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SPANNING COLUMN RANKS OF NON-BINARY BOOLEAN MATRICES AND THEIR PRESERVERS

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Abstract. For any $m \times n$ nonbinary Boolean matrix A, its spanning column rank is the minimum number of the columns of A that spans its column space. We have a characterization of spanning column rank-1 nonbinary Boolean matrices. We investigate the linear operators that preserve the spanning column ranks of matrices over the nonbinary Boolean algebra. That is, for a linear operator T on $m \times n$ nonbinary Boolean matrices, it preserves all spanning column ranks if and only if there exist an invertible nonbinary Boolean matrix P of order m and a permutation matrix Q of order n such that $T(A) = PAQ$ for all $m \times n$ nonbinary Boolean matrix A. We also obtain other characterizations of the (spanning) column rank preserver.

1. Introduction

Let $\mathbb{B}_k$ be the *Boolean algebra* of subsets of a k element set S_k and $\sigma_1, \ldots, \sigma_k$ denote the singleton subsets of S_k. Union is denoted by $+$, and intersection by juxtaposition; 0 denotes the null set and 1 the set S_k. In particular, $\mathbb{B}_1$ is called a *binary Boolean algebra*. For $k \geq 2$, $\mathbb{B}_k$ is called a *non-binary Boolean algebra*.

Let $\mathcal{M}_{m,n}(\mathbb{B}_k)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B}_k$. Addition and multiplication of matrices over $\mathbb{B}_k$ are defined as if it were a field. If $m = n$, we use the notation $\mathcal{M}_n(\mathbb{B}_k)$ instead of $\mathcal{M}_{n,n}(\mathbb{B}_k)$.

There is a great deal of literature on the study of matrix theory over a finite Boolean algebra. But many results are stated only for the binary Boolean matrices. This is due in part, as Kim point out in [4] (Appendix 1), to an isomorphism between the matrices over the Boolean algebra of subsets of a k-element set and the k-tuples of the binary Boolean matrices. This isomorphism allows many questions concerning matrices over an arbitrary finite Boolean

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algebra to be referred to the binary Boolean case. However there are interesting results about the general Boolean matrices that have not been mentioned and they differ somewhat from the binary case. In many instances, the extension of results to the general case is not immediately obvious even though it is not difficult to derive via the isomorphism from the binary case. In [5], Kirkland and Pullman gave a way to derive results in the non-binary Boolean algebra case via the isomorphism from the binary Boolean algebra case by means of a canonical form derived from the isomorphism.

There is much literature on the study of linear operators that preserve the ranks of matrices over several semirings ([1], [3], [5–8] and therein). Boolean matrices also have been the subject of research by many authors.

In 1984, Beasley and Pullman [1] characterized the linear operators that preserve the ranks of matrices over binary Boolean algebra. In 1992, Kirkland and Pullman [5] extended the results of binary Boolean case in [1] to those of non-binary Boolean case. The results are following:

Theorem 1.1. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$. Then T preserves the ranks of rank-1 matrices and rank-2 matrices if and only if T is in the group of operators generated by the congruence (if $m = n$, also the rotation) operators if and only if T preserves the ranks of matrices.*

The definitions of congruence operators and rotation operators in Theorem 1.1 are given in Section 3.

In 1993, Song [6] obtained characterizations of the linear operators that preserve the column ranks of matrices over binary Boolean algebra as following:

Theorem 1.2. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ with $n \geq m \geq 4$. Then the following are equivalent*:

(i) *T preserves column ranks $1, 2$ and 3*;
(ii) *T is a congruence operator*;
(iii) *T preserves the column ranks of matrices.*

Also, Song and Lee [8] gave the same results as Theorem 1.2 over non-binary Boolean algebra $\mathbb{B}_k$ with $\min\{m, n\} \geq 3$. But their results had an error. On the matrices over non-binary Boolean algebra $\mathbb{B}_k$, some congruence operator does not preserve the column rank of a column rank-3 matrix (see Example 4.3). So, we shall present some revised results (see Theorem 4.6).

Recently, Song and Hwang [7] characterized the linear operators that preserve the spanning column ranks of nonnegative matrices.

In this paper, we characterize the linear operators that preserve the spanning column ranks of matrices over non-binary Boolean algebra. From now on, $\mathbb{B}_k$ means non-binary Boolean algebra with $k \geq 2$.

2. Preliminaries

For each $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, the *p-th constituent* [5] of A, A_p, is the matrix in $\mathcal{M}_{m,n}(\mathbb{B}_1)$ whose (i, j)th entry is 1 if and only if $a_{i,j} \supseteq \sigma_p$. Via the constituents, every matrix A can be written uniquely as $\sum_{p=1}^{k} \sigma_p A_p$, which is called the *canonical form* of A.

It follows from the uniqueness of the decomposition, and the fact that the singleton sets are mutually orthogonal idempotents, that for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, all $B, C \in \mathcal{M}_{n,q}(\mathbb{B}_k)$, and all $\alpha \in \mathbb{B}_k$,

(a)$(AB)_p = A_p B_p$;

(b)$(B + C)_p = B_p + C_p$;

(c)$(\alpha A)_p = \alpha_p A_p$

for all $p = 1, \ldots, k$.

The *Boolean rank*, $b(A)$, of a nonzero matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ is defined as the least integer r such that $A = BC$ for some $B \in \mathcal{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathcal{M}_{r,n}(\mathbb{B}_k)$. The Boolean rank of the zero matrix is zero. In the case of $\mathbb{B}_1$, we refer to $b(A)$ as the *binary Boolean rank*, and denote it by $b_1(A)$. For a binary Boolean matrix A, we have $b(A) = b_1(A)$ by definition.

Let $\mathcal{V}$ be a nonempty subset of $\mathbb{B}_k^n = \mathcal{M}_{n,1}(\mathbb{B}_k)$. If $\mathcal{V}$ is closed under addition and multiplication by scalars, then it is called a *vector space* over $\mathbb{B}_k$, and each member of $\mathcal{V}$ is called a *vector*. Lowercase, boldface letters will represent vectors, a vector $\boldsymbol{v}$ is a column vector ($\boldsymbol{v}^t$ is a row vector). The concepts of *subspaces*, *spanning sets*, *bases* and *dimension* of a vector space $\mathcal{V}$ are defined so as to coincide with familiar definition when $\mathbb{B}_k$ were a field.

A subset $\mathbb{X}$ of vectors in $\mathbb{B}_k^n$ is *linearly dependent* if there exists some vector $\boldsymbol{x} \in \mathbb{X}$ such that $\boldsymbol{x}$ is a linear combination of vectors in $\mathbb{X} \setminus \{\boldsymbol{x}\}$. Otherwise, $\mathbb{X}$ is *linearly independent*. Thus, any linearly independent set does not contain the zero vector.

