma 2.6], $y^2 = P$ such that P is a square modulo Q' for only one of the places Q' of k of degree two and it is not a square modulo Q'' where Q'' is a place of k of degree two different from Q'. There exist three distinct place in $\mathbb{P}_{\mathbb{F}_3(x)}$ of degree two. Up to isomorphism ($T \to T + a$, $a \in \mathbb{F}_3^*$), we may assume $Q' = T^2 + T + 2$. Then we have $P = T^5 + T^3 + T + 1$. That is,

$$K = k(y) \text{ where } y^2 + T^5 + T^3 + T + 1 = 0. \tag{4.1}$$

(ii) Let $N_1 = 2$ and $N_2 = 3$. Then $|S_\infty(K)| = 1$ or 2.

• Assume $|S_\infty(K)| = 1$. Then K/k is quadratic, P_∞ and one of the finite places P of k of degree one are ramified and the other places of k of degree one are inert in the extension. Up to isomorphism, let $P = (T)$. Then one of the places Q of k of degree two is ramified and the others are inert. Assume $N = P \cdot Q \prod_{i=1}^{r} P_i$ where $P_i \in \mathbb{P}_k$ of degree greater than two. By Hurwitz's Genus formula for K/k, $\deg P + \deg Q + \sum_{i=1}^{r} \deg P_i = 5$ where $\deg P_i \geq 3$. That is, $\sum_{i=1}^{r} \deg P_i = 2$ and $\deg P_i \geq 3$, which is not possible.

• Assume $|S_\infty(K)| = 2$, then K/k is quartic. It is a well-known fact that $g_{K^+} \leq g_K$. For $g_{K^+} = 2$, by Hurwitz's Genus formula, degree of the different of K/K^+ is -2. Since this is not reasonable, $g_{K^+} = 0$ or 1.

Let $g_{K^+} = 0$, that is $h_{K^+} = 1$. Then by [7, Proposition 4.1], $K^+ \subseteq K_P^+$ with $\deg P = 2$ or $K^+ \subseteq K_{P_1P_2}^+$ with $\deg P_i = 1$.

Assume the first case holds.

If P is ramified in K/K^+, by Hurwitz's Genus formula for K/K^+,

$$2 = -4 + 2 \cdot \deg P_\infty + \deg P + 2\sum_{i=1}^{r} \deg Q_i,$$

where $P \neq Q_i$ are places of k which are also ramified in K/K^+. Then $N = PQ$ where $\deg Q = 1$. If Q splits in K^+/k, then $N_1 \geq 4$, which is a contradiction. Let Q be inert in K^+/k. Since it is ramified in K/K^+ then there exists $\gamma \in \mathbb{P}_K$ lying over Q such that $\deg \gamma = 2$. Since $N_2 = 3$ and γ and the place lying over P are of degree two, there exists a place Q' of k, different from P and Q, with $\deg Q' \leq 2$, which is ramified in K/k. That means Q' divides N, which is a contradiction.

If P is not ramified in K/K^+, by Hurwitz's Genus formula for K/K^+,

$$2 = -4 + 2 \cdot \deg P_\infty + 2\sum_{i=1}^{r} \deg Q_i,$$

where $P \neq Q_i$ are places of k which are ramified in K/K^+. Then $N = PQ$ where $\deg Q = 2$ or $N = PQ_1Q_2$ where $\deg Q_i = 1$. Since $N_1 = 2, N_2 = 3$, using an argument similar to above, there exists another ramified place Q' of k with $\deg Q' \leq 2$. Then Q' divides N, which is not possible.

Assume $K^+ \subseteq K_{P_1P_2}^+$ with $\deg P_i = 1$. Since $N_1 = |S_\infty(K)|$, P_i are inert in K/K^+. By Hurwitz's Genus formula for K/K^+,

$$2 = -4 + 2 \cdot \deg P_\infty + 2\sum_{i=1}^{r} \deg Q_i,$$

where $P_i \neq Q_j$ are places of k which are ramified in K/K^+. We have $N = P_1P_2Q$ with $\deg Q = 2$ or $N = P_1P_2Q_1Q_2$ with $\deg Q_i = 1$. $N = P_1P_2Q_1Q_2$ with $\deg Q_i = 1$ implies $N_1 \geq 4$ or $N_2 \geq 4$, which is a contradiction. $N = P_1P_2Q$ with $\deg Q = 2$ implies $N_2 \geq 4$ or there exists another ramified place of degree less than or equal to two. Since both of them are not reasonable, we skip this case. Hence $g_{K^+} \neq 0$.

