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# ON THE DEGENERATE MAXIMAL SPACELIKE SURFACES

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Abstract. The purpose of this paper is to investigate various kinds of degeneracy of maximal surfaces in $\mathbb{L}^n$ in view of the generalized Gauss map.

## 1. Introduction

There are three important ways in which a maximal surface in $\mathbb{L}^n$ may be degenerate. Many statements that we wish to make hold in full generality only after certain kinds of degeneracy are excluded. On the other hand, the degenerate surfaces are of particular interest in itself.

We adopt the notations in [5]. Denote by $M$ a Riemannian surface, and define a maximal (spacelike) surface $\mathbb{S}$ in $\mathbb{L}^n$ by an immersion (or embedding) $X : M \longrightarrow \mathbb{L}^n$, where local coordinates $u^1$ , $u^2$ on $M$ serve as isothermal parameters for the surface and $z = u^1 + iu^2$ as a complex coordinate on $M$. The Gauss map $\Phi(z) = (\phi_1(z), \phi_2(z), \cdots, \phi_n(z))$ from $M$ into $\mathbb{Q}_+^{n-2}$ is given in local complex coordinate on $M$ as in [5]. Note that the indefinite Fubini-Study metric $ds^2$ on $CP_+^{n-1}$ is given by $ds_o^2 = \pi^* ds^2$. Here $\pi : \mathbb{C}_1^n(+) \longrightarrow CP_+^{n-1}$ and

$$ds_o^2 = 2\frac{\sum_{j<k} \epsilon_j \mid z_j dz_k - z_k dz_j \mid^2}{(-z_1\overline{z_1} + z_2\overline{z_2} + \cdots z_n\overline{z_n})^2} \quad , \tag{1}$$

where $\epsilon_1 = -1$, and $\epsilon_j = 1$ otherwise.

We adopt terminologies about character of subspace of $CP^{n-1}$ naturally so that the image of a spacelike subspace $H$ of $\mathbb{C}_1^n$ under the natural projection $\pi : \mathbb{C}_1^n \longrightarrow CP^{n-1}$ is also called a spacelike subspace of $CP^{n-1}$, and so on.

## 2. On the Degenerate Maximal Surfaces

**Definition 1.** The maximal surface $\mathbb{S}$ *lies fully* in $\mathbb{L}^n$ if the image $X(M)$ does not lie in any proper affine subspace of $\mathbb{L}^n$, and *degenerate of the first kind* if its Gaussian image $\Phi(M)$ lies fully in a spacelike subspace of $CP^{n-1}$, *degenerate of*

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*the second kind* if its Gaussian image $\Phi(M)$ lies fully in a timelike subspace of $CP^{n-1}$, *degenerate of the third kind* if its Gaussian image $\Phi(M)$ lies fully in a null subspace of $CP^{n-1}$, and is *k-degenerate* if $k$ is the largest integer such that the image under Gauss map $\Phi(M)$ lies in a projective subspace of codimension $k$ in $CP^{n-1}$. The surface $\mathbb{S}$ is *decomposable* if , with respect to some orthonormal basis in $\mathbb{L}^n$, the functions $\phi_j$ satisfy

$$-\phi_1^2 + \phi_2^2 + \cdots \phi_k^2 \equiv 0, \phi_{k+1}^2 + \cdots + \phi_n^2 \equiv 0 \tag{2}$$

for some $k$, $1 \leq k < n$, and *h-decomposable* if $h$ is the smallest number of $k$ for which (2) holds after suitable change of coordinates. Especially, the composability is called *the first kind* if $-|\phi_1|^2 + |\phi_2|^2 + \cdots |\phi_k|^2 > 0$, *the second kind*, otherwise. The Gauss map is said to lie in a *real spacelike* hyperplane $\sum_k \epsilon_k a_k z_k = 0$ if the vector $A = (a_1, \cdots, a_n)$ may be chosen to be real, timelike, and *lie in a tangent hyperplane* if the vector $A = (a_1, \cdots, a_n)$ satisfies $\sum_{k=1}^n \epsilon_k {a_k}^2 = 0$.

