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# EXTENDING THE APPLICABILITY OF INEXACT GAUSS–NEWTON METHOD FOR SOLVING UNDERDETERMINED NONLINEAR LEAST SQUARES PROBLEMS

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Abstract. The aim of this paper is to extend the applicability of Gauss-Newton method for solving underdetermined nonlinear least squares problems in cases not covered before. The novelty of the paper is the introduction of a restricted convergence domain. We find a more precise location where the Gauss-Newton iterates lie than in earlier studies. Consequently the Lipschitz constants are at least as small as the ones used before. This way and under the same computational cost, we extend the local as well the semilocal convergence of Gauss-Newton method. The new developmentes are obtained under the same computational cost as in earlier studies, since the new Lipschitz constants are special cases of the constants used before. Numerical examples further justify the theoretical results.

## 1. Introduction

We consider the *nonlinear least squares* problem

$$\min_{x\in\Omega}\ \zeta(x) := \frac{1}{2}F(x)^T F(x), \tag{1}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, $F : \Omega \to \mathbb{R}^m$ is a continuously differentiable nonlinear function. A wide variety of applications can be found in mathematical programming literature, see for example [5]–[20].

It is not hard to see that finding the stationary points of $\zeta$ is equivalent to solving the following nonlinear equation

$$\nabla\zeta(x) = F'(x)^T F(x) = 0. \tag{2}$$

Thus, Newton's method for solving (2) can be used to solve (1). However, Newton's method for solving (2) requires the computation of the Hessian matrix of $\zeta$ at each iteration, and this may be difficult especially for large scale problems (see for instance [8]). A generalization of the Newton method called

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the Gauss-Newton method (GN), can be used to solve the problem (1). This iterative algorithm computes the sequence

$$x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \qquad k = 0, 1, \dots,$$

where $F'(x_k)^\dagger$ denotes the Moore-Penrose inverse of the linear operator $F'(x_k)$. Many authors have studied the local as well as semi-local convergence of the Gauss-Newton method; see for instance [4, 6, 11, 15, 22].

The study of convergence of iterative algorithms is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms; while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points.

In [6] Bao *et al.*, considered the case when $m \leq n$, i.e., the problem (1) is underdetermined. They proposed some approximate Gauss-Newton methods for solving (1), and they studied the convergence of their method under the full row rank assumption. They noted for that purpose that, in the case when $F'(x_k)$ is of full row rank, $F'(x_k)^\dagger = F'(x_k)^T(F'(x_k)F'(x_k)^T)^{-1}$. Thus, solving the Gauss-Newton step, $d_k := -F'(x_k)^\dagger F(x_k)$, is equivalent to solving the equation

$$F'(x_k)F'(x_k)^T s_k = -F(x_k), \tag{3}$$

and setting the step $d_k := F'(x_k)^T s_k$.

Bao *et al*, [6], proposed the truncated Gauss-Newton method for solving (1) which solves (3) inexactly and compute $d_k$ by $d_k := F'(x_k)^T s_k$. They considered the assumption that the Fréchet derivatives are Lipschitz continuous and of full row rank, and established Kantorovich convergence criteria for their algorithm.

A usual assumption to obtain quadratic convergence of Newton's method, is the Lipschitz continuity of $F'$ in a neighborhood of the solution, see [1,2,6,9,16, 17]. Indeed, ensuring control of the derivative is an important consideration in the convergence analysis of Newton's method. On the other hand, a couple of studies have dealt with the issue of convergence analysis of Newton's method, by relaxing the assumption of Lipschitz continuity of $F'$, see for example [3–5, 19].

Here, we extend the convergence domain of truncated Gauss-Newton method even further than [5–7, 10–13, 15, 19–22] using our new idea of restricted convergence domains. To achieve this goal, we first introduce the center-Lipschitz condition which determines a subset of the original domain for the mapping containing the iterates. The classical Kantorovich's condition is then related to the subset instead of the original domain. This way, the center-Lipschitz condition is more precise than if they were depending on the original domain of the mapping as in earlier studies. The new technique leads to: weaker sufficient convergence conditions, tighter error bounds on the distance involved and an at least as precise information on the location of the solution. These advantages are obtained under the same computational cost as in earlier studies, since in practice the new conditions are special cases of the Kantorovich's condition. The idea introduced in this paper can be used on other iterative methods. Numerical examples are also provided to show that our results apply to solve equations but not earlier ones [3,4].

The remainder of this paper is organized as follows. In Section 2, some notations and important results used throughout the paper are presented. In Sections 3 and 4, the convergence analysis is obtained for algorithm TGNU (I) and TGNU (II). Section 5 contains numerical examples showing the superiority of the new results. Finally, the paper ends with a conclusion section.

## 2. Majorizing sequences

The convergence analysis that follows in Section 3 is based on some scalar functions. Define function $f_\beta(\cdot) : \mathbb{R} \cup \{0\} \to \mathbb{R}$ by

$$f_\beta(t) = \alpha t^2 - \beta t + \mu, \tag{4}$$

where $\alpha > 0$, $\beta > 0$ and $\mu \geq 0$.

Suppose that

$$4\alpha\mu \leq \beta^2, \tag{5}$$

then $f_\beta$ has two distinct positive zeros if (5) is a strict inequality and one positive zero if equality holds in (5).