Let $g_{K^+} = 1$. It is known that $h_{K^+} \mid h_K = 3$, then $h_{K^+} = 1$ or 3. Since $[K^+ : k] = 2$ and $q = 3$, by [6, Theorem 3.4], $K^+ = k(\sqrt{P_1P_2})$ with $\deg P_1 = 1$ and $\deg P_2 = 3$. Then $h_{K^+} = 3$ and $h^-_K = 1$. By [7, Theorem 3.9], $K = k(\sqrt{-P_1}, \sqrt{-P_2})$ and $P_2 = T^3 + 2T + 1$. In this case, $h_K = 5$ and we arrive a contradiction. □

5. Imaginary extensions with relative class number three

Let K/k be an imaginary extension with relative class number $h^-_K = 3$. If $h_K = 3$, then $h_{K^+} = 1$ and K is one of the function fields given in Theorems 3.5, 3.7 and 4.3. In the following theorem, we list all imaginary abelian extensions with $h_K = 3h_{K^+} > 3$.

Theorem 5.1. *Let K be an imaginary function field with relative class number three such that $h_K > 3$. Then, up to isomorphism, K satisfies one of the following conditions*:

(1) $q = 3$, $K = k(\sqrt{-P}, \sqrt{-(T^3 + 2T^2 + 1)})$ *and* $K^+ = k(\sqrt{P(T^3 + 2T^2 + 1)})$ *where* $\deg P = 1$.
(2) $q = 3$, $K = k(\sqrt{-P}, \sqrt{-(T^5 + T^3 + T + 1)})$ *and* $K^+ = k(\sqrt{P(T^5 + T^3 + T + 1)})$ *where* $\deg P = 1$.
(3) $q = 3$, $K = k(\sqrt{-P_1}, \sqrt{-P_2})$ *with* $h_K = 63$ *and* $K^+ = k(\sqrt{P_1P_2})$ *with* $h_{K^+} = 21$ *where* $P_1 = T^3 + 2T + 1$ *and* $P_2 = T^3 + 2T^2 + 1$.
(4) $q = 3$, $K = k(\sqrt{-P_1}, \sqrt{-P_2})$ *with* $h_K = 399$ *and* $K^+ = k(\sqrt{P_1P_2})$ *with* $h_{K^+} = 133$ *where* $P_1 = T^3 + 2T + 1$ *and* $P_2 = T^5 + T^3 + T + 1$.
(5) $q = 5$, $K = k(\sqrt{-P}, \sqrt{-(T^3 + 4T + 2)})$ *and* $K^+ = k(\sqrt{P(T^3 + 4T + 2)})$ *where* $\deg P = 1$.
(6) $q = 5$, $K = k(\sqrt{-P}, \sqrt{-(T^3 + 4T + 3)})$ *and* $K^+ = k(\sqrt{P(T^3 + 4T + 3)})$ *where* $\deg P = 1$.
(7) $q = 7$, $K = k(\sqrt{-P}, \sqrt{-(T^3 + 3)})$ *and* $K^+ = k(\sqrt{P(T^3 + 3)})$ *where* $\deg P = 1$.

