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EVALUATION OF THE ZETA FUNCTIONS OF TOTALLY REAL NUMBER FIELDS AND ITS APPLICATION

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Abstract. In this paper, we are interested in the evaluation of special values of the Dedekind zeta function of a totally real number field. In particular, we revisit Siegel method for values of the zeta function of a totally real number field at negative odd integers and explain how this method is applied to the case of non-normal totally real number field. As one of its applications, we give divisibility property for the values in the special case.

1. Introduction

There are several well known techniques to compute special values at non-positive integers of the Dedekind zeta functions of totally real number fields. There are polyhedral and cohomological methods(Shintani[17], Sczech[6]) that use exact arithmetic, and there are approximate methods(Lavrik-Friedman[3, 19]) that use floating-point arithmetic. Finally, there is a method due to Siegel[18] that uses the representation of the special values as constant terms of Eisenstein series, then uses relations among modular forms to find an explicit formula. However, no matter what method we use, to explicitly compute the special values is not easy. Zagier[21] gave an elementary expression for $\zeta_K(1-2b)$ by using Siegel method, where K is a real quadratic field and b is a positive integer. Many authors[4, 8, 10, 12] used Siegel method to compute special values at negative odd integer of the Dedekind zeta functions of some cubic, quartic fields. Halbritter and Pohst[7] developed a method of expressing special values of the partial zeta functions of totally real cubic fields as a finite sum involving norm, trace, and 3-fold Dedekind sums. Their result has been exploited by Byeon[1] and Lee[11] to give an explicit formula for the values of the partial zeta functions of the simplest cubic fields and some non-normal totally real cubic fields. In order to compute the exact value of the Dedekind zeta function of totally real cubic fields, Louboutin[13] used information on the size

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of its denominator. In this paper, we revisit Siegel method and explain how this method is applied to the case of non-normal totally real number field.

Now, we follow notations of [12]. Let K be an algebraic number field and $\mathcal{O}_K$ the ring of integers of K. For an ideal I of $\mathcal{O}_K$, we define the sum of ideal divisors function $\sigma_r(I)$ by

$$\sigma_r(I) = \sum_{J|I} N_{K/\mathbb{Q}}(J)^r, \tag{1}$$

where J runs over all ideals of $\mathcal{O}_K$ which divide I. Note that, if $K = \mathbb{Q}$ and $I = (n)$, our definition coincides with the usual sum of divisors function

$$\sigma_r(n) = \sum_{\substack{d|n \\ d>0}} d^r. \tag{2}$$

Now let K be a totally real algebraic number field. For $l, b = 1, 2, \ldots$, we define

$$S_l^K(2b) = \sum_{\substack{\nu \in \delta_K^{-1} \\ \nu \gg 0 \\ \mathrm{Tr}_{K/\mathbb{Q}}(\nu) = l}} \sigma_{2b-1}((\nu)\delta_K), \tag{3}$$

where δ_K is the different of K. Later we shall study the sum (3) intensively. We just call three conditions in (3)(i.e., $\nu \in \delta_K^{-1}$, $\nu \gg 0$, $\mathrm{Tr}_{K/\mathbb{Q}}(\nu) = l$) the Siegel conditions. At this moment, we remark that this is a finite sum. We now state Siegel's formula.

Theorem 1.1. (Siegel [18]) *Let b be a natural number, K a totally real algebraic number field of degree n, and $h = 2bn$. Then*

$$\zeta_K(1-2b) = 2^n \sum_{l=1}^{r} b_l(h) S_l^K(2b). \tag{4}$$

The numbers $r \geq 1$ and $b_1(h), \ldots, b_r(h) \in \mathbb{Q}$ depend only on h. In particular,

$$r = \dim_{\mathbb{C}} \mathcal{M}_h, \tag{5}$$

where $\mathcal{M}_h$ denotes the space of modular forms of weight h. Thus by a well-known formula,

$$r = \begin{cases} [\frac{h}{12}] & \textit{if} \quad h \equiv 2 \pmod{12} \\ [\frac{h}{12}] + 1 & \textit{if} \quad h \not\equiv 2 \pmod{12}. \end{cases}$$

Proof. See [18] or [21]. □

Zagier [21] computed the values of $b_l(h)$ for $4 \leq h \leq 40$, and we obtain:

Corollary 1.2. (i) *Let K be a totally real cubic number field. Then*

$$\zeta_K(-1) = 2^3 \cdot (-\frac{1}{504}) \cdot S_1^K(2). \tag{6}$$

(ii) *Let K be a totally real quartic number field. Then*

$$\zeta_K(-1) = 2^4 \cdot \frac{1}{480} \cdot S_1^K(2). \tag{7}$$

By Corollary 1.2, for a totally real cubic number field(resp. a totally real quartic number field), $S_1^K(2) = -63\zeta_K(-1)$ (resp. $S_1^K(2) = 30\zeta_K(-1)$) is an integer.