The smaller zero denoted by $t_*$ is given by

$$t_* = \frac{\beta - \sqrt{\beta^2 - 4\alpha\mu}}{2\alpha}. \tag{6}$$

Let $K > 0$, $L > 0$, $\delta \geq 0$ and $\omega \in [0, 1)$ be parameters with $K \leq L$. Define

$$\alpha(M, \omega, \delta) = \frac{M(1+\omega)}{2[1 + M\omega\delta(1 + \omega\delta(1+\omega))]},$$
$$\beta(M, \omega, \delta) = 1 - \frac{\omega[1 - M\delta(1+\omega)]}{1 + M\omega\delta(1 + \omega\delta(1+\omega)},$$
$$\mu = \delta(1+\omega),$$

where $M = K$ or $L$. Set $\alpha := \alpha(K, \omega, \delta)$, $\beta := \beta(K, \omega, \delta)$, $\alpha_1 := \alpha(L, \omega, \delta)$ and $\beta_1 := \beta(L, \omega, \delta)$.

Similarly function $f_{\beta_1}(\cdot) : \mathbb{R} \cup \{0\} \to \mathbb{R}$ is defined by

$$f_{\beta_1}(t) = \alpha_1 t^2 - \beta_1 t + \mu \tag{7}$$

and if

$$4\alpha_1\mu \leq \beta_1^2, \tag{8}$$

we can set

$$t^* = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\alpha_1\mu}}{2\alpha_1} \tag{9}$$

to be the smallest zero of function $f_{\beta_1}$.

It was shown in [6, Lemma 3.1] that, if

$$L\delta \leq H \tag{10}$$

holds, so does (8), where

$$H = \frac{(1-\omega)^2}{(1+\omega)(\sqrt{(2\omega^2-\omega+1)^2 + 2\omega(1-\omega)^3} + 2\omega^2 - \omega + 1)}. \tag{11}$$

Clearly, by replacing $\alpha_1$, $\beta_1$, $L$ by $\alpha$, $\beta$, $K$, respectively in the proof of the lemma, we have that, if

$$K\delta \leq H \tag{12}$$

holds, so does (5).

Condition (10) is the sufficient convergence criterion for the semi-local convergence of method TGNU I, II. We shall show that (10) can be replaced by (12). Notice that

$$L\delta \leq H \Rightarrow K\delta \leq H. \tag{13}$$

Implication (13) does not imply

$$K\delta \leq H \Rightarrow L\delta \leq H$$

unless, if $K = L$. Hence, the applicability of the methods TGNU I, II studied in Section 3 and Section 4 is extended. Moreover, the error bounds are also improved. Similarly, it is easy to see that

$$4\alpha_1\mu \leq \beta_1^2 \Rightarrow 4\alpha\mu \leq \beta^2, \tag{14}$$

since $\alpha \leq \alpha_1$ and $\frac{\beta_1^2}{\alpha_1^2} \leq \frac{\beta^2}{\alpha^2}$.

Moreover, the implication (14) does not imply

$$4\alpha\mu \leq \beta^2 \Rightarrow 4\alpha_1\mu \leq \beta_1^2, \tag{15}$$

unless, if $K = L$.

We need to define some majorizing sequences for method TGNU I, II. First, we define sequences used in [6]:

$$t_0 = 0,\ t_{k+1} = t_k - \frac{f_{\beta_1}(t_k)}{f'_{1_1}(t_k)} = t_k - \frac{\alpha_1(t_k - t_{k-1})^2 + (1-\beta_1)(t_k - t_{k-1})}{2\alpha_1 t_{k-1}}, \tag{16}$$

$$\hat{t}_0 = 0,\ \hat{t}_{k+1} = \hat{t}_k - \frac{f_{\beta_1}(\hat{t}_k)}{f'_{\beta_1}(t_k)} = \hat{t}_k - \frac{\alpha_1(\hat{t}_k - \hat{t}_{k-1})^2 + (1-\beta_1)(\hat{t}_k - \hat{t}_{k-1})}{2\alpha_1 \hat{t}_k - \beta_1}, \tag{17}$$

where $f_{1_1}(t) = \alpha_1 t^2 - t + \mu$.

The following result was given in [6].

**Lemma 1.** *Let $t_*$, $\{t_k\}$, $\{\hat{t}_k\}$ be given, respectively by* (9), (16) *and* (17). *Suppose that* (8) *holds. Then, the following items hold for each $k = 0, 1, \ldots$*

$$t_k < t_{k+1} < t_*, \quad \hat{t}_k < \hat{t}_{k+1} < t_*, \tag{18}$$

*$\{t_k\}$ and $\{\hat{t}_k\}$ increasingly converge to $t_*$. Moreover,*

$$\hat{t}_{k+1} - \hat{t}_k \le \frac{\mu}{\beta_1} \quad \text{for each} \quad k = 0, 1, \ldots \tag{19}$$

*and for*

$$\gamma_1 = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\alpha_1\mu}}{\beta_1 + \sqrt{\beta_1^2 - 4\alpha_1\mu}}$$

$$t_* - \hat{t}_k = \frac{\gamma_1^{2^k - 1}}{\sum_{i=0}^{2^k-1} \gamma_1^i} t_* \quad \text{for each} \quad k = 0, 1, \ldots. \tag{20}$$

Next, we define the new sequences that shall be shown to be majorizing for method TGNU I, II for $0 \le K_0 \le K$ :

$$\bar{r}_0 = 0, \quad \bar{r}_{k+1} = \bar{r}_k - \frac{(1 - K\bar{r}_k)[\alpha(\bar{r}_k - \bar{r}_{k-1})^2 + (1-\beta)(\bar{r}_k - \bar{r}_{k-1})]}{(1 - K_0\bar{r}_k)(2\alpha_1\bar{r}_k - 1)}, \tag{21}$$

$$r_0 = 0, \quad r_{k+1} = r_k - \frac{\alpha(r_k - r_{k-1})^2 + (1-\beta)(r_k - r_{k-1})}{2\alpha_1 r_k - 1)} = r_k - \frac{f_\beta(r_k)}{f'_\beta(r_k)}, \tag{22}$$

$$\bar{\hat{s}}_0 = 0, \quad \bar{\hat{s}}_{k+1} = \bar{\hat{s}}_k - \frac{(1 - K\bar{\hat{s}}_k)[\alpha(\bar{\hat{s}}_k - \bar{\hat{s}}_{k-1})^2 + (1-\beta)(\bar{\hat{s}}_k - \bar{\hat{s}}_{k-1})]}{(1 - K_0\bar{\hat{s}}_k)(2\alpha\bar{\hat{s}}_k - \beta)} \tag{23}$$

and

$$\hat{s}_0 = 0, \quad \hat{s}_{k+1} = \hat{s}_k - \frac{\alpha(\hat{s}_k - \hat{s}_{k-1})^2 + (1-\beta)(\hat{s}_k - \hat{s}_{k-1})}{2\alpha\hat{s}_k - \beta} = \hat{s}_k - \frac{f_\beta(\hat{s}_k)}{f'_\beta(\hat{s}_k)}. \tag{24}$$