For any matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, the *column space* of A is the vector space over $\mathbb{B}_k$ that spanned by all columns of A.

Since $\mathbb{B}_1$ is canonically identified with the subsemiring $\{0, 1\}$ of $\mathbb{B}_k$, any binary Boolean matrix can be considered as a matrix over both $\mathbb{B}_1$ and $\mathbb{B}_k$.

The *column rank*, $c(A)$, of $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ is the dimension of the column space of A. The *spanning column rank*, $sc(A)$, of $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ is the minimum cardinality of the columns of A that span its column space. As with Boolean rank, the zero matrix is assigned column rank and spanning column rank 0. For the binary Boolean algebra, we denote $c(A)$ and $sc(A)$ by $c_1(A)$ and $sc_1(A)$, respectively for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$.

It follows that

$$0 \leq b(A) \leq c(A) \leq sc(A) \leq n \tag{2.1}$$

for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Furthermore, we have

$$b(A) = b(X), \quad c(A) = c(X) \quad \text{and} \quad sc(A) = sc(X) \tag{2.2}$$

for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ and $X = \left[\begin{smallmatrix} A & O_1 \\ O_2 & O_3 \end{smallmatrix}\right]$, where the O_i are zero matrices of suitable sizes.

Let $\mu(\mathbb{B}_k, m, n)$ be the largest integer r such that for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $b(A) = c(A)$ if $b(A) \leq r$. Beasley and Pullman [2] determined the value of μ

over $\mathbb{B}_1$ as follow:

$$\mu(\mathbb{B}_1, m, n) = \begin{cases} 1 & \text{if } \min(m,n) = 1; \\ 3 & \text{if } m \geq 3 \textit{ and } n = 3; \\ 2 & \textit{otherwise.} \end{cases} \tag{2.3}$$

Also, Song and Lee [8] determined the value of μ over $\mathbb{B}_k$ as follow:

$$\mu(\mathbb{B}_k, m, n) = \begin{cases} 2 & \text{if } 2 = n \leq m; \\ 1 & \textit{otherwise.} \end{cases} \tag{2.4}$$

Lemma 2.1. *If the columns of $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ are linearly independent, then $sc(A) = n$.*

Proof. The proof follows from the definition of linearly independent. □

But, Example 2.3 (below) shows that even if the columns of $B \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ are linearly independent, $c(B) < n$ is possible.

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be matrices in $\mathcal{M}_{m,n}(\mathbb{B}_k)$. Then we say that the matrix A *absorbs* the matrix B if and only if $a_{i,j} \supseteq b_{i,j}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, and we write $A \supseteq B$ or $B \subseteq A$. If $A \subseteq B$ and $B \subseteq A$, then we have $A = B$. Similarly, for vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{B}_k^n$, $\boldsymbol{a} \supseteq \boldsymbol{b}$ can be defined because $\boldsymbol{a}$ and $\boldsymbol{b}$ are considered as matrices in $\mathcal{M}_{n,1}(\mathbb{B}_k)$.

For any matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, we write $A = [\boldsymbol{a}_1\, \boldsymbol{a}_2\, \cdots \boldsymbol{a}_n]$, where $\boldsymbol{a}_j$ is the jth column of A for $j = 1, \ldots, n$.

Theorem 2.2. *For any matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$, we have $c_1(A) = sc_1(A)$.*

Proof. Let $A = [\boldsymbol{a}_1\, \boldsymbol{a}_2\, \cdots \boldsymbol{a}_n]$, $c_1(A) = r$ and $sc_1(A) = s$. By (2.1), we have $r \leq s$. So, we suffice to show that $r \geq s$. Let $\mathcal{V}$ be the column space of A. Since $sc_1(A) = s$, there exists a linearly independent set $X = \{\boldsymbol{a}_{h1}, \ldots, \boldsymbol{a}_{hs}\}$, where the $\boldsymbol{a}_{hi}$ are columns of A such that $\mathcal{V} = \text{span}\{\boldsymbol{a}_{h1}, \ldots, \boldsymbol{a}_{hs}\}$. Since $c_1(A) = r$, there exists a basis $Y = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r\}$ of $\mathcal{V}$. It follows that

$$\text{span}\{\boldsymbol{a}_{h1}, \ldots, \boldsymbol{a}_{hs}\} = \text{span}\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_r\}. \tag{2.5}$$

Let $\boldsymbol{a}_{ht}$ be arbitrary in X. It follows from (2.5) that

$$\boldsymbol{a}_{ht} = \sum_{i=1}^{r} \alpha_i \boldsymbol{b}_i \quad \text{and} \quad \boldsymbol{b}_i = \sum_{j=1}^{s} \beta_j\, \boldsymbol{a}_{hj} \tag{2.6}$$

for some $\alpha_i, \beta_j \in \mathbb{B}_k$. Thus, we have

$$\boldsymbol{a}_{ht} = \sum_{i=1}^{r} \alpha_i \Big(\sum_{j=1}^{s} \beta_j\, \boldsymbol{a}_{hj} \Big) = \sum_{i=1}^{r} (\alpha_i\, \beta_t) \boldsymbol{a}_{ht} + \sum_{j=1, j \neq t}^{s} \Big(\sum_{i=1}^{r} \alpha_i\, \beta_j \Big) \boldsymbol{a}_{hj}.$$

Since X is linearly independent, we have $\sum_{i=1}^{r} (\alpha_i\, \beta_t) \neq 0$. That is, there exists an index $l \in \{1, \ldots, r\}$ such that $\alpha_l \beta_t \neq 0$ so that $\alpha_l = \beta_t = 1$. It follows from (2.6) that $\boldsymbol{a}_{ht} \supseteq \boldsymbol{b}_l$ and $\boldsymbol{b}_l \supseteq \boldsymbol{a}_{ht}$, equivalently $\boldsymbol{a}_{ht} = \boldsymbol{b}_l$. This shows that $X \subseteq Y$, and hence $r \geq s$. Therefore $c_1(A) = sc_1(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$. □

But, for the non-binary Boolean matrix, the column rank and the spanning column rank may differ, and so the inequalities in (2.1) may be strict (see below Example 2.3).

Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ and $B \in \mathcal{M}_{p,q}(\mathbb{B}_k)$. We define the *direct sum* of A and B, denoted $A \oplus B$, as the block-diagonal matrix of the form $\left[\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right]$. Then we have $b(A \oplus B) = b(A) + b(B)$, $c(A \oplus B) = c(A) + c(B)$ and $sc(A \oplus B) = sc(A) + sc(B)$.

