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BOUNDARY VALUE PROBLEM FOR ONE-DIMENSIONAL ELLIPTIC JUMPING PROBLEM WITH CROSSING $n-$EIGENVALUES

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ABSTRACT. This paper is dealt with one-dimensional elliptic jumping problem with nonlinearities crossing n eigenvalues. We get one theorem which shows multiplicity results for solutions of one-dimensional elliptic boundary value problem with jumping nonlinearities. This theorem is that there exist at least two solutions when nonlinearities crossing odd eigenvalues, at least three solutions when nonlinearities crossing even eigenvalues, exactly one solutions and no solution depending on the source term. We obtain these results by the eigenvalues and the corresponding normalized eigenfunctions of the elliptic eigenvalue problem and Leray-Schauder degree theory.

1. Introduction

Let $\Omega = (c, d) \subset R$, $c < d$, $m \in N$, $m < \infty$. Let $L^{2m}(\Omega, R)$ be $2m-$Lebesgue space and $W^{1,2m}(\Omega, R)$ be the Lebesgue Sobolev space. We know that the eigenvalue problem

$$-(u')' = \lambda u \qquad \text{in} \;\; \Omega = (c, d),$$
$$u = 0 \qquad \text{on} \;\; \Omega,$$

has infinitely many positive positive eigenvalues λ_j, $j = 1, 2, \cdots$, $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ and the corresponding normalized eigenfunctions ϕ_j, $j = 1, 2, \cdots$ and the first eigenfunction ϕ_1 is positive. We note that the elliptic eigenvalue problem

$$-(|u'|^{2m-2}u')' = \Lambda |u|^{2m-2}u \qquad \text{in} \;\; \Omega$$

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$$u = 0 \qquad \text{on } \Omega$$

has infinitely many eigenvalues $\Lambda_j = \lambda_j^m$, $0 < \Lambda_1 = \lambda_1^m < \Lambda_2 = \lambda_2^m \leq \cdots \leq \Lambda_k = \lambda_k^m \leq \cdots$ and the corresponding normalized eigenfunctions ϕ_j, $j = 1, 2, \cdots$, where the first eigenfunction ϕ_1 is positive.

In this paper we consider multiplicity of solutions $u \in W^{1,2m}(\Omega, R)$ for the following one-dimensional elliptic Dirichlet boundary value problem with jumping nonlinearities;

$$-(|u'|^{2m-2}u')' = b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1} \qquad \text{in } \Omega, \tag{1.1}$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

where $s \in R$, $u^+ = \max\{u, 0\}$ and $u^- = -\min\{u, 0\}$.

$p-$Laplacian boundary value problems with $p-$growth conditions arise in applications of nonlinear elasticity theory, electro rheological fluids, non-Newtonian fluid theory in a porous medium (cf. [7], [12]). Our problems are characterized as a jumping problem. Jumping problem was first suggested in the suspension bridge equation as a model of the nonlinear oscillations in differential equation

$$u_{tt} + K_1 u_{xxxx} + K_2 u^+ = W(x) + \epsilon f(x, t), \tag{1.2}$$
$$u(0, t) = u(L, t) = 0, \qquad u_{xx}(0, t) = u_{xx}(L, t) = 0.$$

This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression. Choi and Jung (cf. [2], [4], [5]) and McKenna and Walter (cf.[10]) investigate the existence and multiplicity of solutions for the single nonlinear suspension bridge equation with Dirichlet boundary condition. In [3], the authors investigate the multiplicity of solutions of a semilinear equation

$$Au + bu^+ - au^- = f(x) \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \Omega,$$

where Ω is a bounded domain in R^n, $n \geq 1$, with smooth boundary $\partial\Omega$ and A is a a second order linear partial differential operator when the forcing term is a multiple $s\phi_1$, $s \in R$, of the positive eigenfunction and the nonlinearity crosses eigenvalues.