*Remark* 1. (a) It follows from the definition of sequences $\{r_k\}$, $\{\hat{s}_k\}$, $\{\bar{r}_k\}$, $\{\bar{\hat{r}}_k\}$ and a simple inductive argument that for each $k = 1, 2, \ldots$

$$0 \le \bar{r}_k \le r_k, \qquad 0 \le \bar{r}_{k+1} - \bar{r}_k \le r_{k+1} - r_k, \tag{25}$$

$$0 \le \bar{\hat{s}}_{k+1} \le \bar{\hat{s}}_k, \quad \text{and} \quad 0 \le \bar{\hat{s}}_{k+1} - \bar{\hat{s}}_k \le \hat{s}_{k+1} - \hat{s}_k, \tag{26}$$

so sequences $\{\bar{r}_k\}$, $\{\bar{\hat{r}}_k\}$ under the hypotheses of next lemma increasingly converge to their unique least upper bounds $\bar{r}^* = \lim_{k\to+\infty} \bar{r}_k$ and $\bar{\hat{s}}^* = \lim_{k\to+\infty} \bar{\hat{s}}_k$ such that

$$\bar{r}^* \le t^* \qquad \text{and} \qquad \bar{\hat{s}}^* \le t^*. \tag{27}$$

So far sequences $\{\bar{r}_k\}$, $\{\bar{\hat{s}}_k\}$ were shown to converge under the same hypotheses (see (5)) as $\{r_k\}$, $\{\hat{s}_k\}$. However, these sequences may converge under even weaker hypotheses than (5). Such results can be found in [2]–[5], [17].

(b) If we only suppose (8), then as noted above (5) also holds. A simple inductive argument again, the definition of the sequences lead to (25) and (26). Then, clearly we have the analog of Lemma 1 for sequence $\{r_k\}$.

**Lemma 2.** *Let $t_*$, $\{r_k\}$, $\{\hat{r}_k\}$ be given, respectively by* (6)*,* (22) *and* (24)*. Suppose that* (5) *holds. Then, the following items hold*

$$r_k < r_{k+1} < t_*, \qquad \hat{s}_k < \hat{s}_{k+1} < t_*, \tag{28}$$

*$\{r_k\}$ and $\{\hat{s}_k\}$ increasingly converge to $t_*$. Moreover,*

$$\hat{s}_{k+1} - \hat{s}_k \le \frac{\mu}{\beta} \quad \text{for each} \quad k = 0, 1, \ldots \tag{29}$$

*and for* $\gamma = \frac{\beta - \sqrt{\beta^2 - 4\alpha\mu}}{\beta + \sqrt{\beta^2 - 4\alpha\mu}}$

$$t_* - \hat{s}_k = \frac{\gamma^{2^k - 1}}{\sum_{i=0}^{2^k - 1} \gamma^i}. \tag{30}$$

*Proof.* Simply replace $\alpha_1$, $\beta_1$, $\gamma_1$, $t_k$, $\hat{t}_k$, $t^*$, $f_{\beta_1}$, (8) by $\alpha$, $\beta$, $\gamma$, $r_k$, $\hat{s}_k$, $t_*$, $f_\beta$, (5), in the proof of Lemma 1, respectively. □

*Remark* 2. In view of Remark 1 we have

$$0 \le \bar{r}_k \le r_k \le t_k, \quad 0 \le \bar{r}_{k+1} - \bar{r}_k \le r_{k+1} - r_k \le t_{k+1} - t_k \tag{31}$$

and

$$0 \le \bar{\bar{s}}_{k+1} \le \hat{s}_k \le \hat{t}_k, \quad 0 \le \bar{\bar{s}}_{k+1} - \bar{\bar{s}}_k \le \hat{s}_{k+1} - \hat{s}_k \le \hat{t}_{k+1} - \hat{t}_k. \tag{32}$$

Hence, in view of (30)–(32) not only the sufficient convergence criteria are weaker under our new approach (see (5), (12), replacing (8), (10), respectively) but also the error bounds on $\|x_{k+1} - x_k\|$, $\|x_k - x^*\|$ are also improved as well as the information on the location of the solution $x^*$. It is very important to notice that these advantages are obtained under the same computational cost as in [6]. Indeed in practice the computation of parameter $L$ requires the computation of parameters $K_0$ and $K$ as special cases.

## 3. Convergence analysis for algorithm TGNU (I)

We present the semi-local and local convergence analysis of the truncated GN method denoted by Algorithm TGNU ($\{\epsilon_k\}$), for solving problem (1) under condition (45) that follows.