Proof. Let $h_K = 3h_{K^+} > 3$, then $K^+ \neq k$ and K/k is not totally imaginary. By [12, Equation (2.a)]

$$h^-_K = (\frac{\delta^s_K}{Q})h^-(O_K),$$

where δ_K denotes the order of the Galois group $\mid Gal(K/K^+) \mid$ of K over K^+. Let O_K and O_{K^+} denote the integral closure of $\mathbb{A}$ in K and K^+ and let O^*_K and $O^*_{K^+}$ be the unit groups of O_K and O_{K^+} respectively. Then $Q = [O^*_K : O^*_{K^+}]$ is the unit index and we know that Q divides δ_K. Also $s = [K^+ : k] - 1$ and $h^-(O_K)$ is the relative ideal class number of K. Since $(\frac{\delta^s_K}{Q})$ and $h^-(O_K)$ are positive integers and $h^-_K = 3$, we have two possibilities:

In the first one, $(\frac{\delta^s_K}{Q}) = 1$ and $h^-(O_K) = 3$, so $\delta^s_K = Q$. Since Q is a divisor of δ_K, we have $s = 0$ or $s = 1$. Assume $s = 0$. Then $Q = 1$. So $K^+ = k$ and

$h_{K^+} = 1$, which is a contradiction. So, $s = 1$ and $\delta_K = Q$. So K^+/k is a real quadratic extension.

In the second one, $(\frac{\delta_K^s}{Q}) = 3$ and $h^-(O_K) = 1$. Then $s = 1$ or $s = 2$. If $s = 1$, $\delta_K = 3Q$ and K^+/k is real quadratic. If $s = 2$, $\delta_K = 3 = Q$ and K^+/k is a real cubic extension.

Hence when $h_K^- = 3h_{K^+} > 1$, we have the following cases:

(1) $[K : K^+] = [O_K^* : O_{K^+}^*]$ and K^+/k is a real quadratic extension.
(2) $[K : K^+] = 3[O_K^* : O_{K^+}^*]$ and K^+/k is a real quadratic extension.
(3) $[K : K^+] = 3 = [O_K^* : O_{K^+}^*]$ and K^+/k is a real cubic extension.

Since $h_K, h_{K^+} > 1$, we have $g_K, g_{K^+} \geq 1$ and by [2, Propotion 2.4]

$$h_K^- \geq (\sqrt{q} - 1)^{2(\delta_K - 1)(g_{K^+} - 1) + \deg(D(K/K^+))}. \tag{5.1}$$

Since infinite places are tamely ramified in K/K^+, we have

$$\begin{aligned} 2(\delta_K - 1)(g_{K^+} - 1) + \deg(D(K/K^+)) &\geq \deg(\text{Infinite part of } D(K/K^+)) \\ &= [K^+ : k](\delta_K - 1). \end{aligned} \tag{5.2}$$

(1) Let $[K : K^+] = [O_K^* : O_{K^+}^*]$ and K^+/k be a real quadratic extension. Since extension is imaginary, $\delta_K \geq 2$ and $q \neq 2$. By inequalities (5.1) and (5.2), we have $3 \geq (\sqrt{q} - 1)^2$ and hence $3 \leq q \leq 7$.

(i) Let $q = 3$. Then $Q = \delta_K = 2$. Let $N := \prod_{i=1}^{s} P_i^{e_i}$ be the conductor of K and define $N := \prod_{i=1}^{s} P_i$. Since $Q = 2$, we have $s \geq 2$ by [13, Section 4]. Since $[K : k] = 4$, $Gal(K/k)$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Assume the first one and assume $X_K = \langle \chi \rangle$ is Dirichlet characters group of K. By the analytic class number formula,

$$h_K^- = (\Sigma_{A \in M_N} \chi(A))(\Sigma_{A \in M_N} \chi^3(A)). \tag{5.3}$$

Since $|M_N|$ is even and $o(\chi) = 4$, h_K^- is divisible by 2 which is a contradiction. Hence, $Gal(K/k) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Let U, V and K^+ be three quadratic subfields of K associated to the three subgroups of $Gal(K/k)$. By [16, Main Theorem], $h_K = h_U h_V h_{K^+}$. Since $h_K = 3h_{K^+}$, we may assume $h_U = 1$ and $h_V = 3$. By [7, Theorem 3.6 and Theorem 3.8], U is either $k(\sqrt{-P})$ where $\deg P = 1$ or $k(\sqrt{-(T^3 + 2T + 1)})$. By Theorem 3.5 and Theorem 4.3, V is either $k(\sqrt{-(T^3 + 2T^2 + 1)})$ or $k(\sqrt{T^5 + T^3 + T + 2})$. Then K satisfies one of the following conditions:

