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ANALYSIS OF L^1-WEIGHTS IN ONE-DIMENSIONAL MINKOWSKI-CURVATURE PROBLEMS

Rui Yang and Yong-Hoon Lee*

Abstract. L^1-weight functions are investigated to give necessary conditions on the existence of nontrivial solutions for various types of scalar equations and systems of one-dimensional Minkowski-curvature problems.

1. Introduction

In this paper, we first consider the following one-dimensional Minkowski-curvature problem

$$(P) \qquad \begin{cases} -\Big(\phi(u'(t))\Big)' = r(t)u(t), & t \in (a,b), \\ u(a) = 0 = u(b), \end{cases}$$

where $\phi(y) = \frac{y}{\sqrt{1-|y|^2}}$, $y \in (-1,1)$, function $r \in L^1(a,b)$ satisfies $r(t) \geq 0$, $r(t) \not\equiv 0$ in any compact subinterval of $[a,b]$.

The weight functions play a critical role in determining the eigenvalues and eigenfunctions. One may refer to [2, 7, 8, 9, 10, 11] for the existence, nonexistence and smoothness of solutions of p-Laplace problem ($\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$) and [1, 12, 13, 14] for other types of differential equations.

From the properties of Minkowski-curvature operator, we notice that it is hard to analyze eigenvalues and eigenfunctions of the boundary valve problem (abbreviated to BVP) of the operator directly. However the existence and multiplicity of solutions for the problems with continuous or L^1-class weight functions can be studied regarding the eigenvalues corresponding to the second order linear problem, see [4, 5, 6]. For the BVP of second order linear ordinary differential equation, Lyapunov [1] proved that $\frac{4}{b-a} \leq \int_a^b r(t)dt$ is a necessary

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*Corresponding author.

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condition for the existence of a positive solution for the linear ordinary differential equation

$$\begin{cases} -u'' = r(t)u, & t \in (a,b), \\ u(a) = 0 = u(b), \end{cases}$$

where $r \in C([a,b],[0,\infty))$.

For the BVP of one-dimensional p-Laplacian equation, readers can refer to [2, 3]. Specially, Sim-Lee [3] estimated Lyapunov inequalities for a single equation, a cycled system and a coupled system with weight functions which are beyond $L^1(a,b)$. The authors only dealt with the case of positive solutions in [3] but here in this paper, we could handle any nontrivial solutions with help of C^1-regularity of solutions. It is the advantage of L^1 condition on the weight function.

In this paper, we aim at exploring L^1-weight functions to give necessary conditions for the existence of nontrivial solutions for scalar equations as well as systems of one-dimensional Minkowski-curvature problems.

The rest of this paper is organized as follows. We analyze L^1-weight functions for a scalar equation, a cycled system and a strongly coupled system of one-dimensional Minkowski-curvature problems in Section 2, 3, and 4, respectively.

2. Scalar equation

We say u a solution of problem (P) if $u \in C[a,b] \cap C^1(a,b)$, $|u'(t)| < 1$ for $t \in (a,b)$, and $\phi\big(u'(t)\big)$ is absolutely continuous in any compact subinterval of (a,b), and u satisfies the equation and the boundary conditions in problem (P). The following proposition shows that all solutions u of problem (P) are in $C^1[a,b]$ and satisfy $\|u'\|_\infty < 1$.

Proposition 2.1. *If u is a nontrivial solution of problem (P), then $u \in C^1[a,b]$, $|u'(t)| < 1$ for $t \in [0,1]$.*

Proof. Let u be a nontrivial solution of problem (P). Then it suffices to prove that $u \in C^1[a,b]$ and $|u'(a)| < 1$ and $|u'(b)| < 1$. Since $u \in C[a,b]$, there exists a $t^* \in (a,b)$ such that $u'(t^*) = 0$. Thus we obtain

$$u'(t) = \phi^{-1}\left(\int_t^{t^*} r(\tau)u(\tau)d\tau\right). \tag{1}$$

$r \in L^1(a,b)$ implies that $u'(a)$ exists and thus $|u'(a)| < 1$. Similarly we get $u'(b)$ exists and $|u'(b)| < 1$. This completes the proof. $\square$

Theorem 2.2. *If problem (P) has a nontrivial solution, then one has*

$$\frac{4}{b-a} < \int_a^b r(t)dt. \tag{2}$$

Proof. Let u be a nontrivial solution of problem (P). By Proposition 2.1, $u \in C^1[a,b]$. For $t \in [a,b]$, we have

$$|u(t)| \leq \int_a^t |u'(\tau)| d\tau,$$

and

$$|u(t)| \leq \int_t^b |u'(\tau)| d\tau.$$

Applying Hölder's inequality, we have

$$2|u(t)| \leq \int_a^b |u'(\tau)| d\tau \leq (b-a)^{1/2} \left(\int_a^b |u'|^2 d\tau \right)^{1/2},$$

that is

$$|u(t)|^2 \leq \frac{b-a}{4} \left(\int_a^b |u'|^2 d\tau \right). \tag{3}$$

Multiplying both sides of (3) by $r(t)$, we get

$$r(t)|u(t)|^2 \leq \frac{b-a}{4} r(t) \left(\int_a^b |u'|^2 d\tau \right). \tag{4}$$