**Algorithm TGNU** ($\{\epsilon_k\}$)
Choose an initial point $x_0 \in \Omega \subseteq \mathbb{R}^n$. For each $k = 0, 1, \ldots$, until convergence, do:
Step 1: Compute $F'(x_k)$.
Step 2: Solve (3) to find $s_k$ such that the

$$\lambda_k := F'(x_k)F'(x_k)^T s_k + F(x_k) \tag{33}$$

satisfies

$$\|F'(x_0)^\dagger \lambda_k\| \leq \epsilon_k. \tag{34}$$

Step 3: Set $x_{k+1} = x_k + F(x_k)^T s_k$.
Let $Q : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator or an $m \times n$ matrix. We denote by $Q^T$ the adjoint of $Q$. We say that a mapping $Q^\dagger : \mathbb{R}^m \to \mathbb{R}^n$ or an $n \times m$ matrix $Q^\dagger$ is the Moore-Penrose inverse of $Q$, if the following estimates are satisfied:

$$QQ^\dagger Q = Q, \quad Q^\dagger QQ^\dagger = Q^\dagger, \quad (QQ^\dagger)^T = QQ^T, \quad (Q^\dagger Q)^T = Q^\dagger Q.$$

Moreover, if $Q$ is of full row rank, then $Q^\dagger = Q^T(QQ^T)^{-1}$ and $QQ^\dagger = I_{\mathbb{R}^m}$, where $I_{\mathbb{R}^m}$ stands for the $m \times m$ identity matrix. Furthermore, it follows easily from the definition of the Moore-Penrose inverse, that $(Q^\dagger P)^\dagger = P^\dagger Q$ if $P$ and $Q$ are full rank.

From now on we shall simply say full rank instead of simply full row rank. More details about the properties of Moore-Penrose inverse can be found in [7, 20, 21]. In the rest of the paper, we suppose that $m \leq n$, unless otherwise stated.

Let $U(w, \delta)$ stand for the open ball in $\mathbb{R}^n$ with center $w$ and of radius $\delta > 0$. Then, $\bar{U}(w, \delta)$ stands for its closure.

From now on, we shall often use the identity

$$x_{k+1} = x_k - F'(x_k)^\dagger F(x_k) + F'(x_k)^\dagger \lambda_k \tag{35}$$

implied by (33) and (34) provided that $F'(x_k)$ is of full rank.

Let $x_0 \in \mathbb{R}^n$. Define $R := \sup\{t \geq 0 : U(x_0, t) \subseteq \Omega\}$. Let also $\|\cdot\|$ be the Euclidean vector norm or the induced matrix norm and set $\Omega_0 = U(x_0, R)$.

We need the definition of the center Lipschitz condition.

**Definition 1.** A nonlinear operator $G(\cdot) : \Omega_0 \to \mathbb{R}^m$ is said to satisfy the center Lipschitz condition at $x_0$ with center Lipschitz constant $K_0$ on $\Omega_0$, if

$$\|G(x) - G(x_0)\| \leq K_0\|x - x_0\| \quad \text{for each} \quad x, y \in \Omega_0. \tag{36}$$

Define $U_0 := \Omega_0 \cap U(x_0, \frac{1}{K_0})$.

**Definition 2.** A nonlinear operator $G(\cdot) : \Omega_0 \to \mathbb{R}^m$ is said to satisfy the restricted Lipschitz condition with Lipschitz constant $K$ on $U_0$, if

$$\|G(x) - G(y)\| \leq K\|x - y\| \quad \text{for each} \quad x, y \in U_0. \tag{37}$$

In earlier studies [6]–[22] the following condition was used instead of the combination of Definition 1 and Definition 2 that we shall use in the present study.

**Definition 3.** A nonlinear operator $G(\cdot) : \Omega_0 \to \mathbb{R}^m$ is said to satisfy the Lipschitz condition with Lipschitz constant $L$ on $\Omega_0$, if

$$\|G(x) - G(y)\| \leq L\|x - y\| \quad \text{for each} \quad x, y \in \Omega_0. \tag{38}$$

Notice, however that $U_0 \subseteq \Omega_0$. Hence, we have that

$$K \leq L. \tag{39}$$

We also have by Definition 1 and Definition 3 that

$$K_0 \leq L \tag{40}$$

and $\frac{L}{K_0}$ can be arbitrarily large [3]–[5].

In the rest of the paper, we shall assume that

$$K_0 \leq K. \tag{41}$$

Otherwise, i.e., if $K \leq K_0$ the results to follow will also hold with $K_0$ replacing $K$. Let us see why the new results improve the earlier results in [6]:

Let $x_0 \in \Omega_0$ be such that $F'(x_0)$ is of full row rank. Set $G(x) = F'(x_0)^\dagger F'(x)$. We need the auxiliary result.

**Lemma 3.** *Let $x_0 \in \Omega_0$ be such that $F'(x_0)$ is of full row rank. Suppose that $F'(x_0)^\dagger F'(x)$ satisfies the Lipschitz condition with Lipschitz constant $L$ on $\Omega_0$. Then, for each $x \in U_0$, $F'(x)$ is of full row rank and*

$$\|F'(x)^\dagger F'(x_0)\| \leq \frac{1}{1 - L\|x - x_0\|}. \tag{42}$$

If we simply use the more precise and needed $K_0$ instead of $L$ used in the proof of the preceding lemma, we obtain:

**Lemma 4.** *Let $x_0 \in \Omega_0$ be such that $F'(x_0)$ is of full row rank. Suppose that $F'(x_0)^\dagger F'(x)$ satisfies the center-Lipschitz condition with Lipschitz constant $K_0$ on $\Omega_0$. Then, for each $x \in U_0$, $F'(x)$ is of full row rank and*

$$\|F'(x)^\dagger F'(x_0)\| \leq \frac{1}{1 - K_0\|x - x_0\|}. \tag{43}$$

Notice that in view of (40), (43) is more precise than (42) on the norm $\|F'(x)^\dagger F'(x_0)\|$. That exchange of upper bounds in the proofs of the results leads to a tighter convergence analysis. On the other hand, $K$ can replace $L$ on the upper bounds on $\|F'(x_0)^\dagger F'(x_k)\|$ leading again to more precise uppers bounds on this norm. Then, in view of (41), we can reproduce all the proofs of the semi-local results in [6] with simply $K$, $K_0$ replacing $L$, $L_0$, respectively.