(a) $K = k(\sqrt{-P}, \sqrt{-(T^3 + 2T^2 + 1)})$ and $K^+ = k(\sqrt{P(T^3 + 2T^2 + 1)})$ where $\deg P = 1$. In this case, $h_K^- = 3$.
(b) $K = k(\sqrt{-P}, \sqrt{-(T^5 + T^3 + T + 1)})$ and
$K^+ = k(\sqrt{P(T^5 + T^3 + T + 1)})$ where $\deg P = 1$.
In this case, $h_K^- = 3$.
(c) $K = k(\sqrt{-(T^3 + 2T + 1)}, \sqrt{-(T^3 + 2T^2 + 1)})$ and $h_K = 63$,
$K^+ = k(\sqrt{(T^3 + 2T + 1)(T^3 + 2T^2 + 1)})$ and $h_{K^+} = 21$.

(d) $K = k(\sqrt{-(T^3+2T+1)}, \sqrt{-(T^5+T^3+T+1)})$ and $h_K = 399$, $K^+ = k(\sqrt{(T^3+2T+1)(T^5+T^3+T+1)})$ and $h_{K^+} = 133$.

(ii) Let $q = 4$. Then $Q = \delta_K = 3$. Let $N := \prod_{i=1}^{s} P_i^{e_i}$ be the conductor of K and define $N' := \prod_{i=1}^{s} P_i$. Since $Q = 3$, we have $s \geq 2$. Since $[K : k] = 6$, N is not a square-free polynomial, then $[K : K \cap K_{N'}] > 1$. Since $[K : K \cap K_{N'}] = [KK_{N'} : K_{N'}]$ is a divisor of $[K_N : K_{N'}] = 2^l$ for some integer l, we have $[K : K \cap K_{N'}] = 2$ and $[K \cap K_{N'} : k] = 3$. By [10, Theorem 3.1], any finite places of K^+ are unramified in K, so any finite places of k is unramified in $K \cap K_{N'}$. Hence, $K \cap K_{N'} = k$ and we arrive a contradiction. So there exists no solution for this case.

(iii) Let $q = 5$. Then $Q = \delta_K = 2$ or 4.

• Let $\delta_K = 4$. Then by inequalities (5.1) and (5.2), we have $3 \geq (\sqrt{5}-1)^6 \approx 3.56$ which is a contradiction.

• Let $Q = \delta_K = 2$ and let $N := \prod_{i=1}^{s} P_i^{e_i}$ be the conductor of K and define $N := \prod_{i=1}^{s} P_i$. Since $Q = 2$, we have $s \geq 2$. Since $[K : k] = 4$, $Gal(K/k)$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Assume the first one and assume $X_K = \langle \chi \rangle$ is Dirichlet characters group of K. By the analytic class number formula,

$$h_K^- = (\Sigma_{A \in M_N} \chi(A))(\Sigma_{A \in M_N} \chi^3(A)). \tag{5.4}$$

Since $|M_N|$ is even and $o(\chi) = 4$, h_K^- is divisible by 2 which is a contradiction. Hence, $Gal(K/k) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Let U, V and K^+ be three quadratic subfields of K associated to the three subgroups of $Gal(K/k)$. By [16, Main Theorem], $h_K = h_U h_V h_{K^+}$. Since $h_K = 3h_{K^+}$, we may assume $h_U = 1$ and $h_V = 3$. By [7, Theorem 3.6], U is $k(\sqrt{-P})$ where $\deg P = 1$. By Theorem 3.5, V is either $k(\sqrt{-(T^3+4T+2)})$ or $k(\sqrt{-(T^3+4T+3)})$. Then K satisfies one of the following conditions:

(a) $K = k(\sqrt{-P}, \sqrt{-(T^3+4T+2)})$ and $K^+ = k(\sqrt{P(T^3+4T+2)})$ where $\deg P = 1$. In this case, $h_K^- = 3$.

(b) $K = k(\sqrt{-P}, \sqrt{-(T^3+4T+3)})$ and $K^+ = k(\sqrt{P(T^3+4T+3)})$ where $\deg P = 1$. In this case, $h_K^- = 3$.