*Remark* 1. $\mathbb{S}$ is degenerate of the first kind if there exists a nonzero timelike vector $A = (a_1, \cdots, a_n)$ in $C_1^n$ such that

$$\sum_{j=1}^n \epsilon_j a_j \phi_j \equiv 0 \quad . \tag{3}$$

Furthermore, $\mathbb{S}$ is k-degenerate of the first kind if we can find exactly k-orthonormal vectors $A_1, A_2, \cdots, A_k$ in $\mathbb{C}_1^n$ for which such an equation holds, where $A_1$ is timelike.

**Proposition 2.1.** *Let $\mathbb{S}$ be a maximal surface in $\mathbb{L}^n$, and $\hat{\mathbb{S}}$ its image under the Gauss map.*

(1) *The following statements are equivalent:*
   (a) *$\mathbb{S}$ lies fully in a spacelike affine hyperplane of $\mathbb{L}^n$.*
   (b) *$\mathbb{S}$ is 1-decomposable (of the second-kind).*
   (c) *$\mathbb{S}$ is degenerate with $\hat{\mathbb{S}}$ lying fully in a real spaceike hyperplane.*
(2) *If $\mathbb{S}$ is 2-decomposable (of the second kind), then $\mathbb{S}$ cannot lie fully in $\mathbb{L}^n$, and $\mathbb{S}$ is degenerate, with $\hat{\mathbb{S}}$ lying in a real null tangent hyperplane.*
(3) *$\mathbb{S}$ is 2-decomposable if and only if there exists a direct a direct sum decompositions of $\mathbb{L}^n$ into $\mathbb{L}^2 \oplus \mathbb{R}^{n-2}$ with respect to which $\mathbb{S}$ becomes the direct sum of a lightlike line in $\mathbb{L}^2$ and a minimal surface in $\mathbb{R}^{n-2}$.*

*Proof.* (1) Suppose $\mathbb{S}$ is defined by an immersion $X : M \longrightarrow \mathbb{L}^n$. When $\mathbb{S}$ lies fully in a spacelike affine hyperplane of $\mathbb{L}^n$, there is a timelike vector $A = (a_1, \cdots, a_n) \in \mathbb{L}^n$ such that

$$\sum_{k=1}^n \epsilon_k a_k x_k \equiv constant. \tag{4}$$

Under an orthonormal change of coordinates so that $A = e_1$, the hyperplane can be put in the form $\tilde{x}_1 \equiv \textit{constant}$, which is equivalent to $\phi_1 \equiv 0$. Hence (a) is equivalent to (b).

In terms of a local isothermal parameter $u = u^1 + iu^2$ on $M$, we have

$$x_k = Re \int \phi_k du \, , \tag{5}$$

in oter words, $\phi_k = \frac{\partial x_k}{\partial u^1} - i\frac{\partial x_k}{\partial u^2}$. Hence $\sum \epsilon_k a_k \phi_k \equiv 0$ if and only if $\sum \epsilon_k a_k \frac{\partial x_k}{\partial u^1} \equiv \sum \epsilon_k a_k \frac{\partial x_k}{\partial u^2} \equiv 0$ which is equivalent to the statement $\sum \epsilon_k a_k x_k \equiv \textit{constant}$. Hence (a) is equivalent to (c).

(2) If $\mathbb{S}$ is 2-decomposable, then $\phi_1 \equiv \phi_2$ or $\phi_1 \equiv -\phi_2$. Hence $x_1 + x_2$ or $x_1 - x_2$ are constant and $\mathbb{S}$ should lie on a null affine hyperplane of $\mathbb{L}^n$. Since $(1, 1, 0, \cdots, 0)$ and $(1, -1, 0, \cdots, 0)$ are in $\mathbb{Q}^{n-2}$, clearly $\mathbb{S}$ is degenerate, with $\hat{\mathbb{S}}$ lying in a real null tangent hyperplane.