The same technique can be used to improve the local results in [6]. It is convenient for the semi-local convergence analysis that follows to introduce some parameter, sequences and conditions.

Set

$$\rho = \|F'(x_0)^\dagger F(x_0)\|. \tag{44}$$

Let $\{\eta_k\}$ be a non-negative sequence satisfying $0 \leq \eta := \sup_{k\geq 0} \eta_k < 1$ for some $\eta \in [0, 1)$. We shall suppose that

$$\epsilon_k \leq \|F'(x_0)^\dagger F(x_k)\| \tag{45}$$

and

$$\epsilon_k \leq \|F'(x_0)^\dagger F(x_k)\|^2. \tag{46}$$

Next, we present the semi-local convergence analysis of sequence $\{x_n\}$ generated by the Algorithm TGNU ($\{\epsilon_k\}$).

**Theorem 5.** *Suppose that there exists $x_0 \in \Omega_0$ such that $F'(x_0)$ is of full row rank and $U(x_0, R) \subseteq \Omega$. Moreover, suppose that $F'(x_0)^\dagger F'(\cdot)$ satisfies the center-Lipschitz condition at $x_0$ with Lipschitz constant $K_0$ on $\Omega_0$ and the restricted Lipschitz condition with constant $K$ on $U_0$, (12) holds and $r^* \leq R$, where $R$, $K_0$, $K$, $\Omega_0$, $U_0$ and $R^*$ are defined previously. Then, sequence $\{x_k\}$ generated by (35) with starting point $x_0$ by the method with sequence $\{\epsilon_k\}$ satisfying (45) is well defined, remains in $U(x_0, r^*)$ for each $k = 0, 1, \ldots$ and converges to a solution $x_*$ of equation $F(x) = 0$ in $\bar{U}(x_0, r^*)$.*

*Moreover, the following error bound holds*:

$$\|x_k - x_*\| \leq r^* - r_k \quad \textit{for each} \quad k = 0, 1, \ldots. \tag{47}$$

*Proof.* By simply following the proof of [6, Theorem 3.1], replacing sequence $\{t_k\}$ by $\{r_k\}$ and the $L_0$, $L$ by $K_0$, $K$, respectively, we obtain the estimates with the crucial differences (see also Remark 3.7).

The estimates for $j = 0, 1, \ldots, k$

$$\|F'(x_j)^\dagger F'(x_0)\| \leq \frac{1}{1 - K_0\|x_j - x_0\|} \leq \frac{1}{1 - K_0 r_j}, \tag{48}$$

$$\|F'(x_0)^\dagger F'(x_j)\| \leq \frac{(1 - K r_j)(r_{j+1} - r_j}{1 + \omega}, \tag{49}$$

and by (22), (35), (48), (49) and

$$\begin{aligned} \|x_{j+1} - x_j\| &= \|F'(x_j)^\dagger F'(x_0)(-F'(x_0)^\dagger F(x_j) + F'(x_0)^\dagger \lambda_j\| \\ &\leq (1 + K)\|F'(x_j)^\dagger F'(x_0)\|\|F'(x_0)^\dagger F(x_j)\| \\ &\leq \bar{r}_{j+1} - \bar{r}_j \leq r_{j+1} - r_j, \end{aligned} \tag{50}$$

so

$$\|x_{k+1} - x_k\| \leq \bar{r}_{j+1} - \bar{r}_j \leq r_{j+1} - r_j. \tag{51}$$

The rest of the proof follows as in [6, Theorem 3.1] with the noted modifications. □

*Remark* 3. (a) The corresponding estimates in [6] are less tight (see (5), (6), (8), (9), (10), (12), (13)–(15)),

$$\|F'(x_j)^\dagger F'(x_0)\| \leq \frac{1}{1 - L\|x_j - x_0\|} \leq \frac{1}{1 - Lt_j},$$

$$\|F'(x_0)^\dagger F'(x_j)\| \leq \frac{(1 - Lt_j)(t_{j+1} - t_j)}{1 + \omega}$$

and

$$\|x_{j+1} - x_j\| \leq t_{j+1} - t_j.$$

(b) If $m = n$, (45) and (46) reduce to

$$\begin{aligned} &\|F'(x_0)^{-1}\lambda_k\| \leq \epsilon_k \|F'(x_0)^{-1}F(x_k)\| \text{ and} \\ &\|F'(x_0)^{-1}\lambda_k\| \leq \epsilon_k \|F'(x_0)^{-1}F(x_k)\|^2 \quad k = 0, 1, \ldots. \end{aligned} \tag{52}$$

This way we obtain the next corollary that follows for inexact Newton methods [6, 13, 15, 22]. This corollary improves Corollary 3.1 (of Theorem 3.1 in [13]) which in turn improved the results in [13, 15].

**Corollary 6.** *Suppose that $m = n$ and there exists $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $U(x_0, R) \subseteq \Omega$. Moreover, suppose that $F'(x_0)^{-1}F'(\cdot)$ satisfies the center-Lipschitz condition at $x_0$ with center Lipschitz constant $K_0$ on $\Omega_0$ and the restricted Lipschitz condition with Lipschitz constant $K$ on $U_0$,* (12) *holds and $r^* \leq R_0$. Then, sequence $\{x_k\}$ generated by method* (35) (*with $F'(x)^\dagger = F'(x)^{-1}$*) *with sequence $\{\epsilon_k\}$ satisfying* (46) *is well defined, remains in $U(x_0, r^*)$ and converges to a solution $x_*$ of equation $F(x) = 0$ in $\bar{U}(x_0, r^*)$.*

*Moreover, the following error bound holds*:

$$\|x_k - x_*\| \leq r^* - r_k \quad \text{for each} \quad k = 0, 1, \ldots. \tag{53}$$

Next, we present the local convergence analysis of method (35) with (12) satisfied using Theorem 5. To achieve this, we define

$$\hat{K} = \frac{K}{1 - K_0 \varrho} \quad \text{for some} \quad \varrho > 0 \quad \text{to be determined later.} \tag{54}$$

Notice that

$$\hat{L} = \frac{L}{1 - \hat{L}\varrho} \tag{55}$$

was used in [6]. We have that

$$\hat{K} \leq \hat{L}. \tag{56}$$

This modification leads to advantages similar to the semi-local convergence case and under the same computational cost (see also Remark 4 and the numerical section).