(iv) Let $q = 7$. Then $Q = \delta_K = 2, 3$ or 6.

• Let $\delta_K = 3$ or 6. Then by inequalities (5.1) and (5.2), we have $3 \geq (\sqrt{7}-1)^4 \approx 7.33$ which is a contradiction.

• Let $Q = \delta_K = 2$ and let $N := \prod_{i=1}^{s} P_i^{e_i}$ be the conductor of K and define $N := \prod_{i=1}^{s} P_i$. Since $Q = 2$, we have $s \geq 2$. Since $[K : k] = 4$, $Gal(K/k)$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Assume the first one and assume $X_K = \langle \chi \rangle$ is Dirichlet characters group of K. By the analytic class number formula,

$$h_K^- = (\Sigma_{A \in M_N} \chi(A))(\Sigma_{A \in M_N} \chi^3(A)). \tag{5.5}$$

Since $|M_N|$ is even and $o(\chi) = 4$, h_K^- is divisible by 2 which is a contradiction. Hence, $Gal(K/k) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Let U, V and K^+ be three quadratic subfields of K associated to the three subgroups of $Gal(K/k)$. By [16, Main Theorem], $h_K = h_U h_V h_{K^+}$. Since $h_K = 3h_{K^+}$, we may assume $h_U = 1$ and $h_V = 3$.

By [7, Theorem 3.6], U is $k(\sqrt{-P})$ where $\deg P = 1$. By Theorem 3.5, V is $k(\sqrt{-(T^3+3)})$. $K = k(\sqrt{-P}, \sqrt{-(T^3+3)})$ and $K^+ = k(\sqrt{P(T^3+3)})$ where $\deg P = 1$. In this case, $h_K^- = 3$.

(2) Let $[K : K^+] = 3[O_K^* : O_{K^+}^*]$ and K^+/k be a real quadratic extension. Since $\delta_K = 3Q \geq 3$, by inequalities (5.1) and (5.2), we have $3 \geq (\sqrt{q}-1)^4$ and hence $3 \leq q \leq 5$.

(i) Let $q = 3$. Then $\delta_K = 2$, but $\delta_K = 3Q$ is divisible by 3, so we arrive a contradiction.

(ii) Let $q = 4$. Then $\delta_K = 3$ and $Q = 1$. Let $N := \prod_{i=1}^s P_i^{e_i}$ be the conductor of K and define $N' := \prod_{i=1}^s P_i$. Since $[K : k] = 6$, N is not a square-free polynomial, then $[K : K \cap K_{N'}] > 1$. Since $[K : K \cap K_{N'}] = [KK_{N'} : K_{N'}]$ is a divisor of $[K_N : K_{N'}] = 2^l$ for some integer l, we have $[K : K \cap K_{N'}] = 2$ and $[K \cap K_{N'} : k] = 3$. If all finite places of K^+ are unramified in K, then all finite places of k are unramified in $K \cap K_{N'}$. Hence, $K \cap K_{N'} = k$ which is a contradiction. Thus, there exists a place P of K^+ which is ramified in K. Then $\deg(D(K/K^+)) \geq 4 + 2\deg(P) \geq 6$.

By Hurwitz Genus formula,

$$
\begin{aligned}
2g_K - 2 &= (2g_{K^+} - 2)[K : K^+] + \deg(D(K/K^+)) \\
&\geq 3(2g_{K^+} - 2) + 6 \\
&\geq 6g_{K^+}.
\end{aligned} \tag{5.6}
$$

Then $g_K = 3g_{K^+} + y$ where y is a positive integer. By [13, Inequality (3e)],

$$
\begin{aligned}
h_K^- &\geq \frac{4^{3g_{K^+}+y-1}}{3g_{K^+}+y+1}\frac{9}{5}\frac{1}{3^{2g_{K^+}}} \\
&\geq \frac{4^{g_{K^+}+y-1}}{3g_{K^+}+y+1}\frac{9}{5}\frac{4^{2g_{K^+}}}{3^{2g_{K^+}}} \\
&\geq 2\frac{9}{5}(\frac{4}{3})^4 \\
&\geq 10,
\end{aligned} \tag{5.7}
$$

when $g_{K^+} \geq 2$. Since $h_K = 3h_{K^+} > 3$, $g_{K^+} \geq 1$. Hence, $g_{K^+} = 1$.