Since u is a nontrivial solution of problem (P), we have

$$\int_a^b \frac{|u'|^2}{\sqrt{1-|u'|^2}} dt = \int_a^b r(t)|u(t)|^2 dt. \tag{5}$$

Integrating (4) in (a,b), we get

$$\int_a^b r(t)|u(t)|^2 dt \leq \frac{b-a}{4} \int_a^b r(t) dt \left(\int_a^b |u'|^2 d\tau \right). \tag{6}$$

From (5) and (6), it follows that

$$\int_a^b \frac{|u'|^2}{\sqrt{1-|u'|^2}} dt < \frac{b-a}{4} \int_a^b r(t) dt \left(\int_a^b \frac{|u'|^2}{\sqrt{1-|u'|^2}} d\tau \right).$$

And we obtain (2). □

Remark 1. In Theorem 2.2, there is no equality in (2) for the Minkowski-curvature problem compared with the result in Laplacian problem, see [1].

3. Cycled system

In this section, let us consider L^1-weights which can guarantee the existence of nontrivial C^1-class solution for a cycled system

$$(CS)\quad \begin{cases} \Big(\phi(u_1'(t))\Big)' + r_1(t)u_2(t) = 0, \\ \Big(\phi(u_2'(t))\Big)' + r_2(t)u_3(t) = 0, \\ \qquad\qquad\qquad \cdots \\ \Big(\phi(u_{n-1}'(t))\Big)' + r_{n-1}(t)u_n(t) = 0, \\ \Big(\phi(u_n'(t))\Big)' + r_n(t)u_1(t) = 0, \qquad\qquad t \in (a,b), \\ u_1(a) = \cdots = u_n(a) = 0 = u_1(b) = \cdots = u_n(b), \end{cases}$$

where $r_i \in L^1(a,b)$, $1 \le i \le n$. Mainly due to the fact $r_i \in L^1(a,b)$, we may prove a solution $(u_1, u_2, \cdots, u_n)$ of (CS) satisfies $u_i \in C^1[a,b]$, $\|u_i'\|_\infty < 1$ by obvious modification of the proof of Proposition 2.1. We say $(u_1, u_2, \cdots, u_n)$ a solution of problem (CS) if $u_i \in C^1[a,b]$, $\|u_i'\|_\infty < 1$, and $\phi\big(u_i'(t)\big)$ is absolutely continuous in any compact subinterval of (a,b), and u_i satisfies the equations and the boundary conditions in problem (CS), $1 \le i \le n$.

Theorem 3.1. *If problem (CS) has a nontrivial solution, then one has*

$$\left(\frac{4}{b-a}\right)^n < \int_a^b r_1(t)dt \cdots \int_a^b r_n(t)dt. \tag{7}$$

Proof. Let $(u_1, u_2, \cdots, u_n)$ be a nontrivial solution of problem (CS). As in (3), we have

$$|u_i(t)|^2 \le \frac{b-a}{4}\left(\int_a^b |u_i'|^2 dt\right). \tag{8}$$

Multiplying both sides of the first equation in problem (CS) by $u_1(t)$ and then integrating it over (a,b), we obtain

$$\int_a^b \frac{|u_1'|^2}{\sqrt{1-|u_1'|^2}}dt = \int_a^b r_1(t)|u_1(t)||u_2(t)|dt.$$

Together with (8), we get

$$\begin{aligned} \int_a^b |u_1'|^2 dt &< \int_a^b \frac{|u_1'|^2}{\sqrt{1-|u_1'|^2}}dt \\ &= \int_a^b r_1(t)|u_1(t)||u_2(t)|dt \\ &\le \frac{b-a}{4}\int_a^b r_1(t)dt\left(\int_a^b |u_1'|^2 dt\right)^{1/2}\left(\int_a^b |u_2'|^2 dt\right)^{1/2}, \end{aligned}$$

i.e.,

$$\left(\int_a^b |u_1'|^2 dt\right)^{1/2} < \frac{b-a}{4}\int_a^b r_1(t)dt \left(\int_a^b |u_2'|^2 dt\right)^{1/2}.$$