The proofs of the next three results are obtained from the corresponding ones in Lemma 3.2, Theorem 3.2 and Corollary 3.2 in [6], respectively by using the modification already suggested in Theorem 5. So these proofs are omitted (see also Remark 4).

**Lemma 7.** *Suppose there exists $x^* \in \Omega$ solving equation $F(x) = 0$, such that $F'(x^*)$ is of full rank and $U(x^*, R) \subseteq \Omega$. Moreover, suppose that $F'(x^*)^\dagger F'(\cdot)$ satisfies the center-Lipschitz condition at $x^*$ with constant $K_0$ on $\Omega_0$* (*with $x_0 = x^*$*) *and the restricted Lipschitz condition with Lipschitz constant $K$ on $U_0$* (*with $x_0 = x^*$*). *Furthermore, suppose that $0 < \varrho < \{R, \frac{1}{K_0}\}$. Then, the following items hold for each $x \in U(x^*, r)$*:

(1) *$F'(x_0)$ is of full rank and $\|F'(x_0)^\dagger F'(x^*)\| \le \frac{1}{1-K_0\varrho}$;*
(2) *$F'(x_0)^\dagger F'(\cdot)$ satisfies the Lipschitz condition with Lipschitz constant $\hat{K}$ given in (54) on $U(x_0, R-\varrho)$;*
(3) *$\delta \le \frac{K\varrho^2}{2(1-K_0\varrho)} + \varrho$, where $\delta := \|F'(x_0)^\dagger F(x_0)\|$.*

**Theorem 8.** *Suppose that there exists $x^* \in \Omega$ solving equation $F(x) = 0$, so that $F'(x^*)$ is of full rank and $U(x^*, R) \subseteq D$. Moreover, suppose that $F'(x^*)^\dagger F'(\cdot)$ satisfies the center-Lipschitz condition at $x_0$ with constant $K_0$ on $D_0$ and the restricted Lipschitz condition with Lipschitz constant $K$ on $U_0$. Define*

$$\varrho := \min\left\{\frac{(1-\omega)R}{1-\omega+4(1+\omega)^2},\ \frac{1}{K}\left(1-\frac{1}{\sqrt{1+2H}}\right)\right\}. \tag{57}$$

*Then, sequence $\{x_k\}$ generated by method (35) for $x_0 \in U(x^*, \varrho)$ and $\{\epsilon_k\}$ satisfying (45) is well defined, remains in $U(x^*, \varrho)$ for each $k = 0, 1, 2, \ldots$ and converges to a solution $\hat{x}_*$ of equation $F(x) = 0$.*

We also have the following corollary of Theorem 8 for the GN method.

**Corollary 9.** *Suppose that there exists $x^* \in \Omega$ solving equation $F(x) = 0$ such that $F'(x^*)$ is of full rank and $U(x^*, R) \subseteq D$. Suppose that $F'(x^*)^\dagger F'(\cdot)$ satisfies the center-Lipschitz condition with constant $K_0$ on $D_0$ and the restricted Lipschitz condition with Lipschitz constant $K$ on $U_0$. Define*

$$\bar{\varrho} := \min\left\{\frac{R}{5},\ \frac{2-\sqrt{2}}{2K}\right\}. \tag{58}$$

*Then, sequence $\{x_k\}$ generated for $x_0 \in U(x^*, \bar{\varrho})$ by GN method, is well defined, remains in $U(x^*, \bar{\varrho})$ for each $k = 0, 1, 2, \ldots$ and converges to a solution $\hat{x}_*$ of equation $F(x) = 0$.*

*Remark* 4. (a) The radii of convergence in [6] are given respectively by,

$$\varrho_1 := \min\left\{\frac{(1-\omega)R}{1-\omega+4(1+\omega)^2},\ \frac{1}{L}\left(1-\frac{1}{\sqrt{1+2H}}\right)\right\} \tag{59}$$

and

$$\bar{\varrho}_1 := \min\left\{\frac{R}{5},\ \frac{2-\sqrt{2}}{2L}\right\}. \tag{60}$$

Then, we have that

$$\varrho_1 \le \varrho \tag{61}$$

and

$$\bar{\varrho}_1 \le \bar{\varrho}. \tag{62}$$

That is we obtain an at least as large radius of convergence leading to a wider choice of initial points. Moreover, as already shown in Section 2 the error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - \hat{x}_*\|$ are tighter leading to fewer iterations to achieve a desired error tolerance. It is worth noticing that the preceding advantages are obtained under the same computational effort as in [6], since in practice the computation of $L$ requires the computation of $K_0$ and $K$ as special cases.

(b) The results obtained here can be improved even further, if we consider instead of the ball $U_0$ the ball $U_1$ defined by

$$U_1 := \Omega \cap U\left(x_1, \frac{1}{K_0} - \|F'(x_0) \dagger F(x_0)\|\right).$$

Notice that $U_1 \subseteq U_0 \subseteq U_0 \subseteq \Omega$, so the Lipschitz constant $K$ can be replaced by a constant $\hat{K}$ at least at small. Then, $\hat{K}$ can replace $K$ in all preceding results. The iterates $\{x_k\}$ lie in $U_1$ according to the proofs (see also the numerical examples).