2. Computation of $S_1^K(2b)$

Let K be a totally real algebraic number field of degree n and S_K(or simply S) be the set of elements in K which satisfy the Siegel conditions described in (3). Fix an integral basis $\alpha_1, \ldots, \alpha_n$ of K. For $\nu \in K$, we can write

$$\nu = x_1\alpha_1 + \cdots + x_n\alpha_n, \tag{8}$$

where $x_i \in \mathbb{Q}$. Then we have an embedding $\phi : K \longrightarrow \mathbb{R}^n$ given by

$$\phi(\nu) = (x_1, \ldots, x_n). \tag{9}$$

The condition $\nu \in \delta_K^{-1}$ implies that the denominator of x_i, $i = 1, 2, \ldots, n$, is bounded by D_K, where D_K denotes the discriminant of K. The condition $\mathrm{Tr}_{K/\mathbb{Q}}(\nu) = l$ is equivalent to saying that $\phi(\nu)$ lies in the hyperplane

$$a_1x_1 + \cdots + a_nx_n = l, \tag{10}$$

where $a_i = \mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i)$. Finally the condition $\nu \gg 0$ becomes n distinct linear inequalities defined over K in the variables $(x_1, \ldots, x_n)$. Therefore the elements in S_K can be put into one-to-one correspondence with the lattice points in a bounded $(n-1)$-dimensional region under ϕ. We shall call this set(or any set which can be put into one-to-one correspondence with this set under a suitable linear transformation) as a Siegel lattice for K and denote it by T_K(or simply T). Notice that equation (3) expresses $S_l^K(2b)$ as a weighted sum of ideal divisor functions over a Siegel lattice. Hence the description of a Siegel lattice is very important in the computation of $S_l^K(2b)$.

Now, we are ready to explain how to compute $S_l^K(2b)$. First, we need to find the set of elements in K which satisfy the Siegel conditions described in (3). One can think that it is a problem to find lattice points in $(n-1)$-dimensional region (If it is necessary, we can use a suitable linear transformation giving one-to-one correspondence). For example, if n is 3(resp. 4), then the set is lattice points in the region of some triangle(resp. tetrahedron). If K is a Galois extension of $\mathbb{Q}$, we can find a Siegel lattice more efficiently. In fact, we can check that if ν is an element in K which satisfy the Siegel conditions described in (3) and $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, then $\sigma(\nu)$ also satisfies the Siegel conditions. Therefore, examining the movement of ν under Galois action, we can decrease the complexity finding the Siegel lattice by about a ratio $1/n$(cf. [8, 10, 12]). Note that if there exist no elements fixed under action of Galois group, the number of Siegel lattice points is multiple of n. After finding lattice points near appropriate boundary, then we can find the Siegel lattice by looking into movements of conjugates of the points. On the other hand, if K is not a Galois extension of $\mathbb{Q}$, we can not use Galois action. But, by a suitable linear transformation giving one-to-one correspondence, we can find a good-shape region, which means a region to easily understand a Siegel lattice. For example, if appropriate boundary of the region is parallel or close to an axis, it is not difficult to find lattice points in the region(cf. [4]). This procedure is always available in the case that the ring of integers $\mathcal{O}_K$ is known even though K is not Galois extension of $\mathbb{Q}$ if n is less than equal to 4.

Next, we need to investigate the prime ideal decomposition of $(\nu)\delta_K$. Let $N((\nu)\delta_K)$ be the the norm of ideal $(\nu)\delta_K$. If K is a Galois extension of $\mathbb{Q}$, considering the conjugate of $(\nu)\delta_K$, we can determine the prime ideal decomposition. For example, if $N((\nu)\delta_K) = p^2$ where p is prime, we have the type of $(\nu)\delta_K = Q_1^2$ or $(\nu)\delta_K = Q_1Q_2$, where Q_1, Q_2 are prime ideals of $\mathcal{O}_K$ over p. One can determine whether $(\nu)\delta_K = Q_1^2$ or $(\nu)\delta_K = Q_1Q_2$ by computing the product of conjugates of $(\nu)\delta_K$(cf. [10] or [12]). In the case of non-normal extension of $\mathbb{Q}$, we exactly know prime ideals Q of $\mathcal{O}_K$ over p as the form of $\mathbb{Z}$-module by using methods in [15]. Combining the information of the prime ideals, we have the prime ideal decomposition for $(\nu)\delta_K$. Finally, combining the results, we can compute $S_1^K(2b)$.