Example 2.3. Consider matrices

$$A = \begin{bmatrix} \sigma_1 & \sigma_1 & \sigma_1 + \sigma_2 \\ 0 & \sigma_1 + \sigma_2 & \sigma_1 + \sigma_2 \end{bmatrix} \in \mathcal{M}_{2,3}(\mathbb{B}_k) \quad \text{and} \quad B = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \in \mathcal{M}_2(\mathbb{B}_k).$$

Then we have $b(A) = 2$ and $c(A) = 3$ (see Lemma 4.2), and thus $sc(A) = 3$ by (2.1). Since two columns of B are linearly independent, we have $sc(B) = 2$ by Lemma 2.1. But we can easily show that $\{\left[\begin{smallmatrix} \sigma_1 \\ \sigma_2 \end{smallmatrix}\right]\}$ is a basis of the column space of B. Therefore we have $c(B) = 1$. Also, $b(B) = 1$ by (2.1) (in fact, $B = \left[\begin{smallmatrix} \sigma_1 \\ \sigma_2 \end{smallmatrix}\right] \left[\sigma_1 \ \sigma_2\right]$). Let

$$X = A \oplus B \oplus [0] \in \mathcal{M}_{5,6}(\mathbb{B}_k).$$

Then we have $b(X) = 3$, $c(X) = 4$ and $sc(X) = 5$. Therefore we conclude that $0 < b(X) < c(X) < sc(X) < 6$.

Lemma 2.4. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ and $B \in \mathcal{M}_{n,q}(\mathbb{B}_k)$. Then we have $sc(AB) \leq sc(B)$.*

Proof. Let $sc(B) = r$. Then there exist columns $\boldsymbol{b}_{i1}, \ldots, \boldsymbol{b}_{ir}$ of B with minimum cardinality such that $\{\boldsymbol{b}_{i1}, \ldots, \boldsymbol{b}_{ir}\}$ spans the column space of B. Notice that any jth column of AB is of the form $A\boldsymbol{b}_j$ and $\boldsymbol{b}_j = \sum_{l=1}^{r} \alpha_l \boldsymbol{b}_{il}$ for some $\alpha_l \in \mathbb{B}_k$. Hence we have $A\boldsymbol{b}_j = \sum_{l=1}^{r} \alpha_l (A\boldsymbol{b}_{il})$. This shows that any column of AB can be written as a linear combination of $A\boldsymbol{b}_{i1}, \ldots, A\boldsymbol{b}_{ir}$. Hence $sc(AB) \leq r = sc(B)$. □

But, in general, it is possible that $sc(A_1B) > sc(A_1)$ and $sc(A_2B) < sc(A_2)$ for some matrices A_1, A_2 and B over $\mathbb{B}_k$. For example, consider matrices

$$A_1 = [\sigma_1 \ 1 \ 0], \quad A_2 = [\sigma_1 \ \sigma_2 \ \sigma_3] \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

over $\mathbb{B}_3$. Then we can easily show that $sc(A_1) = 1$ and $sc(A_2) = 3$ by Lemma 2.1. But we have $A_1B = A_2B = [\sigma_1 \ \sigma_2 \ 0]$ so that $sc(A_1B) = sc(A_2B) = 2$ by Lemma 2.1 and (2.2). It follows that $sc(A_1B) > sc(A_1)$ and $sc(A_2B) < sc(A_2)$.

Lemma 2.5. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then $sc(A) = 1$ if and only if there exists a nonzero column $\boldsymbol{a}$ of A such that $A = \boldsymbol{a}\boldsymbol{x}^t$ for some nonzero vector $\boldsymbol{x} \in \mathbb{B}_k^n$ with $sc(\boldsymbol{x}^t) = 1$.*

Proof. It is straightforward to see that $sc(\boldsymbol{a}\boldsymbol{x}^t) = 1$ for a nonzero column $\boldsymbol{a}$ of A and a nonzero vector $\boldsymbol{x} \in \mathbb{B}_k^n$ with $sc(\boldsymbol{x}^t) = 1$. Conversely, assume that $sc(A) = 1$ and let $A = [\boldsymbol{a}_1 \, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n]$. Then there exists a nonzero column $\boldsymbol{a}$ of A such that span$\{\boldsymbol{a}\}$ is the column space of A. Thus there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{B}_k$, not all zeros, such that $\boldsymbol{a}_j = \alpha_j \boldsymbol{a}$ for all $j = 1, \ldots, n$. Let $\boldsymbol{x} = [\alpha_1\, \alpha_2 \cdots \alpha_n]^t$. Then we have

$$A = \boldsymbol{a}\boldsymbol{x}^t = [\alpha_1 \boldsymbol{a}\, \alpha_2 \boldsymbol{a} \cdots \alpha_n \boldsymbol{a}].$$

It remains to show that $sc(\boldsymbol{x}^t) = 1$. Since $sc(A) = 1$, there exists a nonzero column $\alpha_r \boldsymbol{a}$ of A such that $\alpha_j \boldsymbol{a} = \beta_j(\alpha_r \boldsymbol{a})$ for all $j = 1, \ldots, n$ and $\beta_j \in \mathbb{B}_k$. Thus, we have $\alpha_j \subseteq \alpha_r$ for all $j = 1, \ldots, n$ so that $\sum_{j=1}^n \alpha_j \subseteq \alpha_r$. Hence $\alpha_r = \sum_{j=1}^n \alpha_j$. This shows that span$\{\alpha_r\}$ is the column space of $\boldsymbol{x}^t = [\alpha_1\, \alpha_2 \cdots \alpha_n]$. Thus, we have $sc(\boldsymbol{x}^t) = 1$. □

Corollary 2.6. *If $A = [\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ has spanning column rank 1, then there exists a nonzero column $\boldsymbol{a}$ of A such that $\boldsymbol{a} = \sum_{j=1}^n \boldsymbol{a}_j$.*

Proof. Since $sc(A) = 1$, by Lemma 2.5 and its proof, there exist a nonzero column $\boldsymbol{a}$ of A and scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{B}_k$, not all zeros, such that $A = [\alpha_1 \boldsymbol{a}\, \alpha_2 \boldsymbol{a} \cdots \alpha_n \boldsymbol{a}]$ and $\alpha_r = \sum_{j=1}^n \alpha_j$ for some $r \in \{1, \ldots, n\}$. It follows that

$$\boldsymbol{a}_j = \alpha_j \boldsymbol{a} \subseteq \alpha_r \boldsymbol{a} \subseteq \boldsymbol{a}$$

for all $j = 1, \ldots, n$, equivalently $\sum_{j=1}^n \boldsymbol{a}_j \subseteq \boldsymbol{a}$. Hence we have $\boldsymbol{a} = \sum_{j=1}^n \boldsymbol{a}_j$ because $\sum_{j=1}^n \boldsymbol{a}_j \supseteq \boldsymbol{a}$. □