## 4. Convergence analysis for algorithm TGNU (II)

We present the semi-local as well as the local convergence analysis of Algorithm TGNU (see method (35)) for solving problem (1.1) under condition (46) along the same lines of Section 4 but some of the parameters are defined differently. We also use sequences $\{\bar{\hat{s}}_k\}$, $\{\hat{s}_k\}$ instead of $\{\bar{\hat{r}}_k\}$, $\{\hat{r}_k\}$, respectively (or $\{t_k\}$, $\{\hat{t}_k\}$, respectively in [6]).

The proofs are obtained as in Section 3 with the above modifications. Therefore, these proofs are omitted.

As in Section 3, it is convenient to introduce some parameters

$$\alpha := \frac{K(1+\omega\delta)}{2[1+K\delta^2\omega(1+\delta\omega)]} + \frac{\omega}{1+\omega(\delta-1)}, \quad \beta := 1 + \frac{\omega}{1+\omega(\delta-1)} \tag{63}$$

and

$$\mu := 1 + \beta\delta(1+\omega\delta). \tag{64}$$

If

$$\delta \leq \frac{1}{\sqrt{(K+2\omega)^2 + 2K\omega} + K + 2\omega}, \tag{65}$$

then (5) holds.

**Theorem 10.** *Suppose that there exists $x_0 \in \Omega$ such that $F'(x_0)$ is of full rank and $U(x_0, R) \subseteq \Omega$. Moreover, suppose that $F'(x_0)^\dagger F'(\cdot)$ satisfies the center-Lipschitz condition at $x_0$ with Lipschitz constant $K_0$ on $\Omega_0$, the restricted Lipschitz condition with constant $K$ on $U_0$,* (65) *holds and $\hat{s}^* \leq R$ and $\{\epsilon_k\}$ satisfies* (46)*. Then, sequence $\{x_k\}$ generated by method* (35) *is well defined, remains in $U(x_0, \hat{s}^*)$ for each $k = 0, 1, \ldots$ and converges quadratically to a* *solution $x_*$ of equation $F(x) = 0$ in $\bar{U}(x_0, \bar{r}^*)$. Moreover, the following error bound holds*:

$$\|x_k - x^*\| \le \frac{\gamma^{2^k-1}}{\sum_{i=0}^{2^k-1} \gamma^i} \bar{r}^*, \tag{66}$$

*where $\bar{r}^*$, $\gamma$ are given in Section 2 for $\alpha$, $\beta$, $\mu$ given in* (61) *and* (62).

*Remark* 5. (a) Let $H(\omega) := H$ be defined by (11) for each $\omega \in (0, 1)$ and let $\hat{H}$ be defined by

$$\hat{H}(\omega, K) = \frac{K}{\sqrt{(K + 2\omega)^2 + 2K\omega} + K + 2\omega} \tag{67}$$

for each $K \in [0, +\infty)$ and $\omega \in (0, 1)$. Then, (12) and (65) are equivalent to

$$K\delta \le \hat{H}(\omega, K) \tag{68}$$

and

$$K\delta \le \hat{H}(\omega), \tag{69}$$

respectively.

(b) If $\epsilon_k \equiv 0$, method (3.3) is reduced to the GN method. Choose $\omega_k \equiv 0$. Then, we have

$$H = \frac{1}{2}, \quad \bar{r}^* = \frac{1 - \sqrt{1 - 2K\delta}}{K}, \quad \text{and} \quad \gamma = \frac{1 - \sqrt{1 - 2K\delta}}{1 + \sqrt{1 - 2K\delta}}. \tag{70}$$

We obtain from Theorem 10 the following improvement of the Kantorovich-like theorem for the GN method.

**Corollary 11.** *Suppose that there exists $x_0 \in \Omega$ such that $F'(x_0)$ is of full rank and $U(x_0, R) \subseteq \Omega$. Moreover, suppose that $F'(x_0)^\dagger F'(\cdot)$ satisfies the center-Lipschitz condition with Lipschitz constant $K_0$ on $\Omega_0$, the restricted Lipschitz condition with constant $K$ on $U_0$. Furthermore, suppose that*

$$0 < K\delta \le \frac{1}{2} \quad \text{and} \quad \bar{s}^* \le R_0. \tag{71}$$

*Then, sequence $\{x_k\}$ generated by method* (35) *is well defined, remains in $U(x_0, \bar{s}^*)$ for each $k = 0, 1, \ldots$ and converges quadratically to a solution $x_*$ of equation $F(x) = 0$ in $\bar{U}(x_0, \bar{s}^*)$. Moreover, estimate* (66) *holds, where $\bar{s}^*$ and $\gamma$ are given in* (70).

Next, we present the local convergence analysis of method (35) under condition (46).

**Theorem 12.** *Suppose there exists $x^* \in \Omega$ solving equation $F(x) = 0$ such that $F'(x^*)$ is of full rank and $U(x^*, R) \subseteq \Omega$. Moreover, suppose that $F'(x^*)^\dagger F'(\cdot)$*

*satisfies the center-Lipschitz condition with Lipschitz constant $K_0$ on $\Omega_0$, and the restricted Lipschitz condition with constant $K$ on $U_0$. Define*

$$\varrho := \min\left\{\frac{2}{11}R,\ \frac{1}{3(\sqrt{(K+\omega)^2+K\omega}+K+\omega)}\right\}. \tag{72}$$

*Then, sequence $\{x_k\}$ generated for $x_0 \in U(x^*,\varrho)$ by method* (35) *with sequence $\{\epsilon_k\}$ satisfying* (46) *is well defined, remains in $U(x^*,\varrho)$ for each $k = 0,1,\ldots$ and converges quadratically to a solution $\hat{x}_*$ of equation $F(x) = 0$.*

*Remark* 6. (a) The results of this section improve the corresponding results Lemma 3.3, Theorem 3.3, Remark 3.4, Corollary 3.3 and Theorem 3.4 in [6] along the same lines of the arguments and comparison made in Remarks of Section 2, Section 3 and this section.

