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THE ASYMPTOTIC BEHAVIOUR OF THE AVERAGING VALUE OF SOME DIRICHLET SERIES USING POISSON DISTRIBUTION

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Abstract. We investigate the averaging value of a random sampling of a Dirichlet series with some condition using Poisson distribution.

Our result is the following: Let $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series that converges absolutely for $\mathrm{Re}(s) > 1$. If X_t is an increasing random sampling with Poisson distribution and there exists a number $0 < \alpha < \frac{1}{2}$ such that $\sum_{n \le u} a_n \ll u^{\alpha}$, then we have

$$\mathbb{E}L(1/2 + iX_t) = O(t^{\alpha}\sqrt{\log t}),$$

for all sufficiently large t in $\mathbb{R}$.

As a result, we get the behaviour of $L(\frac{1}{2} + it)$ such that L is a Dirichlet L-function or a modular L-function, when t is sampled by the Poisson distribution.

1. Introduction

The Lindelöf Hypothesis is an important conjecture about behaviour of the Riemann zeta function along the $\mathrm{Re}(z) = \frac{1}{2}$. The conjecture states the absolute value of $\zeta(\frac{1}{2} + it)$ is less than t^{ϵ} as $t \to \infty$. (cf. [5], [6]) Naturally, the Lindelöf Hypothesis can be extended other Dirichlet series including Dirichlet L-function and modular L-function.

In regard to the Lindelöf Hypothesis, there are many attempts using probabilitic methods. Lifshits and Weber [4] researched the behaviour of the Riemann zeta function $\zeta(\frac{1}{2} + it)$ using the Cauchy random walk. After that, Jo and Yang [3] researched the behaviour of the Riemann zeta function $\zeta(\frac{1}{2} + it)$ using the Gamma distribution. In the previous paper, Jo [6] studied the behaviour of the Riemann zeta function $\zeta(s)$ along the critical strip $s = 1/2 + it$, when t is sampled by the Poisson distribution.

In this paper, we study the behaviour of Dirichlet series with some conditions using the Poisson distribution. From this, we get the behaviour of $L(\frac{1}{2} + it)$ such

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that L is a Dirichlet L-function or a modular L-function, when t is sampled by the Poisson distribution.

The following is the main result of this paper.

Theorem 1.1. *Let X_t denote the Poisson process with $\mathbb{E}(X_t) = t$ and $\mathrm{Var}(X_t) = t$. Suppose that the Dirichlet series*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > 1$. If there exists a number $0 < \alpha < \frac{1}{2}$ such that

$$A(u) = \sum_{n \le u} a_n \ll u^{\alpha},$$

then for all sufficiently large t,

$$\mathbb{E}L(1/2 + iX_t) = O(t^{\alpha}\sqrt{\log t}).$$

From this theorem, we have the following corollaries:

Corollary 1.2. *Let X_t denote the Poisson process with $\mathbb{E}(X_t) = t$ and $\mathrm{Var}(X_t) = t$. And let χ be a non-principal Dirichlet character with modulo N and $L_\chi(s)$ be the corresponding Dirichlet L-function such that*

$$L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Then we have for all sufficiently large t,

$$\mathbb{E}L_\chi(1/2 + iX_t) = O(\sqrt{\log t}).$$

Corollary 1.3. *Let X_t denote the Poisson process with $\mathbb{E}(X_t) = t$ and $\mathrm{Var}(X_t) = t$. And let f be a cusp form of weight k over $\mathrm{SL}_2(\mathbb{Z})$ such that*

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} \exp(2\pi i n z)$$

and $L_f(s)$ be the corresponding L-function such that

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Then we have for all sufficiently large t,

$$\mathbb{E}L_f(1/2 + iX_t) = O(t^{\frac{1}{3}}\sqrt{\log t}).$$

Because the Poisson process is increasing with mean value t and its variance t, we can use this process to explain the behaviour of $L(\frac{1}{2}+it)$ as $t \to \infty$. In this paper, we use the Landau notation $f = O(g)$, which means that $|f(x)| \le Cg(x)$ for some constant C and the Vinogradov notation $f \ll g$ which is equivalent to $f = O(g)$.

2. Preliminaries

2.1. Poisson process

The Poisson distribution is the discrete probability distribution of the number of events that occur in an interval time period.