Lemma 2.7. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. If all entries of A are 0 or 1, then $sc(A) = sc_1(A)$.*

Proof. Since $\mathbb{B}_1$ can be considered as a subsemiring of $\mathbb{B}_k$, we have $sc(A) \leq sc_1(A)$. Now, we will show that $sc(A) \geq sc_1(A)$. If $sc(A) = r$, then there exist columns $\boldsymbol{a}_{i1}, \ldots, \boldsymbol{a}_{ir}$ of A with minimum cardinality such that span$\{\boldsymbol{a}_{i1}, \ldots, \boldsymbol{a}_{ir}\}$ is the column space of A. Then the p-th constituents $(\boldsymbol{a}_{i1})_p, \ldots, (\boldsymbol{a}_{ir})_p$ generate all columns of A_p over $\mathbb{B}_1$. Hence $sc_1(A_p) \leq r$. But $A = A_p$ for all $p = 1, \ldots, k$. Hence $sc_1(A) \leq r = sc(A)$. □

A matrix $A \in \mathcal{M}_n(\mathbb{B}_k)$ is called *invertible* if there exists a matrix $B \in \mathcal{M}_n(\mathbb{B}_k)$ such that $AB = I_n = BA$, where I_n is the identity matrix. It is well-known [9] that a matrix $A \in \mathcal{M}_n(\mathbb{B}_k)$ is invertible if and only if all its constituents are permutation matrices. In particular, if A is invertible, then $A^{-1} = A^t$, where A^t denotes the transpose of A. Furthermore, permutation matrices are the only invertible members of $\mathcal{M}_n(\mathbb{B}_1)$.

Let $Q = [q_{i,j}] \in \mathcal{M}_n(\mathbb{B}_k)$ be invertible. Since all constituents of Q are permutation matrices, it follows that $\sum_{l=1}^n q_{i,l} = 1$ and $\sum_{l=1}^n q_{l,j} = 1$ for all $i, j = 1, \ldots, n$. In addition, for all $i, j, r, s = 1, \ldots, n$, we have

$$(2.7) \qquad q_{i,r}q_{i,s} = \begin{cases} q_{i,r} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases} \quad \text{and} \quad q_{r,j}q_{s,j} = \begin{cases} q_{r,j} & \text{if } r = s \\ 0 & \text{if } r \neq s \end{cases}$$

because all elements of $\mathbb{B}_k$ are idempotents.

Lemma 2.8. *If $P \in \mathcal{M}_m(\mathbb{B}_k)$ is invertible, then $sc(PA) = sc(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$.*

Proof. If $\boldsymbol{a}_{i1}, \ldots, \boldsymbol{a}_{ir}$ be any columns of A, then we can easily show that $\{\boldsymbol{a}_{i1}, \ldots, \boldsymbol{a}_{ir}\}$ spans the column space of A if and only if $\{P\boldsymbol{a}_{i1}, \ldots, P\boldsymbol{a}_{ir}\}$ spans the column space of PA. Thus, the Lemma follows. □

But, even if $Q \in \mathcal{M}_n(\mathbb{B}_k)$ is invertible, then $sc(AQ) = sc(A)$ may not be true as the following example shows:

Example 2.9. Let a and b be elements in $\mathbb{B}_k$ such that $a + b = 1$ and $ab = 0$. Consider a matrix

$$Q = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \oplus I_{n-2} \in \mathcal{M}_n(\mathbb{B}_k),$$

where $n \geq 2$. Then $QQ^t = I_n$ so that Q is invertible. But, for a matrix

$$A = [a\ b] \oplus O \in \mathcal{M}_{m,n}(\mathbb{B}_k),$$

we have $sc(A) = 2$ by Lemma 2.1 and (2.2), while

$$sc(AQ) = sc([1\ 0] \oplus O) = 1.$$

Hence we have $sc(AQ) \neq sc(A)$.

3. Spanning column rank preservers

In this section, we have characterizations of the linear operators that preserve the spanning column ranks of matrices over non-binary Boolean algebra $\mathbb{B}_k$.

Suppose that T is an operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$. Say that

(i) T is *linear* if $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for all $\alpha, \beta \in \mathbb{B}_k$ and for all $X, Y \in \mathcal{M}_{m,n}(\mathbb{B}_k)$,
(ii) T is a *congruence operator* if there exist invertible matrices $P \in \mathcal{M}_m(\mathbb{B}_k)$ and $Q \in \mathcal{M}_n(\mathbb{B}_k)$ such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$,
(iii) T *preserves spanning column rank* if $sc(T(X)) = sc(X)$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$,
(iv) T *preserves spanning column rank* r if $sc(T(X)) = r$ whenever $sc(X) = r$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$.

Boolean rank (respectively, column rank) preservers are defined in a manner similar to (iii) and (iv).

Since the column rank and the spanning column rank of matrices over $\mathbb{B}_1$ are the same by Theorem 2.2, we can apply Theorem 1.2 for the column rank of matrices over $\mathbb{B}_1$ to the case of the spanning column. Thus we obtain the following:

Theorem 3.1. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ with $n \geq m \geq 4$. Then the following are equivalent*:

(i) *T preserves spanning column ranks* $1, 2$ *and* 3;
(ii) *T is a congruence operator*;
(iii) *T preserves spanning column rank.*

If $n \leq 3$, then the characterizations of linear operators that preserve (spanning) column rank on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ are the same as those of linear operators that preserve Boolean rank on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ by (2.3), which were characterized in [1].

But, as shown in Example 2.3, the column rank and the spanning column rank may differ over $\mathcal{M}_{m,n}(\mathbb{B}_k)$. Thus, the characterizations of linear operators that preserve spanning column rank of matrices over $\mathbb{B}_k$, may not be the same as those over $\mathbb{B}_1$. The following Example shows that some congruence operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ does not preserve spanning column rank 2.

Example 3.2. Let Q be the matrix in Example 2.9. Define a linear operator T on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ by $T(X) = XQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then we have that T is a congruence operator since Q is invertible. Consider the matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ in Example 2.9. Then $sc(A) = 2$, while $sc(T(A)) = sc(AQ) = 1$ as we showed in Example 2.9. Therefore, T does not preserve spanning column rank 2.

If T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$, for each $p = 1, \ldots, k$, define its *p-th constituent operator*, T_p, by $T_p(X) = (T(X))_p$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. By the linearity of T, we have

$$T(X) = \sum_{p=1}^{k} \sigma_p T_p(X_p)$$

for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$.