(3) This is clear from Theorem 2.2 [6].

□

Before we state the following proposition, we have to mention what the constant Gauss map means. Note the following equivalent statements:

(1) $\mathbb{S}$ lies on a plane in $\mathbb{L}^n$.
(2) $\mathbb{S}$ has the same tangent plane everywhere.
(3) Gauss map is constant.
(4) $\frac{\phi_k}{\phi_j} = \textit{constant}$ for any $j$, $k$.

Futhermore, when $\mathbb{S}$ is a maximal surface in $\mathbb{L}^3$ with $K \equiv 0$, then the self-adjoint shape operator $A_p : T_pM \longrightarrow T_pM$ is zero everywhere, which means $\mathbb{S}$ is in fact a portion of a spacelike plane in $\mathbb{L}^3$.

The following proposition follows immediately from the above observation together with Proposition 2.1.

**Proposition 2.2.** *In $\mathbb{L}^3$, the following statements are equivalent:*

(1) $\mathbb{S}$ *is 1-decomposable.*
(2) $\mathbb{S}$ *id degenerate of the first kind.*
(3) $\mathbb{S}$ *has a constant Gauss map.*
(4) $\mathbb{S}$ *lies in a spacelike plane.*
(5) *The Gaussian curvature $K$ vanishes everywhere.*

*Remark* 2. In $\mathbb{L}^3$, $\mathbb{S}$ cannot be 2-decomposable since $-\phi_1^2 + \phi_2^2 = 0$, $\phi_3^2 = 0$ implies $-|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 = 0$, a contradiction to the fact that $-|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0$.

On the other hand, in either $\mathbb{L}^4$ or $\mathbb{L}^5$ a decomposable maximal surface must be degenerate, which can be proved easily.

However, in $\mathbb{L}^6$ there exists a decomposable surface that is not degenerate. Consider the direct sum of a maximal surface $X$ in $\mathbb{L}^3$ induced from the Gauss map $\Phi(z) = \left(\frac{z^2+1}{2}, \frac{z^2-1}{2}, z\right)$ and a minimal surface $\tilde{X}$ in $\mathbb{R}^3$ induced from the

Gauss map $\tilde{\Phi}(z) = \left(\frac{z^3(1-z^4)}{2}, \frac{iz^3(1+z^4)}{2}, z^5\right)$ defined on the upper-half plane of $\mathbb{C}$. Then we can easily see that the corresponding Gauss map $\Phi$, $\tilde{\Phi}$ do not satisfy any nontrivial linear equation of the form $\sum \epsilon_k a_k \phi_k + \sum b_k \tilde{\phi}_k \equiv 0$.

Let $\mathbb{S}$ be an (orientable) spacelike surface defined by an immersion $X : M \longrightarrow \mathbb{L}^n$ when $M$ is a Riemann surface. Locally we can define the Gauss map by $\phi(u) = \frac{\partial x_k}{\partial u^1} - i\frac{\partial x_k}{\partial u^2}$ for an isothermal parameter $u = u^1 + iu^2$. If $v = v^1 + iv^2$ is another isothermal parameter, then $\tilde{\phi}_k(v) = \phi_k(u)\frac{du}{dv}$, where $\tilde{\phi}(v) = \frac{\partial x_k}{\partial v^1} - i\frac{\partial x_k}{\partial v^2}$. Hence $\phi_k(u)du = \tilde{\phi}_k(v)dv$, which will give us global differentials $\alpha_k = \phi_k(u)du$ on $\mathbb{S}$.