(b) The rest of the results in Section 4 in [6] are also immediately improved along the same lines. However, we leave the details to the motivated reader.

## 5. Numerical examples

We present numerical examples to show that the earlier results do not apply or if they apply, our results also apply and can do better. For simplicity we choose $m = n$ in the next two examples. The first example is given for the semi-local case.

**Example 1.** Let $m = n = 1$, $\Omega_0 = \Omega = U(0,1)$, $\omega = 0$, $\lambda_k = 0$, $x_0 = 1$, $R = 1$. Define function $F$ on $\Omega_0$ by

$$F(x) = x^3 - h \quad \text{for some} \quad h \in (0, \frac{1}{2}). \tag{73}$$

Then, we have $\delta = \frac{1}{3}(1-h)$, $K_0 = 3-h$, $L = 2(2-h)$, $\alpha_1 = \frac{L}{2} = 2-h$, $\beta_1 = 1$ and $\mu = \delta$. Then, old condition (8) is not satisfied, since

$$4\alpha_1\mu > \beta_1^2 \quad \text{for each} \quad h \in (0, \frac{1}{2}). \tag{74}$$

Notice also that condition (8) in this special case is the famous for its simplicity and clarity Kantorovich sufficient convergence criterion for the convergence of Newton's method [6,15–17] (see also (71) with $K = L$). Therefore, there is no guarantee under the old results that Newton's method converges to $x^* = \sqrt[3]{h}$ starting at $x_0 = 1$.

Using our results, we have

Case: $U_0 = \Omega_0 \cap U\left(x_0, \frac{1}{K_0}\right) = U\left(x_0, \frac{1}{K_0}\right)$.

Then, $F'(x_0)^{-1}F'(\cdot)$ is restricted Lipschitz with $K = 2\left(\frac{4-h}{3-h}\right)$. We also have that $\alpha = \frac{K}{2}$, $\beta = 1$ and $\mu = \delta$.

Then, condition (5) becomes

$$\frac{4}{3}\frac{(4-h)(1-h)}{3-h} \le 1 \tag{75}$$

which is satisfied for $h \in I_0 := [.46198316\ldots, \frac{1}{2})$. The same range for $h$ is obtained, if we use (71).

Case: $U_1 = \Omega_0 \cap U\left(x_1, \frac{1}{K_0} - \delta\right)$.

We need the computation

$$\begin{aligned}\|F'(x_0)^{-1}(F'(x) - F'(y))\| &\le (\|x - x_1\| + \|y - x_1\| + 2\|x_1\|)\|x - y\| \\ &\le 2\left[\left(\frac{1}{K_0} - \delta\right) + \frac{2+h}{3}\right]\|x - y\|,\end{aligned}$$

so

$$\hat{K} = \frac{-2(2h^2 - 5h - 6)}{3 - h}.$$

Using these values, (5) is satisfied for all $h \in I := \left[.44137239, \frac{1}{2}\right]$ which extends the interval found in preceding case. Notice also that $\delta K_0 < 1$ for all $h \in I$. Therefore, if $h \in I_0$ or $h \in I$ our results guarantee the convergence of Newton's method to $x^*$ starting at $x_0 = 1$.

Finally, as already noted in Section 2, if both (5) and (8) hold, then our results provide tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ and a more precise information on the location of the solution $x^*$. Notice also that $K_0 < L$ and $\hat{K} < L$ for all $h \in [0, \frac{1}{2})$ and

$$K_0 < \hat{K} \quad \text{if} \quad h > 2 - \sqrt{3},$$
$$K_0 > \hat{K} \quad \text{if} \quad h < 2 - \sqrt{3}$$

and

$$K_0 = \hat{K} \quad \text{if} \quad h = 2 - \sqrt{3}.$$

The second example concerns the local case.

**Example 2.** Let $m = n = 3$ and $\Omega = U(x^*, 1)$, so $R = 1$ and $\Omega_0 = \Omega$. Choose $\lambda_k = 0$, so $\omega = 0$. Define mapping $F$ on $\Omega$ for $w = (x, y, z)^T$ by

$$F(w) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T. \tag{76}$$

Then, we have for $x^* = (0, 0, 0)^T$ that the Fréchet-derivative is given by

$$F'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{77}$$

We obtain $L = e$, $K_0 = e - 1$, and $U_0 = \Omega_0 \cap U\left(x^*, \frac{1}{K_0}\right) = U\left(x^*, \frac{1}{K_0}\right)$, $K = e^{\frac{1}{K_0}}$. By using (57)–(60), we get that

$$\varrho = \bar{\varrho} = .16366659$$

and

$$\varrho_1 = \bar{\varrho}_1 = .10172259,$$

so $\varrho_1 = \bar{\varrho}_1 < \varrho = \bar{\varrho}$. Hence, our convergence radii are larger than the ones in [6]. It is worth noticing that larger radius of convergence implies a wider choice of initial points and fewer iterations to achieve a desired error tolerance. These improvements are important in computational mathematics.

## 6. Conclusion

In this paper, semi-local as well as local convergence results of Gauss-Newton method by using a restricted convergence domain have been obtained for solving problem (1), extending the applicability of the method under the same computational cost as in [6]. This idea has been used to obtain a tighter local convergence analysis and an at least as precise complexity for Newton iteration as in earlier studies [2]–[5]. The idea can be used on other iterative methods [2]–[22].

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