Lemma 3.3. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$. If T preserves spanning column rank r, then each constituent operator T_p preserves spanning column rank r on $\mathcal{M}_{m,n}(\mathbb{B}_1)$.*

Proof. Assume that A is a matrix in $\mathcal{M}_{m,n}(\mathbb{B}_1)$ with $sc_1(A) = r$. Then by Lemma 2.7, we have $sc(A) = r$ and $sc(\sigma_p A) = r$ for each $p = 1, \ldots, k$. Since T preserves spanning column rank r, we obtain that $sc(T(A)) = r$ and $sc(T(\sigma_p A)) = r$. It follows that

$$\begin{aligned} sc_1(T_p(A)) = sc(T_p(A)) = sc(\sigma_p T_p(A)) &= sc\left(\sigma_p\left(\sum_{j=1}^{k} \sigma_j T_j(A_j)\right)\right) \\ &= sc(\sigma_p T(A)) = sc(T(\sigma_p A)) = r \end{aligned}$$

for all $p = 1, \ldots, k$. Therefore each constituent operator T_p preserves spanning column rank r on $\mathcal{M}_{m,n}(\mathbb{B}_1)$. ☐

But the converse of Lemma 3.3 is not true. For example, consider a linear operator T on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ in Example 3.2. Then we can easily show that all constituent operators T_p on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ preserve spanning column rank 2, while T on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ does not preserve spanning column rank 2.

Lemma 3.4 ([8])**.** *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$. If each constituent operator of T preserves binary Boolean rank r, then T preserves Boolean rank r on $\mathcal{M}_{m,n}(\mathbb{B}_k)$.*

Lemma 3.5. *Let $n \geq 2$, and let Q be invertible in $\mathcal{M}_n(\mathbb{B}_k)$. Define a linear operator T on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ by $T(X) = XQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then T preserves spanning column ranks* 1 *and* 2 (*if and*) *only if Q is a permutation matrix.*

Proof. Suppose that T preserves spanning column ranks 1 and 2. Since $Q = [q_{i,j}]$ is invertible, we lose no generality in assuming that $q_{j,j} \neq 0$ for all $j = 1, \ldots, n$.

Claim. Let $j \in \{1, \ldots, n\}$ be arbitrary. Then we claim that $q_{i,j} = 0$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$. If this is true, then $q_{j,j} = 1$. Since j is arbitrary, it follows that $Q = I_n$ is a permutation matrix.

Proof of Claim. First, we show that $q_{2,1} = q_{3,1} = \cdots = q_{n,1} = 0$. Let i be an arbitrary index in $\{2, \ldots, n\}$. Let

$$X_1 = [q_{1,1}\ 0 \cdots 0\ q_{i,1}\ 0 \cdots 0]$$

be an $1 \times n$ matrix, and let $Y_1 = \left[\begin{smallmatrix} X_1 \\ O \end{smallmatrix}\right] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ so that $sc(X_1) = sc(Y_1)$ by (2.2). It follows from (2.7) that

$$T(Y_1) = [q_{1,1} + q_{i,1}] \oplus O,$$

and thus $sc(T(Y_1)) = 1$. If $q_{i,1} \neq 0$, then $\{q_{1,1}, q_{i,1}\}$ are linearly independent since $q_{1,1}q_{1,i} = 0$ by (2.7). Hence by Lemma 2.1, $sc(X_1) = sc(Y_1) = 2$. This contradicts to the fact that T preserves spanning column rank 2. Hence $q_{i,1} = 0$. Since $i \in \{2, \ldots, n\}$ is arbitrary, we have $q_{2,1} = q_{3,1} = \cdots = q_{n,1} = 0$. This implies that $q_{1,1} = 1$. □

Next, suppose that j is an arbitrary index in $\{1, \ldots, n\}$. Now, we will show that $q_{i,j} = 0$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$. Let i be arbitrary in $\{1, \ldots, n\} \setminus \{j\}$, and let $X = [x_1\, x_2 \cdots x_n]$ be an $1 \times n$ matrix, where

$$x_t = \begin{cases} q_{i,j} & \text{if } t = i; \\ q_{j,j} & \text{if } t = j; \\ 0 & \text{elsewhere.} \end{cases}$$

Let $Y = \left[\begin{smallmatrix} X \\ O \end{smallmatrix}\right] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ so that $sc(X) = sc(Y)$. Then the only $(1, j)$th entry of $T(Y)$ is nonzero and $q_{j,j} + q_{i,j}$ so that $sc(T(Y)) = 1$. If $q_{i,j} \neq 0$, then $\{q_{j,j}, q_{i,j}\}$ are linearly independent since $q_{j,j}q_{i,j} = 0$ by (2.7). Hence by Lemma 2.1, $sc(X) = sc(Y) = 2$, a contradiction to the fact that T preserves spanning column rank 2. Hence $q_{i,j} = 0$. Since $i \in \{1, \ldots, n\} \setminus \{j\}$ is arbitrary, we have $q_{i,j} = 0$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$, and thus $q_{j,j} = 1$. Since j is arbitrary, it follows that $Q = I_n$. □

Let σ^* denote the complement of σ for each $\sigma \in \mathbb{B}_k$. For each $p = 1, \ldots, k$, we define the *p-th rotation operator*, $R^{(p)}$ on $\mathcal{M}_n(\mathbb{B}_k)$ by

$$R^{(p)}(X) = \sigma_p X_p^t + \sigma_p^* X$$

for all $X \in \mathcal{M}_n(\mathbb{B}_k)$. Then we see that $R^{(p)}$ has the effect of transposing X_p while leaving the remaining constituents unchanged. Each rotation operator is linear on $\mathcal{M}_n(\mathbb{B}_k)$ and their product is the *transposition operator*, $X \to X^t$ for all $X \in \mathcal{M}_n(\mathbb{B}_k)$.

Example 3.6. Let $R^{(p)}$ be any p-th rotation operator on $\mathcal{M}_n(\mathbb{B}_k)$ with $n \geq 2$. Now, we will show that $R^{(p)}$ does not preserve spanning column rank 1. Consider a matrix

$$A = \begin{bmatrix} \sigma_p & 1 \\ 0 & 0 \end{bmatrix} \oplus O \in \mathcal{M}_n(\mathbb{B}_k).$$

Then $sc(A) = 1$. But $R^{(p)}(A) = \left[\begin{smallmatrix} \sigma_p & \sigma_p^* \\ \sigma_p & 0 \end{smallmatrix}\right] \oplus O$ has spanning column rank 2 by Lemma 2.1. Hence $R^{(p)}$ does not preserve spanning column rank 1.