In general, when $1 < p < \infty$, the eigenvalue problem

$$-(|u'|^{p-2}u')' = \nu|u|^{p-2}u \qquad \text{in } \Omega, \tag{1.3}$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

has a nondecreasing sequence of nonnegative eigenvalues ν_j obtained by the Ljusternik-Schnirelman principle tending to ∞ as $j \to \infty$, where the first eigenvalue ν_1 is simple and only eigenfunctions associated with ν_1 do not change sign, the set of eigenvalues is closed, the first eigenvalue ν_1 is isolated. Thus there are two sequences of eigenfunctions $(\beta_j)_j$ and $(\mu_j)_j$ corresponding to the eigenvalues ν_j such that the first eigenfunction $\beta_1 > 0$ in the sequence $(\beta_j)_j$ and the first eigenfunction $\mu_1 < 0$ in the sequence $(\mu_j)_j$, which was proved in [8].

Let us set the operator $-L_{2m}$ by

$$-L_{2m}u = -(|u'|^{2m-2}u')'.$$

Then (1.1) is equivalent to the equation

$$u = (-L_{2m})^{-1}(b|u|^{2m-2}u^{+} - a|u|^{2m-2}u^{-} + s\phi_1^{2m-1}).$$

Our main theorem is as follows:

Theorem 1.1. *Let $m \in N$, $m < \infty$, $a < b$, $-\infty < a < \lambda_1^m, \cdots, \lambda_n^m < b < \lambda_{n+1}^m$ and $s \in R$. Then*
(i) if $m < \infty$ and $s > 0$, then (1.1) has no solution
(ii) if $m < \infty$ and $s = 0$, then (1.1) has a unique trivial solution $u = 0$.
(iii) if $m < \infty$, there exists $s_1 < 0$ such that for any s with $s_1 < s < 0$, (1.1) has at least two solutions if n is odd, and three solutions if n is even.

For the proof of Theorem 1.1 we estimate a priori bound and calculate the Leray-Schauder degree of $u-(-L_{2m})^{-1}(b|u|^{2m-2}u^{+}-a|u|^{2m-2}u^{-}+s\phi_1^{2m-1})$ in the neighborhood of positive solution, in the neighborhood of negative solutions and in the whole solution bounded subspace, respectively. The outline of the proof of Theorem 1.1 is as follows: In Section 2, we introduce some preliminaries. In Section 3, we prove Theorem 1.1 by using direct computations and Leray-Schauder degree theory.

2. Preliminaries

Let $L^p(\Omega, R)$ be the Lebesgue space defined by

$$L^p(\Omega, R) = \{u|\ u : \Omega \to R \text{ is measurable, } \int_\Omega |u|^p dx < \infty\}$$

which is endowed with the norm

$$\|u\|_{L^p(\Omega)} = \inf\{\lambda > 0|\ \int_\Omega |\frac{u(x)}{\lambda}|^p \leq 1\},$$

and $W^{1,p}(\Omega, R)$ be the Lebesgue Sobolev space defined by

$$W^{1,p}(\Omega, R) = \{u \in L^p(\Omega, R)|\ u' \in L^p(\Omega, R)\}$$

which is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega,R)} = [\int_\Omega |u'(x)|^p dx]^{\frac{1}{p}} + [\int_\Omega |u(x)|^p dx]^{\frac{1}{p}}].$$

Let $1 < p < \infty$ and $h \in L^r(\Omega)$, $r > 1$. Then the problem

$$-(|u'|^{p-2}u')' = h(x) \qquad \text{in } \ \Omega, \tag{2.1}$$
$$u = 0 \qquad \partial\Omega$$

has a unique solution $u \in C^1(\bar{\Omega})$ which is of the form

$$u(x) = \int_\Omega g_p^{-1}\big(c_f - \int_\Omega h(\tau)d\tau\big)dy, \tag{2.2}$$

where $g_p(t) = |t|^{p-2}t$ for $t \neq 0$, $g_p(0) = 0$ and its inverse g_p^{-1} is $g_p^{-1}(t) = t^{\frac{1}{p-1}}$ if $t > 0$ and $g_p^{-1}(t) = -|t|^{\frac{1}{p-1}}$ if $t < 0$ and c_f is the unique constant such that $u = 0$ on $\partial\Omega$. By [[9], Lemma 2.1 or [10], Lemma 4.2], the solution operator S satisfies that $S : L^p(\Omega) \to C^1(\bar{\Omega})$ is continuous and by [[13], Corollary 2.3], the embedding $S : L^p(\Omega) \to C(\bar{\Omega})$ is continuous and compact. We also have *Poincar'e*-type inequality.