**Proposition 2.3.** *(1) A hyperplane $H$ through the origin in $\mathbb{C}^n_1$ is degenerate if and only if $H$ is defined by a linear equation*

$$z_1 = \sum_{k=2}^{n} a_k z_k \ , \tag{6}$$

*where $\sum_{k=2}^n |a_k|^2 = 1$.*

*(2) $H$ is degenerate hyperplane defined by an equation (6) if and only if there exists an element in $U(1, n-1)$ such that $H$ is mapped onto the hyperplane defined by $z_1 = z_2$ under the transformation.*

*Proof.* Degenerate hyperplane $H$ can be decompose into the direct sum $\mathbb{C}^{n-2} \oplus span\{\xi\}$,where lightlike $\xi$ is orthogonal to the $\mathbb{C}^{n-2}$, by the modification of theorem (1.1) [4]. Under a suitable transformation of $U(1, n-1)$, $\xi$ can be transformed to $(1, 1, \cdots, 0)$. Hence $H$ can be defined by $z_1 = z_2$ in $\mathbb{C}^n_1$. □

*Remark* 3. Every maximal surface in $\mathbb{L}^n$ is locally isometric to a complex curve in $\mathbb{C}^n_1$. Namely, if $X : M \longrightarrow \mathbb{L}^n$ defines a maximal surface $\mathbb{S}$ in isothermal parameters, and if $D$ is a simply-connected domain in $M$, then the coordinate functions $x_k$ are harmonic on $M$. Hence there exist analytic functions $f_k$ on $D$ such that $x_k = Re f_k$, $k = 1, \cdots, n$. The metric of index 2 induced from $\mathbb{C}^n_1$ on the analytic curve

$$C : \frac{1}{\sqrt{2}}(f_1, \cdots, f_n) \tag{7}$$

then coincides with the metric on the original surface. To see this, adopt an isothermal coordinate $u = (u^1, u^2)$ on $D$, and compare $g\left(\frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j}\right)$ in $\mathbb{L}^n$ with $g\left(\frac{\partial C}{\partial u^i}, \frac{\partial C}{\partial u^j}\right)$ in $\mathbb{R}^{2n}_2$.

If $\mathbb{S}$ is isometric to a complex analytic curve lying fully in $\mathbb{C}^n_1$, then $\frac{1}{\sqrt{2}}(f_1, \cdots, f_n)$ coincides with the curve in $\mathbb{C}^n_1$. Therefore, for no $(a_1, \cdots, a_n)$ in $\mathbb{C}^n_1$

$$\sum \epsilon_k a_k f_k \equiv constant \ . \tag{8}$$

(8) implies the derivative of $(f_1, f_2, \cdots, f_n)$, which coincides with the Gaussian image of $\mathbb{S}$, should lie fully in $CP^{n-1}_+$.

Now we turn our attention to the Gauss map of degenerate maximal surfaces in $\mathbb{L}^n$.

**Proposition 2.4.** *Let $\mathbb{S}$ be a maximal surface in $\mathbb{L}^n$ which is degenerate of the first kind. Then there exists an orthonormal basis of $\mathbb{L}^n$ with respect to which the function $\phi_k$ defining the Gauss map in terms of local isothermal parameters satisfy*

$$\begin{array}{l} \phi_1 = c\phi_2 \ , \\ \sum_{k=3}^n \phi_k^2 = (c^2-1)\phi_2^2 \ , \\ (1-|c|^2)|\phi_2|^2 + \sum_{k=3}^n |\phi_k|^2 > 0 \ . \end{array} \tag{9}$$

*(In fact, $c = it, 0 \le t < 1$.)*

*Conversely, given any complex constant $c$ with $|c| < 1$ and analytic functions $\phi_1, \cdots, \phi_n$ satisfying (9), the corresponding maximal surface $\mathbb{S}$ is degenerate of the first kind. Its image under the Gauss map lies in the hyperplane $z_1 - cz_2 = 0$.*

*Proof.* Since $\hat{\mathbb{S}}$ lies in a spacelike hyperplane $\sum_{k=1}^n \epsilon_k a_k z_k \equiv 0$, where $A = (a_1, \cdots, a_n)$ is timelike, we can find $M \in SO(1, n-1)$ such that the hyperplane can be transformed into $\tilde{z}_1 - c\tilde{z}_2 = 0$ for $\tilde{Z} = MZ$ by Proposition 2.6 [5]. The conclusion comes immediately from this fact.