Theorem 3.7. *Suppose that T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ with $n \geq 2$. Then the following statements are equivalent*:

(i) *T preserves spanning column rank*;
(ii) *T preserves spanning column ranks* 1 *and* 2;
(iii) *there exist an invertible matrix $P \in \mathcal{M}_m(\mathbb{B}_k)$ and an $n \times n$ permutation matrix Q such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$.*

Proof. It is obvious that (i) implies (ii). Assume that T preserves spanning column ranks 1 and 2. By Lemma 3.3, each constituent operator T_p on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ preserves spanning column ranks 1 and 2. Since the column rank and the spanning column rank over $\mathcal{M}_{m,n}(\mathbb{B}_1)$ are equal by Theorem 2.2, each T_p on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ preserves column ranks 1 and 2. It follows from (2.3) that each T_p on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ preserves binary Boolean ranks 1 and 2. Then T preserves Boolean ranks 1 and 2 on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ by Lemma 3.4. Thus, by Theorem 1.1, T is in the group of operators generated by the congruence (if $m = n$, also the rotation) operators. But any rotation operator does not preserve spanning column rank 1 by Example 3.6. Hence T should be a congruence operator. That is, there exist invertible matrices $P \in \mathcal{M}_m(\mathbb{B}_k)$ and $Q \in \mathcal{M}_n(\mathbb{B}_k)$ such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Assume that Q is not a permutation matrix. By Lemma 3.5, there exists a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ with $sc(A) = 1$ or 2 such that $sc(A) \neq sc(AQ)$. Since P is invertible in $\mathcal{M}_m(\mathbb{B}_k)$, it follows from Lemma 2.8 that $sc(AQ) = sc(PAQ)$, and hence $sc(A) \neq sc(PAQ)$. Thus, T does not preserve spanning column rank 1 or 2, and therefore Q must be a permutation matrix. This shows that (ii) implies (iii). It follows from Lemma 2.8 that (iii) implies (i) because the spanning column rank of a matrix in $\mathcal{M}_{m,n}(\mathbb{B}_k)$ is unchanged post-multiplication by a permutation matrix. □

Thus we have characterizations of the linear operators that preserve the spanning column ranks of matrices over non-binary Boolean algebra.

4. Column rank preservers

In this section, we obtain characterizations of the linear operators that preserve the column ranks of matrices over non-binary Boolean algebra.

Song [6] characterized the linear operators on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ that preserve column rank (see Theorem 1.2). Furthermore, Song and Lee [8] gave a characterization of the linear operators on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ preserving column ranks $1, 2$ and 3 as following:

Theorem 4.1. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ with $m \geq 2$ and $n \geq 3$. Then T preserves column ranks $1, 2$ and 3 if and only if T is a congruence operator.*

But we assert that for matrices over non-binary Boolean algebra, some congruence operator does not preserve column rank 3 (see Example 4.3), which shows that Theorem 4.1 does not hold. So, we present the revised characterizations on the above Theorem (see Theorem 4.6).

Lemma 4.2. *Let $A = \begin{bmatrix} a & a & b \\ 0 & b & b \end{bmatrix} \in \mathcal{M}_{2,3}(\mathbb{B}_k)$, where $a \neq 0$, $a \subseteq b \subseteq 1$ but $a \neq b$. Then we have $c(A) = 3$.*

Proof. Let $\mathcal{V}$ be the column space of A. Then any vector in $\mathcal{V}$ is of the form

$$x \begin{bmatrix} a \\ 0 \end{bmatrix} + y \begin{bmatrix} a \\ a \end{bmatrix} + z \begin{bmatrix} a \\ b \end{bmatrix} + w \begin{bmatrix} b \\ b \end{bmatrix},$$

where $x, y \subseteq a$ and $z, w \subseteq b$. Let Ω be any subset of $\mathcal{V}$ generating $\mathcal{V}$. If $\begin{bmatrix} a \\ 0 \end{bmatrix} \notin \Omega$, then

$$\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} ya + za + wb \\ ya + zb + wb \end{bmatrix} \tag{4.1}$$

for some $y \subseteq a$ and $z, w \subseteq b$. But then $a = ya + za + wb \subseteq ya + zb + wb = 0$ by the second row in (4.1), which is impossible. Thus, $\begin{bmatrix} a \\ 0 \end{bmatrix} \in \Omega$. If $\begin{bmatrix} a \\ b \end{bmatrix} \notin \Omega$, then

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} xa + ya + wb \\ ya + wb \end{bmatrix} \tag{4.2}$$

for some $x, y \subseteq a$ and $w \subseteq b$. But then $a = xa + ya + wb = xa + b \supseteq b$ by the second row in (4.2), a contradiction to the fact that $a \subseteq b$ but $a \neq b$. Therefore, $\begin{bmatrix} a \\ b \end{bmatrix} \in \Omega$. If $\begin{bmatrix} b \\ b \end{bmatrix} \notin \Omega$, then

$$\begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} (x + y + z)a \\ ya + zb \end{bmatrix}$$

for some $x, y \subseteq a$ and $z \subseteq b$. But then $b = (x + y + z)a \subseteq a$, which is impossible because $a \subseteq b$ but $a \neq b$. Hence $\begin{bmatrix} b \\ b \end{bmatrix} \in \Omega$. Therefore we have concluded that $\left\{ \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix} \right\} \subseteq \Omega$, and hence $c(A) = 3$. □

Example 4.3. Let $m \geq 2$ and $n \geq 3$. Consider a matrix

$$Q = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus I_{n-3} \in \mathcal{M}_n(\mathbb{B}_k),$$

where $a, b \in \mathbb{B}_k$ with $a, b \neq 0, 1$, $a+b = 1$ and $ab = 0$. Define a linear operator T on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ by $T(X) = XQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then T is a congruence operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ because $QQ^t = I_n$ and hence Q is invertible. Consider a matrix $A = \left[\begin{smallmatrix} a & a & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right] \oplus O \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. It follows from Lemma 4.2 and (2.2) that $c(A) = 3$. But the column rank of

$$T(A) = AQ = \begin{bmatrix} a & a & 1 \\ b & a & 1 \end{bmatrix} \oplus O$$

is at most 2 since the first and third columns of $T(A)$ spans the column space of $T(A)$. Thus, T does not preserve column rank 3.