3. Divisibility of $\zeta_K(-1)$

In this section, we discuss the divisibility of $\zeta_K(-1)$. The value is closely related to Birch-Tate conjecture associated to algebraic K-theory. More explicitly, the order of tame kernel $\sharp K_2\mathcal{O}_K$ is related to the value $\zeta_K(-1)$.

Definition 1. (cf. [9]) For any number field K of finite degree over $\mathbb{Q}$ and for $n \in \mathbb{Z}_{\geq 0}$, let $w_n(K)$ be as follows:

$$w_n(K) := 2^{n(2)+1} \prod_p p^{n(p)}, \quad n(p) := \max\{m | [K(\zeta_{p^m}) : K] \leq n\}.$$

Definition 2. (cf. [9]) For any totally real number field F of finite degree over $\mathbb{Q}$ and for any even integer $n \in \mathbb{Z}_{\geq 0}$, we define

$$\xi_n(K) := w_n(K)\zeta_K(1-n).$$

Note that $\zeta_K(1-n) \in \mathbb{Q}$ is known by Siegel-Klingen, so $\xi_n(K) \in \mathbb{Q}$.

Serre[16] proved the following:

Theorem 3.1. *For any totally real number field K, $\xi_2(K) \in \mathbb{Z}$.*

Now, we state Birch-Tate conjecture.

Theorem 3.2. *For any algebraic number field K,*

$$\sharp K_2\mathcal{O}_K = w_2(K)|\zeta_K(-1)|,$$

where $K_2\mathcal{O}_K$ is the tame kernel of the ring of integers of K.

This conjecture is confirmed for any abelian extension K of $\mathbb{Q}$ by Mazur and Wiles([14], [20]). For arbitrary totally real number fields K, the 2-part of the Birch-Tate conjecture is confirmed by Wiles[20].

There exist the formulas for $w_2(K)$[5]. If we apply the formulas to cyclic cubic extension K of $\mathbb{Q}$, then we have $w_2(K) = 24$ except two cases[2]:

$$w_2(K) = 3 \cdot 24 \quad \text{for} \quad K = \mathbb{Q}(\zeta_9)^+,$$
$$w_2(K) = 7 \cdot 24 \quad \text{for} \quad K = \mathbb{Q}(\zeta_7)^+.$$

Since $w_2(K)\zeta_K(-1)$ is an integer, for every cyclic cubic number field K, the value $24\zeta_K(-1)$ is an integer except two cases $\mathbb{Q}(\zeta_9)^+, \mathbb{Q}(\zeta_7)^+$. We see that $\mathbb{Q}(\zeta_7)^+$ is one of an infinite family of simplest cubic fields. By an argument after Corollary 1.2, $S_1^K(2) = -63\zeta_K(-1)$ is an integer. Therefore, if K is the simplest cubic field, the value $-63\zeta_K(-1)$ is divisible by 7 except for $\mathbb{Q}(\zeta_7)^+$ and is always divisible by 3. Furthermore, we know that there exist no elements fixed under action of Galois group in simplest cubic fields(cf. [8]). That means that $-63\zeta_K(-1)$ is divisible by 3 from a different point of view. Noting that $w_2(K) = 24$ for non-normal totally real cubic fields, $-63\zeta_K(-1)$ is divisible by 21 for every non-normal totally real cubic field. Finally, we consider the simplest quartic field, which defined by the polynomial over $\mathbb{Q}$

$$P_t(X) = x^4 - tx^3 - 6x^2 + tx + 1, \tag{11}$$

where t is a natural number such that $t^2 + 16$ is not divisible by an odd square. In this case, we have $w_2(K) = 24$ except two cases[12]:

$$w_2(K) = 4 \cdot 24 \quad \text{for} \quad t = 4,$$
$$w_2(K) = 5 \cdot 24 \quad \text{for} \quad t = 2, 8.$$

By a similar argument, for simplest quartic fields, $30\zeta_K(-1)$ is divisible by 5 except above two cases. Moreover, if t is odd, there exist no elements fixed under action of Galois group[12]. Therefore, in this case, $30\zeta_K(-1)$ is divisible by 20. We summarize the above computation in the following theorem.

Theorem 3.3. (1) *If K is the simplest cubic field, the value $-63\zeta_K(-1)$ is divisible by 7 except for $\mathbb{Q}(\zeta_7)^+$ and is always divisible by 3.*

(2) *If K is non-normal totally real cubic field, the value $-63\zeta_K(-1)$ is divisible by 21.*

(3) *If K is the simplest quartic field, the value $30\zeta_K(-1)$ is divisible by 5 except two cases. In particular, if K is the simplest quartic field, where t is odd in (11), the value $30\zeta_K(-1)$ is divisible by 20.*

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