Conversely, suppose that $\phi_1, \cdots, \phi_n$ satisfy (9). Then $\hat{\mathbb{S}}$ lies in a spacelike hyperplane $z_1 - cz_2 = 0$ of $CP_+^{n-1} \subset CP^{n-1}$. Since any subspace that is included in the hyperplane is also spacelike, $\mathbb{S}$ should be degenerate of the the first kind. □

**Proposition 2.5.** *Let $\mathbb{S}$ be a maximal surface in $\mathbb{L}^n$ which is degenerate of the third kind. Then there exists an orthonormal basis of $\mathbb{L}^n$ with respect to which the functions $\phi_k$ defining the Gauss map in terms of local isothermal parameters satisfy*

$$\begin{array}{l} \phi_1 = i\phi_2 \ , \\ 2\phi_2^2 + \phi_3^2 + \cdots + \phi_n^2 = 0 \ , \\ \sum_{k=3}^n |\phi_k|^2 > 0 \ , \end{array} \tag{10}$$

*or*

$$\begin{array}{l} \phi_1 = \phi_2 \ , \\ \phi_2^2 + \phi_3^2 + \cdots + \phi_n^2 = 0 \ , \\ \sum_{k=3}^n |\phi_k|^2 > 0 \ , \end{array} \tag{11}$$

*Conversely, let $\phi_1, \cdots, \phi_n$ be analytic functions satisfying one of the above. If there is no timelike vector $A \in \mathbb{C}_1^n$ satisfying (3), then the corresponding surface $\mathbb{S}$ is a maximal surface which is degenerate of the third kind. Its image under the Gauss map lies in the hyperplane either $z_1 - iz_2 = 0$ or $z_1 - z_2 = 0$.*

*Proof.* Since $\hat{\mathbb{S}}$ lies in a null hyperplane $\sum_{k=1}^n \epsilon_k a_k z_k \equiv 0$, where $A = (a_1, \cdots, a_n)$ is lightlike, we can find $M \in SO(1, n-1)$ such that the hyperplane can be transformed into $\tilde{z}_1 - i\tilde{z}_2 = 0$ or $\tilde{z}_1 - \tilde{z}_2 = 0$ for $\tilde{Z} = MZ$ by Proposition 2.6 [5]. Since the Gaussian image lies in $\mathbb{Q}_+^{n-2}$, direct calculation gives us the conclusion. Conversely, if $\phi_1, \cdots, \phi_n$ satisfy (10) or (11), then $(\phi_1, \cdots, \phi_n)$ lies in a degenerate hyperplane $H^{n-1}$ of $\mathbb{C}_1^n$. The possible subspaces of $H^{n-1} \subset \mathbb{C}_1^n$ are

$H^m$ or $\mathbb{C}^m$. Since there is no timelike vector $A \in \mathbb{C}^n_1$ satisfying (3), $(\phi_1, \cdots, \phi_n)$ cannot lie fully in $\mathbb{C}^m$. Hence $\mathbb{S}$ should be degenerate of the third kind. □