Lemma 4.4. *Let $m \geq 2$ and $n \geq 3$. Suppose that Q is invertible in $\mathcal{M}_n(\mathbb{B}_k)$. Define a linear operator T on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ by $T(X) = XQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then T preserves column ranks $1, 2$ and 3 (if and) only if Q is a permutation matrix.*

Proof. Suppose that T preserves column ranks $1, 2$ and 3. Assume that Q is not a permutation matrix. Then there exists a row or a column of $Q = [q_{i,j}]$ such that it has at least two nonzero elements different from 1. We lose no generality in assuming that $q_{1,1}$, $q_{1,2} \neq 0, 1$ with $\sigma_1 \subseteq q_{1,1}$ and $\sigma_2 \subseteq q_{1,2}$. Furthermore, we may assume that $q_{2,1} \neq 0, 1$ because $q_{1,1} \neq 0, 1$ and $\sum_{i=1}^{n} q_{i,1} = 1$. Let

$$A_i = \begin{bmatrix} \sigma_i & \sigma_i & \sigma_1 + \sigma_2 & 0 & \cdots & 0 \\ 0 & \sigma_1 + \sigma_2 & \sigma_1 + \sigma_2 & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{M}_{2,n}(\mathbb{B}_k),$$

where $i = 1$ or 2, and let $B = \left[\begin{smallmatrix} A_i \\ O \end{smallmatrix}\right] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. By Lemma 4.2 and (2.2), we have $c(A_i) = c(B) = 3$. Now, we will show that $c(T(B)) \leq 2$. Thus, it suffices to consider the case $m = 2$. Let $\mathcal{V}$ be the column space of $T(A_i)$. To prove $c(T(A_i)) \leq 2$, we will consider two cases:

Case 1. Let $\sigma_2 \subseteq q_{2,1}$. Then there are two possibilities: either $\sigma_1 \subseteq q_{2,2}$ or $\sigma_1 \not\subseteq q_{2,2}$. First, suppose that $\sigma_1 \subseteq q_{2,2}$. Then we have

$$\begin{bmatrix} \sigma_1 & \sigma_2 & q_{1,3} & \cdots & q_{1,n} \\ \sigma_2 & \sigma_1 & q_{2,3} & \cdots & q_{2,n} \\ q_{3,1} & q_{3,2} & q_{3,3} & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} & \cdots & q_{n,n} \end{bmatrix} \subseteq Q.$$

Hence (2.7) ensures that

$$\sigma_1 + \sigma_2 \not\subseteq q_{3,i} \quad \text{and} \quad \sigma_1 + \sigma_2 \not\subseteq q_{i,j} \tag{4.3}$$

for all $i = 1, 2$ and for all $j \geq 3$. It follows that

$$T(A_1) = \begin{bmatrix} \sigma_1 & \sigma_1 & (\sigma_1+\sigma_2)q_{3,3} & \cdots & (\sigma_1+\sigma_2)q_{3,n} \\ \sigma_2 & \sigma_1 & (\sigma_1+\sigma_2)q_{3,3} & \cdots & (\sigma_1+\sigma_2)q_{3,n} \end{bmatrix}.$$

Since $\sum_{l=1}^n q_{3,l} = 1$ and $\sigma_1 + \sigma_2 \not\subseteq q_{3,1} + q_{3,2}$ by (4.3), we have $\sigma_1 + \sigma_2 \subseteq \sum_{l=3}^n q_{3,l}$. Hence $\left[\begin{smallmatrix}\sigma_1+\sigma_2\\ \sigma_1+\sigma_2\end{smallmatrix}\right] \in \mathcal{V}$ and $\left\{\left[\begin{smallmatrix}\sigma_1\\ \sigma_2\end{smallmatrix}\right], \left[\begin{smallmatrix}\sigma_1+\sigma_2\\ \sigma_1+\sigma_2\end{smallmatrix}\right]\right\}$ spans $\mathcal{V}$. But then $c(T(A_1)) \leq 2$, a contradiction since $c(A_1) = 3$. Next, suppose that $\sigma_1 \not\subseteq q_{2,2}$. Then we may assume that $\sigma_1 \subseteq q_{2,3}$ and $\sigma_1 \subseteq q_{3,2}$ because $\sigma_1 \not\subseteq q_{1,2} + q_{2,1} + q_{2,2}$. It follows that

$$T(A_1) = \begin{bmatrix} \sigma_1 & \sigma_1 & \sigma_1+\sigma_2 q_{3,3} & \sigma_2 q_{3,4} & \cdots & \sigma_2 q_{3,n} \\ \sigma_2 & \sigma_1 & \sigma_1+\sigma_2 q_{3,3} & \sigma_2 q_{3,4} & \cdots & \sigma_2 q_{3,n} \end{bmatrix}.$$

Since $\sigma_2 \subseteq q_{2,1}$ and $\sigma_2 \subseteq q_{1,2}$, we have $\sigma_2 \not\subseteq q_{3,1}+q_{3,2}$ and hence $\sigma_2 \subseteq \sum_{l=3}^n q_{3,l}$. This implies that $\left[\begin{smallmatrix}\sigma_1+\sigma_2\\ \sigma_1+\sigma_2\end{smallmatrix}\right] \in \mathcal{V}$. As shown in the above, $c(T(A_1)) \leq 2$, a contradiction.

Case 2. Let $\sigma_2 \not\subseteq q_{2,1}$. Since $\sigma_2 \not\subseteq q_{1,1}$ and $\sum_{l=1}^n q_{l,1} = 1$, we may assume that $\sigma_2 \subseteq q_{3,1}$. We can consider two possibilities: either $\sigma_1 \subseteq q_{2,2}$ or $\sigma_1 \not\subseteq q_{2,2}$. First, suppose that $\sigma_1 \subseteq q_{2,2}$. Then we may assume that $\sigma_2 \subseteq q_{2,3}$ since $\sigma_2 \not\subseteq q_{2,1} + q_{2,2}$. It follows that

$$T(A_2) = \begin{bmatrix} \sigma_2 & \sigma_2 & \sigma_2+\sigma_1 q_{3,3} & \sigma_1 q_{3,4} & \cdots & \sigma_1 q_{3,n} \\ \sigma_2 & \sigma_1 & \sigma_2+\sigma_1 q_{3,3} & \sigma_1 q_{3,4} & \cdots & \sigma_1 q_{3,n} \end{bmatrix}.$$