We have $h_{K^+} = N_1(K^+) \geq |S_\infty(K^+)| = 2$. By [13, Inequality (3b)],

$$
h_{K^+} \leq 3^2. \tag{5.8}
$$

Thus $2 \leq h_{K^+} \leq 9$.

Then by [6, Theorem 3.3], K^+ satisfies one of the following conditions:

(a1) K^+ is a quadratic extension of k with $N = P^4$ and $\deg(P) = 1$.

(b1) K^+ is a quadratic extension of k with $N = P^2$ and $\deg(P) = 2$.

(c1) K^+ is a quadratic extension of k with $N = P_1^2P_2^2$ and $\deg(P_i) = 1$, $i = 1, 2$.

Assume $X_K = \langle\chi_1\rangle \times \langle\chi_2\rangle$ is Dirichlet characters group of K where $o(\chi_1) = 3$ and $o(\chi_2) = 2$. Let E be the subfield of K associated to the subgroup $\langle\chi_1\rangle$.

By analytic class number formula,

$$h_E = h_E^- = (\Sigma_{A\in M_N}\chi_1(A))(\Sigma_{A\in M_N}\chi_1^2(A)) \tag{5.9}$$

and

$$h_K^- = (\Sigma_{A\in M_N}\chi_1(A))(\Sigma_{A\in M_N}\chi_1^2(A))(\Sigma_{A\in M_N}\chi_1(A)\chi_2(a)) \\ (\Sigma_{A\in M_N}\chi_1^2(A)\chi_2(a)). \tag{5.10}$$

Let $S := \{A \in M_N : \chi_2(A) = -1\}$ be a subset of M_N. Then

$$\begin{aligned} h_K^- = &((\Sigma_{A\in M_N\setminus S}\chi_1(A)) + (\Sigma_{A\in S}\chi_1(A))) \\ &((\Sigma_{A\in M_N\setminus S}\chi_1^2(A)) + (\Sigma_{A\in S}\chi_1^2(A))) \\ &((\Sigma_{A\in M_N\setminus S}\chi_1(A)) - (\Sigma_{A\in S}\chi_1(A))) \\ &((\Sigma_{A\in M_N\setminus S}\chi_1^2(A)) - (\Sigma_{A\in S}\chi_1^2(A))). \end{aligned} \tag{5.11}$$

Since $\chi_1^2(A) = \overline{\chi_1(A)}$ is the complex conjugate of $\chi_1(A)$ for all $A \in M_N$ and order of $\chi_1 = 3$, $((\Sigma_{A\in M_N\setminus S}\chi_1(A)) = (x+\sqrt{3}yi)/2$ and $\Sigma_{A\in S}\chi_1(A)(z+\sqrt{3}ti)/2$ for $x, y, z, t \in \mathbb{Z}$. Thus, $h_E = ((x+z)^2 + 3(y+t)^2)/4$ is a positive integer, then $\alpha = ((x-z)^2 + 3(y-t)^2)/4$ is also a positive integer and $h_K^- = h_E\alpha$ implies $h_E = 1$ or 3. By Theorem 3.5 and by [7, Theorem 3.6 and Theorem 3.8].

(a2) $E = k(y)$ such that $y^3 = T(T + w)$ where w is a generator of $\mathbb{F}_4^*$ or

(b2) E is a subfield of K_P with $\deg(P) = 1$ or

(c2) $E = k(\sqrt[3]{T^2 + T + w})$ where w is a generator of $\mathbb{F}_4^*$.

$K = EK^+$, however, we have no solution for these possible values of E and K^+.

(iii) Let $q = 5$. Then $\delta_K = 2$ or 4, which contradicts that $\delta_K = 3Q$ is divisible by 3.