If the probability mass function of X is given by

$$P(X_t = k) = \frac{t^k e^{-t}}{k!} \tag{1}$$

for $k = 0, 1, 2, \cdots$, then we say that a discrete random variable X_t has a Poisson distribution with parameter $t > 0$.

Using (1), we can get the followings:

$$\mathbb{E}(X_t) = \sum_{k=0}^{\infty} k \frac{t^k e^{-t}}{k!} = t$$

$$V(X_t) = \mathbb{E}|X_t|^2 - |\mathbb{E}X_t|^2 = t$$

$$\mathbb{E}(e^{iuX_t}) = \exp(t(e^{iu} - 1)) \tag{2}$$

$$\mathbb{E}(X_t e^{iuX_t}) = te^{iu} \exp(t(e^{iu} - 1)). \tag{3}$$

2.2. Dirichlet character and Dirichlet L-function

A Dirichlet character with modulo N is a function χ from $\mathbb{Z}$ to $\mathbb{C}$ with conditions:

- $\chi(n) = \chi(n + N)$ for all $n \in \mathbb{Z}$.
- If $\gcd(n, N) > 1$, then $\chi(n) = 0$. If $\gcd(n, N) = 1$, then $\chi(n) \neq 0$.
- $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{Z}$.

A Dirichlet character χ with modulo N is called principal character if $\chi(n) = 1$ for all $n \in \mathbb{Z}$ such that $\gcd(n, N) = 1$.
Note that if χ is a non-principal character with modulo N, then

$$\sum_{a=1}^{N} \chi(a) = 0. \tag{4}$$

A Dirichlet L-function is a function of the following form: for $\mathrm{Re}(s) > 1$,

$$L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where χ is a Dirichlet character. By analytic continuation, $L_\chi(s)$ can be extended to whole complex plane and if χ is a non-principal character, then $L_\chi(s)$ can be extended to an entire function.

2.3. Holomorphic cusp form

A holomorphic cusp form of weight k on

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

is a complex-valued function f on $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ satisfying the following conditions:

- f is a holomorphic function on $\mathfrak{H}$.
- For any $z \in \mathfrak{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$
- f is holomorphic and goes to zero as $z \to i\infty$.

Let f be a cusp form of weight k over $\mathrm{SL}_2(\mathbb{Z})$ such that

$$f(z) = \sum_{n=1}^{\infty} a(n) n^{\frac{k-1}{2}} \exp(2\pi i n z).$$

It is well known that the corresponding L-function

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > 1$. And for the partial sum $A_f(u) = \sum_{n \le u} a(n)$, we have a bound

$$A_f(u) \ll u^{\frac{1}{3}} \tag{5}$$

by Hafner and Ivić [1].

3. Proof of Theorem

We start the following lemma about partial sum.

Lemma 3.1. *Suppose that the Dirichlet series*

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > 1$*. If there exists a number* $0 < \alpha < 1$ *such that*

$$A(u) = \sum_{n \le u} a_n \ll u^{\alpha},$$

then $L(s)$ *can be extended to* $\mathrm{Re}(s) > \alpha$ *as follows:*

$$L(s) = s \int_1^{\infty} A(u) u^{-s-1} du.$$

Proof. Note that

$$\sum_{n=1}^{M} \frac{a_n}{n^s} = \int_{1^-}^{M} x^{-s} dA(x) = \left[A(x)x^{-s}\right]_{1^-}^{M} - \int_{1^-}^{M} A(x)d(x^{-s})$$
$$= \frac{A(M)}{M^s} + s\int_{1}^{M} A(x)x^{-s-1}dx.$$

Therefore we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s\int_{1}^{\infty} A(x)x^{-s-1}dx$$

for $\mathrm{Re}(s) > 1$. If $\mathrm{Re}(s) > \alpha$, then

$$\int_{1}^{\infty} A(x)x^{-s-1}dx \ll \int_{1}^{\infty} x^{-\mathrm{Re}(s)+\alpha-1}dx = \frac{1}{\mathrm{Re}(s)-\alpha}.$$