Since $\sigma_1 \not\subseteq q_{3,1} + q_{3,2}$, we have $\sigma_1 \subseteq \sum_{l=3}^n q_{3,l}$. This shows that $\left[\begin{smallmatrix}\sigma_1+\sigma_2\\ \sigma_1+\sigma_2\end{smallmatrix}\right] \in \mathcal{V}$ and $\left\{\left[\begin{smallmatrix}\sigma_2\\ \sigma_1\end{smallmatrix}\right], \left[\begin{smallmatrix}\sigma_1+\sigma_2\\ \sigma_1+\sigma_2\end{smallmatrix}\right]\right\}$ spans $\mathcal{V}$ and hence $c(T(A_2)) \leq 2$, a contradiction since $c(A_2) = 3$. Next, suppose that $\sigma_1 \not\subseteq q_{2,2}$. Since $\sigma_1 \not\subseteq q_{2,1}$, we may assume that $\sigma_1 \subseteq q_{2,3}$. Then we have

$$\begin{bmatrix} \sigma_1 & \sigma_2 & q_{1,3} & q_{1,4} & \cdots & q_{1,n} \\ q_{2,1} & q_{2,2} & q_{2,3} & q_{2,4} & \cdots & q_{2,n} \\ \sigma_2 & q_{3,2} & q_{3,3} & q_{3,4} & \cdots & q_{3,n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ q_{n,1} & q_{n,2} & q_{n,3} & q_{n,4} & \cdots & q_{n,n} \end{bmatrix} \subseteq \Omega.$$

It follows that

$$T(A_1) = \begin{bmatrix} \sigma_1+\sigma_2 & \sigma_1 q_{3,2} & \sigma_1 & \sigma_1 q_{3,4} & \cdots & \sigma_1 q_{3,n} \\ \sigma_2 & \sigma_1 q_{3,2} & \sigma_1+\sigma_2 q_{2,3} & \sigma_2 q_{2,4}+\sigma_1 q_{3,4} & \cdots & \sigma_2 q_{2,n}+\sigma_1 q_{3,n} \end{bmatrix}.$$

Clearly $\left[\begin{smallmatrix}\sigma_1\\ \sigma_1\end{smallmatrix}\right] \in \mathcal{V}$ since $\left[\begin{smallmatrix}\sigma_1\\ \sigma_1\end{smallmatrix}\right] = \sigma_1 \left[\begin{smallmatrix}\sigma_1\\ \sigma_1+\sigma_2 q_{2,3}\end{smallmatrix}\right]$. If $\sigma_2 \subseteq q_{2,3}$, then $\left[\begin{smallmatrix}\sigma_1\\ \sigma_1+\sigma_2\end{smallmatrix}\right] \in \mathcal{V}$. But if $\sigma_2 \not\subseteq q_{2,3}$, we may assume that $\sigma_2 \subseteq q_{2,4}$ since $\sigma_2 \not\subseteq q_{2,1} + q_{2,2}$. Hence the 4th column of $T(A_1)$ is either $\left[\begin{smallmatrix}0\\ \sigma_2\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}\sigma_1\\ \sigma_1+\sigma_2\end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix}\sigma_1\\ \sigma_1+\sigma_2\end{smallmatrix}\right] = \left[\begin{smallmatrix}\sigma_1\\ \sigma_1\end{smallmatrix}\right] + \left[\begin{smallmatrix}0\\ \sigma_2\end{smallmatrix}\right]$, we have $\left[\begin{smallmatrix}\sigma_1\\ \sigma_1+\sigma_2\end{smallmatrix}\right] \in \mathcal{V}$. Consequently, $\left\{\left[\begin{smallmatrix}\sigma_1+\sigma_2\\ \sigma_2\end{smallmatrix}\right], \left[\begin{smallmatrix}\sigma_1\\ \sigma_1+\sigma_2\end{smallmatrix}\right]\right\}$ spans $\mathcal{V}$ so that $c(T(A_1)) \leq 2$, a contradiction.

Therefore Q must be a permutation matrix. □

Lemma 4.5 ([8])**.** *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$. If T preserves column rank r, then each constituent T_p on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ preserves column rank r.*

Now, we give a revised statement of Theorem 4.1 as following:

Theorem 4.6. *Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ with $m \geq 2$ and $n \geq 3$. Then the following are equivalent:*

(i) *T preserves column rank;*
(ii) *T preserves column ranks $1, 2$ and 3;*
(iii) *there exist an invertible matrix $P \in \mathcal{M}_m(\mathbb{B}_k)$ and an $n \times n$ permutation matrix Q such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$.*

Proof. Obviously (i) implies (ii). Assume that T preserves column ranks $1, 2$ and 3. By Lemma 4.5, each constituent operator T_p on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ preserves column ranks $1, 2$ and 3. It follows from (2.3) that each T_p on $\mathcal{M}_{m,n}(\mathbb{B}_1)$ preserves binary Boolean ranks 1 and 2. Then T preserves Boolean ranks 1 and 2 on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ by Lemma 3.4. Thus, by Theorem 1.1, T is in the group of operators generated by the congruence (if $m = n$, also the rotation) operators. But, we claim that any rotation operator $R^{(p)}$ on $\mathcal{M}_n(\mathbb{B}_k)$ does not preserve column rank 3. Consider a matrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_p & \sigma_p & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus O \in \mathcal{M}_n(\mathbb{B}_k).$$

Then $c(A) = 3$ by Lemma 4.2 and (2.2). But $R^{(p)}(A) = A^t$ has column rank 2. Thus T should be a congruence operator on $\mathcal{M}_{m,n}(\mathbb{B}_k)$. That is, there exist invertible matrices $P \in \mathcal{M}_m(\mathbb{B}_k)$ and $Q \in \mathcal{M}_n(\mathbb{B}_k)$ such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Now it remains to show that Q is a permutation matrix. Define a linear operator L on $\mathcal{M}_{m,n}(\mathbb{B}_k)$ by $L(X) = P^{-1}T(X) = XQ$. Then we can easily show that T preserves column ranks $1, 2$ and 3 if and only if so do L. Also, Lemma 4.4 shows that L preserves column ranks $1, 2$ and 3 if and only if Q is a permutation matrix. Thus, (ii) implies (iii).

Assume (iii). Since the column rank of a matrix in $\mathcal{M}_{m,n}(\mathbb{B}_k)$ is unchanged by post-multiplication by a permutation matrix, we lose no generality in assuming that $Q = I_n$. That is, there exists an invertible matrix $P \in \mathcal{M}_m(\mathbb{B}_k)$ such that $T(X) = PX$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Consider any matrix A in $\mathcal{M}_{m,n}(\mathbb{B}_k)$. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ be arbitrary column vectors in $\mathbb{B}_k^m$. Then we can easily show that $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$ is a basis of the column space of A if and only if $\{P\boldsymbol{x}_1, \ldots, P\boldsymbol{x}_r\}$ is a basis of the column space of PA. This shows that $c(A) = c(T(A))$ for all $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Hence (i) is satisfied. □

If $n \leq 2$, then the linear operators that preserve column ranks of matrices over $\mathbb{B}_k$ are the same as the Boolean rank-preservers, which were characterized in Theorem 1.1.

Thus we have characterized the linear operators that preserve the column ranks of matrices over non-binary Boolean algebra $\mathbb{B}_k$.

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