(3) Let $[K : K^+] = 3 = [O_K^* : O_{K^+}^*]$ and K^+/k is a real cubic extension. Since $\delta_K = 3$, by inequalities (5.1) and (5.2), we have $3 \geq (\sqrt{q} - 1)^6$ and hence $3 \leq q \leq 4$.

(i) Let $q = 3$. Then $\delta_K = 2$, which contradicts that $\delta_K = 3$.

(ii) Let $q = 4$. Then $Q = \delta_K = 3$. Then any finite places of K^+ are unramified in K. Let $N := \prod_{i=1}^s P_i^{e_i}$ be the conductor of K and $e(P_i, K/k) = e(P_i, K^+/k) = 3$. Thus the conductor of K^+ is N and $\deg(D(K/K^+)) = 6$. Since $Q \neq 1$, we have $s \geq 2$.

By Hurwitz Genus formula,

$$\begin{aligned} 2g_K - 2 &= (2g_{K^+} - 2)[K : K^+] + \deg(D(K/K^+)) \\ &= 3(2g_{K^+} - 2) + 6 \\ &= 6g_{K^+}. \end{aligned} \tag{5.12}$$

Then $g_K = 3g_{K^+} + 1$. By [13, Inequality (3e)],

$$h_K^- \geq \frac{4^{3g_{K^+}}}{3g_{K^+} + 2}\frac{9}{5}\frac{1}{3^{2g_{K^+}}} \tag{5.13}$$

$$\geq \frac{4^{g_{K^+}}}{3g_{K^+}+2}\frac{9}{5}\frac{4^{2g_{K^+}}}{3^{2g_{K^+}}}$$
$$\geq 2\frac{9}{5}(\frac{4}{3})^4$$
$$\geq 10,$$

when $g_{K^+} \geq 2$. Since $h_K = 3h_{K^+} > 1$, $g_{K^+} \geq 1$. Hence, $g_{K^+} = 1$.

We have $h_{K^+} = N_1(K^+) \geq\mid S_\infty(K^+) \mid= 3$. By [13, Inequality (3b)],

$$h_{K^+} \leq 3^2. \quad (5.14)$$

Thus $3 \leq h_{K^+} \leq 9$.

We know the conductor of K^+ is N and $s \geq 2$. Then by [6, Theorem 3.4], K^+ satisfies one of the following conditions:

(a) $K^+ = k(\sqrt[3]{-P_1^2P_2^2})$ with $\deg(P_1) = 1$ and $\deg(P_2) = 2$, where $N = P_1P_2$. In this case, $Gal(K_N/k) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Hence, $Gal(K/k) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.

(b) $K^+ = k(\sqrt[3]{P_1P_2P_3})$ with $\deg(P_i) = 1$ for $i = 1, 2, 3$ where $N = P_1P_2P_3$. In this case, $Gal(K_N/k) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence, $Gal(K/k) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.

Hence, $Gal(K/k) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$. Let U_1, U_2, U_3 and K^+ be four cubic subfields of K associated to the four subgroups of $Gal(K/k)$ of order 3. By [16, Main Theorem], $h_K = h_{U_1}h_{U_2}h_{U_3}h_{K^+}$. Since $h_{U_1}h_{U_2}h_{U_3} = 3$, we may assume $h_{U_2} = 1 = h_{U_3}$ and $h_{U_1} = 3$. By Theorem 3.5, $U_1 = k(y)$ such that $y^3 = T(T + w)$ where $\langle w\rangle = \mathbb{F}_4^*$ is the cubic function field with class number three. We have $K = U_1K^+$, but there exists no function field K with relative class number three satisfying these conditions. □

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Mehpare Bİlhan
Department of Mathematics
Middle East Technical University
İnönü Bulvari, 06531, Ankara, Turkey
Email address: mehpare@metu.edu.tr

Dİlek Buyruk
Department of Mathematics
Abant İzzet Baysal University
Gölköy, 14280, Bolu, Turkey
Email address: buyruk_d@ibu.edu.tr

Ferruh Özbudak
Department of Mathematics and Institute of Applied Mathematics
Middle East Technical University
İnönü Bulvari, 06531, Ankara, Turkey
Email address: ozbudak@metu.edu.tr