**Proposition 2.6.** *Let $\mathbb{S}$ be a maximal surface in $\mathbb{L}^n$ which is degenerate of the second kind. Then there exists an orthonormal basis of $\mathbb{L}^n$ with respect to which the functions $\phi_k$ defining the Gauss map in terms of local isothermal parameters satisfy one of the following :*

$$\begin{array}{l} \phi_n = c\phi_{n-1} \ , |c| \leq 1 \ , \\ \sum_{k=1}^{n-2} \epsilon_k \phi_k^2 = -(1+c^2)\phi_{n-1}^2 \ , \\ \sum_{k=1}^{n-2} \epsilon_k |\phi_k|^2 + (1+|c|^2)|\phi_{n-1}|^2 > 0 \ ; \end{array} \tag{12}$$

$$\begin{array}{l} \phi_2 = -c\phi_1 \ , |c| < 1, \\ \sum_{k=3}^{n} \phi_k^2 = (1-c^2)\phi_1^2 \ , \\ \sum_{k=3}^{n} |\phi_k|^2 > (1-|c|^2)|\phi_1|^2 \ ; \end{array} \tag{13}$$

$$\begin{array}{l} \phi_3 = \frac{i}{\sqrt{2}}(\phi_2 - \phi_1) \ , \\ -3\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2 + 2\sum_{k=4}^{n} \phi_k^2 = 0 \ , \\ -|\phi_1|^2 + 3|\phi_2|^2 - 2Re(\phi_1\bar{\phi_2}) + 2\sum_{k=4}^{n} |\phi_k|^2 > 0 \ . \end{array} \tag{14}$$

*Conversely, let $\phi_1, \cdots, \phi_n$ be analytic functions satisfying one of the above. If none of their transformations $\tilde{\phi_1}, \cdots, \tilde{\phi_n}$ by the induced $SO(1, n-1)$ -action satisfy (9), (10), (11), then the corresponding surface $\mathbb{S}$ is a degenerate maximal surface of the second kind. Its image under the Gauss map lies one of the hyperplanes given in Proposition (2.6) [5].*

*Proof.* If $\mathbb{S}$ is a maximal surface in $\mathbb{L}^n$ of the second kind, then $\hat{\mathbb{S}}$ must lies in the hyperplane which can be transformed to the hyperplanes in Proposition 2.6 [5] under the $SO(1, n-1)$-action. If $\phi_1, \cdots, \phi_n$ satisfy one of the above, then it is surely degenerate. But the hypothesis tells us that $\hat{\mathbb{S}}$ cannot lie in a spacelike or a null hyperplane of $CP^{n-1}$, which means it cannot be degenerate of the first or the third kind. □

## References

[1] Abe, K. and Magid, M., *Indefinite Rigidity of Complex Submanifold and Maximal Surfaces*, Mathematical Proceedings of Cambridge Philosophical Society **106** (1989), no. 3, 481–494

[2] Akutagawa, K. and Nishigawa, S., *The Gauss Map and Spacelike Surfaces with Prescribed Mean Curvature in Minkowski 3-Space*, Tohoku Mathematical Journal **42** (1990), no. 1, 67–82.

[3] Asperti, Antonio C. and Vilhena, Jose Antonio M., *Spacelike Surfaces in $L^4$ with Degenerate Gauss Map*, Results in Mathematics **60** (2011), no. 1, 185–211.

[4] Graves, L., *Codimension One Isometric Immersions between Lorentz Spaces*, Ph.D. Thesis, Brown University, 1977.

[5] Hong, SK., *On the Indefinite Quadric $\mathbb{Q}_+^{n-2}$*, East Asian Math. Journal **32** (2016), no. 1, 93–100.

[6] ———, *On the Weierstrass Theorem of a Maximal Spacelike Surface*, East Asian Math. Journal **33** (2017), no. 1, 115–121.

[7] Kobayasi, O., *Maximal Surfaces in the 3-dimensional Minkowski Space* $L^3$, Tokyo J. Math. **6** (1983), 297–309.
[8] Milnor, T. K., *Harmonic Maps and Classical Surface Theory in Minkowski 3-space*, Trans. of AMS **280** (1983), 161–185.
[9] O'Neil, B., *Semi-Riemannian Geometry*, Academic Press, New York, 1983.
[10] Osserman, R.,*A Survey of Minimal Surfaces*, Dover, New York, 1986

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