Hence $L(s)$ can be extended to $\mathrm{Re}(s) > \alpha$. □

Proof of Theorem 1.1. By Lemma 3.1, (2) and (3), we have

$$\begin{aligned}
&\mathbb{E}L(1/2+iX_t)\\
&= \mathbb{E}\left((1/2+iX_t)\int_{1}^{\infty} A(u)u^{-1/2-iX_t-1}du\right)\\
&= \frac{1}{2}\int_{1}^{\infty} A(u)u^{-3/2}\mathbb{E}(u^{-iX_t})du + i\int_{1}^{\infty} A(u)u^{-3/2}\mathbb{E}(X_t u^{-iX_t})du\\
&= \frac{1}{2}\int_{1}^{\infty} A(u)u^{-3/2}\exp(t(u^{-i}-1))du + it\int_{1}^{\infty} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du\\
&=: A+B.
\end{aligned}$$

First, we estimate the integral A.
Note that

$$\exp(t(u^{-i}-1)) = \exp\Big(t\big(\cos(\log u) - 1 - i\sin(\log u)\big)\Big). \tag{6}$$

Because

$$|\exp(t(u^{-i}-1))| = \exp\Big(t\big(\cos(\log u)-1\big)\Big) \le 1,$$

we have that

$$A = \frac{1}{2}\int_{1}^{\infty} A(u)u^{-3/2}\exp(t(u^{-i}-1))du = O\left(\int_{1}^{\infty} u^{\alpha-\frac{3}{2}}du\right) = O(1).$$

Second, we consider the integral B.
Note that

$$\cos\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right) = 1 - \frac{1}{2}\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right)^2 + O\Big((\log t)^2/t^2\Big)$$

and

$$\exp\left(t\left(\cos\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right)-1\right)\right) = \exp\left(-\frac{t}{2}\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right)^2 + O\Big((\log t)^2/t\Big)\right)$$
$$= \exp(-\log t)\left(1+O\Big((\log t)^2/t\Big)\right)$$
$$= \frac{1}{t} + O\Big((\log t)^2/t^2\Big).$$

Suppose that, for $m \in \mathbb{Z}$,

$$\frac{\sqrt{2\log t}}{\sqrt{t}} \le |\log u - 2\pi m| \le \pi.$$

Then, for $\alpha = \log u - 2\pi m$, we have $0 \le \cos\alpha \le \cos\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right)$. Therefore we have

$$\exp\left(t\left(\cos\left(\log u\right)-1\right)\right) = \exp\left(t\left(\cos\left(2\pi m+\alpha\right)-1\right)\right) = \exp\left(t\left(\cos\alpha-1\right)\right)$$
$$\le \exp\left(t\left(\cos\left(\frac{\sqrt{2\log t}}{\sqrt{t}}\right)-1\right)\right) = \frac{1}{t} + O\Big((\log t)^2/t^2\Big).$$

From (6), we have $|\exp(t(u^{-i}-1))| = \exp\left(t\left(\cos\left(\log u\right)-1\right)\right)$. Hence, for $m \in \mathbb{Z}$, if

$$\frac{\sqrt{2\log t}}{\sqrt{t}} \le |\log u - 2\pi m| \le \pi,$$

then

$$|\exp(t(u^{-i}-1))| \ll t^{-1}. \tag{7}$$

Let

$$S = \bigcup_{m=0}^{\infty}\left\{u \in \mathbb{R}\ \middle|\ u \ge 1,\ |\log u - 2\pi m| < \frac{\sqrt{2\log t}}{\sqrt{t}}\right\}.$$

We divide B into two parts.

$$B = it\int_S A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du + it\int_R A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du$$
$$=: B_1 + E,$$

where $R = [1,\infty) - S$.

case 1) We consider the integral for R. From (7), we can get

$$E = it\int_R A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du = O\left(t\int_1^{\infty} u^{\alpha-\frac{3}{2}}\frac{1}{t}du\right) = O(1).$$

case 2) The integral for S is the following:

$$B_1 = it\sum_{m=1}^{\infty}\int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du.$$

We divide B_1 into two parts as following:

$$\begin{aligned} B_1 &= it \sum_{m< \frac{1}{2\pi}\log t} \int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du \\ &\quad + it \sum_{m\geq \frac{1}{2\pi}\log t} \int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du \\ &=: M_1 + M_2. \end{aligned}$$

First, we calculate the integral M_2.

$$\begin{aligned} M_2 &\ll t \sum_{m\geq \frac{1}{2\pi}\log t} \int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}} u^{\alpha-3/2}du \ll t \sum_{m\geq \frac{1}{2\pi}\log t} e^{-(1-2\alpha)\pi m}\left(\frac{\sqrt{\log t}}{\sqrt{t}}\right) \\ &\ll t^{\alpha}\sqrt{\log t}. \end{aligned}$$