Similarly, we can get

$$\begin{cases} \left(\int_a^b |u_2'|^2 dt\right)^{1/2} < \frac{b-a}{4}\int_a^b r_2(t)dt \left(\int_a^b |u_3'|^2 dt\right)^{1/2}, \\ \qquad\qquad \cdots \\ \left(\int_a^b |u_{n-1}'|^2 dt\right)^{1/2} < \frac{b-a}{4}\int_a^b r_{n-1}(t)dt \left(\int_a^b |u_n'|^2 dt\right)^{1/2}, \\ \left(\int_a^b |u_n'|^2 dt\right)^{1/2} < \frac{b-a}{4}\int_a^b r_n(t)dt \left(\int_a^b |u_1'|^2 dt\right)^{1/2}. \end{cases}$$

Multiplying all inequalities, we obtain (7). □

4. Strongly coupled system

In this section, we study L^1-weights which can guarantee the existence of nontrivial C^1-class solution for a strongly coupled system

$$(SCS)\begin{cases} \Big(\phi(u_1'(t))\Big)' + r_1(t)(u_1(t) + \cdots + u_n(t)) = 0, \\ \qquad\qquad \cdots \\ \Big(\phi(u_n'(t))\Big)' + r_n(t)(u_1(t) + \cdots + u_n(t)) = 0, \qquad t \in (a,b), \\ u_1(a) = \cdots = u_n(a) = 0 = u_1(b) = \cdots = u_n(b), \end{cases}$$

where $r_i \in L^1(a,b)$, $1 \le i \le n$. By obvious manner as before, we may prove a solution $(u_1, u_2, \cdots, u_n)$ of (SCS) satisfies $u_i \in C^1[a,b]$, $\|u_i'\|_\infty < 1$. So we say $(u_1, u_2, \cdots, u_n)$ a solution of problem (SCS) if $u_i \in C^1[a,b]$, $\|u_i'\|_\infty < 1$, and $\phi\big(u_i'(t)\big)$ is absolutely continuous in any compact subinterval of (a,b), and u_i satisfies the equations and the boundary conditions in problem (SCS), $1 \le i \le n$.

Theorem 4.1. *If problem (SCS) has a nontrivial solution, then one has*

$$\frac{4}{b-a} < \int_a^b r_1(t)dt + \cdots + \int_a^b r_n(t)dt. \tag{9}$$

Proof. Let $(u_1, u_2, \cdots, u_n)$ be a nontrivial solution of problem (SCS). As in (3), we have

$$|u_i(t)| \le \left(\frac{b-a}{4}\right)^{1/2} \left(\int_a^b |u_i'|^2 dt\right)^{1/2}. \tag{10}$$

Multiplying both sides of the first equation in problem (SCS) by $u_1(t)$ and integrating it over (a,b), we get

$$\int_a^b \frac{|u_1'|^2}{\sqrt{1-|u_1'|^2}}dt = \int_a^b r_1(t)(|u_1(t)|^2 + \cdots + |u_1(t)||u_n(t)|)dt.$$

Together with (10), we get

$$\begin{aligned}\int_a^b |u_1'|^2 dt &< \int_a^b \frac{|u_1'|^2}{\sqrt{1-|u_1'|^2}}dt \\ &= \int_a^b r_1(t)(|u_1(t)|^2 + \cdots + |u_1(t)||u_n(t)|)dt \\ &\le \frac{b-a}{4}\int_a^b r_1(t)dt \left(\int_a^b |u_1'|^2 dt\right) \\ &\quad + \cdots + \\ &\quad + \frac{b-a}{4}\int_a^b r_1(t)dt \left(\int_a^b |u_1'|^2 dt\right)^{1/2} \left(\int_a^b |u_n'|^2 dt\right)^{1/2},\end{aligned}$$

i.e.,

$$\left(\int_a^b |u_1'|^2 dt\right)^{1/2} < \frac{b-a}{4}\int_a^b r_1(t)dt \left(\left(\int_a^b |u_1'|^2 dt\right)^{1/2} + \cdots + \left(\int_a^b |u_n'|^2 dt\right)^{1/2}\right).$$

Similarly, we can get

$$\begin{cases} \left(\int_a^b |u_2'|^2 dt\right)^{1/2} < \frac{b-a}{4}\int_a^b r_2(t)dt \left(\left(\int_a^b |u_1'|^2 dt\right)^{1/2} + \cdots + \left(\int_a^b |u_n'|^2 dt\right)^{1/2}\right), \\ \qquad\qquad \cdots \\ \left(\int_a^b |u_n'|^2 dt\right)^{1/2} < \frac{b-a}{4}\int_a^b r_n(t)dt \left(\left(\int_a^b |u_1'|^2 dt\right)^{1/2} + \cdots + \left(\int_a^b |u_n'|^2 dt\right)^{1/2}\right). \end{cases}$$

Adding all equalities, we derive (9). □

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Rui Yang
Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea
E-mail address: `yangr1617@gmail.com`

Yong-Hoon Lee
Department of Mathematics, Pusan National University, Busan 46241, Republic of Korea
E-mail address: `yhlee@pusan.ac.kr`