Next, we calculate the integral M_1.
We divide M_1 into the following:

$$\begin{aligned} & it \sum_{m< \frac{1}{2\pi}\log t} \int_{e^{2\pi m-\sqrt{2\log t/t}}}^{e^{2\pi m+\sqrt{2\log t/t}}} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du \\ &= it \sum_{m< \frac{1}{2\pi}\log t} \int_{[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du \\ &\quad + it \sum_{m< \frac{1}{2\pi}\log t} \int_{[e^{2\pi m+\sqrt{2\log t/t}}]}^{e^{2\pi m+\sqrt{2\log t/t}}} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du \\ &\quad - it \sum_{m< \frac{1}{2\pi}\log t} \int_{[e^{2\pi m-\sqrt{2\log t/t}}]}^{e^{2\pi m-\sqrt{2\log t/t}}} A(u)u^{-3/2-i}\exp(t(u^{-i}-1))du \\ &=: M_{1,1} + E^{+} + E^{-}. \end{aligned}$$

From an integration by parts using the equation

$$\frac{d}{du}\exp(t(u^{-i}-1)) = -i\exp(t(u^{-i}-1))tu^{-i-1}, \tag{8}$$

we get

$$E^+ = it\Big(\sum_{n\le[e^{2\pi m+\sqrt{2\log t/t}}]} a(n)\Big)\int_{[e^{2\pi m+\sqrt{2\log t/t}}]}^{e^{2\pi m+\sqrt{2\log t/t}}} u^{-3/2-i}\exp(t(u^{-i}-1))du$$

$$\ll te^{2\pi m\alpha}\int_{[e^{2\pi m+\sqrt{2\log t/t}}]}^{e^{2\pi m+\sqrt{2\log t/t}}} u^{-3/2-i}\exp(t(u^{-i}-1))du$$

$$\ll te^{2\pi m\alpha}\frac{e^{-\pi m}}{t} = e^{(2\alpha-1)\pi m}.$$

Similarly, we have $E^- \ll e^{(2\alpha-1)\pi m}$.
Using (8), we have

$$M_{1,1} = it\sum_{k=[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]-1} A(k)\int_k^{k+1} u^{-3/2-i}\exp(t(u^{-i}-1))du$$

$$= it\sum_{k=[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]-1} A(k)\Bigg(\Big[\frac{u^{-1/2}i}{t}\exp(t(u^{-i}-1))\Big]_k^{k+1} + i\int_k^{k+1}\frac{u^{-3/2}}{2t}\exp(t(u^{-i}-1))du\Bigg)$$

$$\ll t\sum_{k=[e^{2\pi m-\sqrt{2\log t/t}}]}^{[e^{2\pi m+\sqrt{2\log t/t}}]-1} k^\alpha\frac{k^{-\frac{1}{2}}}{t} \ll e^{(2\alpha+1)\pi m}\frac{\sqrt{\log t}}{\sqrt{t}}.$$

From these facts, we have

$$M_1 \ll \sum_{m<\frac{1}{2\pi}\log t}\left(e^{(2\alpha+1)\pi m}\frac{\sqrt{\log t}}{\sqrt{t}} + 2e^{(2\alpha-1)\pi m}\right) \ll t^\alpha\sqrt{\log t}.$$

Because $M_2 \ll t^\alpha\sqrt{\log t}$, we have

$$B_1 = M_1 + M_2 \ll t^\alpha\sqrt{\log t}.$$

Hence, from case 1 and case 2, we can get $B \ll t^\alpha\sqrt{\log t}$.
Therefore we can know

$$\mathbb{E}L(1/2+iX_t) \ll t^\alpha\sqrt{\log t}$$

and the proof is complete. □

Proof of Corollary 1.2. From (4), we know

$$\Big|\sum_{n\le u}\chi(n)\Big| \le N.$$

By Theorem 1.1, we have

$$\mathbb{E}L_\chi(1/2 + iX_t) = O(\sqrt{\log t}).$$

□

Proof of Corollary 1.3. From (5), we know

$$A_f(u) = \sum_{n \le u} a(n) \ll u^{\frac{1}{3}}.$$

By Theorem 1.1, we have

$$\mathbb{E}L_f(1/2 + iX_t) = O(t^{\frac{1}{3}}\sqrt{\log t}).$$

□

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