Lemma 2.1. *Let* $1 < p < \infty$. *Then*
(i) the embedding

$$W^{1,p}(\Omega, R) \hookrightarrow C(\bar{\Omega}, R)$$

is continuous and compact.
Moreover the embedding

$$W^{1,p}(\Omega, R) \hookrightarrow L^p(\bar{\Omega}, R)$$

is continuous and compact and
(ii) there is a constant $C > 0$ *independent of* u *such that*

$$\|u\|_{L^p(\bar{\Omega},R)} \leq C\|u\|_{W^{1,p}(\bar{\Omega},R)}.$$

Proof. (i) By [Chapter 4.1.2 of [1]] and [Chapter 5 and Chapter 6 of [6]], when $\Omega = (a, b) \subset R^1$, the embedding

$$W^{1,p}(\Omega, R) \hookrightarrow C(\bar{\Omega}, R)$$

is continuous and compact. Since $C(\bar{\Omega}, R) \subset L^p(\bar{\Omega}, R)$, it follows that the embedding

$$W^{1,p}(\Omega, R) \hookrightarrow L^p(\bar{\Omega}, R)$$

is continuous and compact.
(ii) By (1.3), we have

$$\begin{aligned}
\|u\|^p_{W^{1,p}(\bar{\Omega},R)} &= [\int_{\bar{\Omega}} |u'(x)|^p dx]^{\frac{1}{p}} + [\int_{\bar{\Omega}} |u(x)|^p dx]^{\frac{1}{p}}] \\
&\geq \nu^{\frac{1}{p}}[\int_{\Omega} |u(x)|^p dx]^{\frac{1}{p}} + [\int_{\Omega} |u(x)|^p dx]^{\frac{1}{p}} \\
&= (\nu^{\frac{1}{p}} + 1)[\int_{\Omega} |u(x)|^p dx]^{\frac{1}{p}} \\
&= (\nu^{\frac{1}{p}} + 1)\|u\|_{L^p(\bar{\Omega},R)}.
\end{aligned}$$

Thus there exists a constant $C > 0$ such that $\|u\|_{L^p(\bar{\Omega},R)} \leq C\|u\|_{W^{1,p}(\bar{\Omega},R)}$. □

3. Proof of Theorem 1.1

Proof of (i) of Theorem 1.1 (For the case $s > 0$)
We assume that $m \in N$, $m < \infty$, $a < b$, $-\infty < a < \lambda_1^m, \cdots, \lambda_n^m < b < \lambda_{n+1}^m$ and $s > 0$. Then (1.1) can be rewritten as

$$-(|u'|^{2m-2}u')' - \lambda_1^m|u|^{2m-2}u = (b - \lambda_1^m)|u|^{2m-2}u^+ - (a - \lambda_1^m)|u|^{2m-2}u^- + s\phi_1^{2m-1}. \tag{3.1}$$

Taking inner product both side of (3.1) with ϕ_1, we have

$$\big\langle -(|u'|^{2m-2}u')' - \lambda_1^m|u|^{2m-2}u, \phi_1\big\rangle = \big\langle (b-\lambda_1^m)|u|^{2m-2}u^+ - (a-\lambda_1^m)|u|^{2m-2}u^- + s\phi_1^{2m-1}, \phi_1\big\rangle. \tag{3.2}$$

The left hand side of (3.1) is equal to 0. On the other hand, the right hand side of (3.1) is positive because $b-\lambda_1^m > 0$, $-(a-\lambda_1^m) > 0$ and $s\phi_1^{2m-1} > 0$ for $s > 0$. Thus if $s > 0$, then there is no solution for (1.1).

Proof of (ii) of Theorem 1.1 (For the case $s = 0$)
If $s = 0$, then (3.2) is reduced to the equation

$$\big\langle -(|u'|^{2m-2}u')' - \lambda_1^m|u|^{2m-2}u, \phi_1\big\rangle = \big\langle (b-\lambda_1^m)|u|^{2m-2}u^+ - (a-\lambda_1^m)|u|^{2m-2}u^-, \phi_1\big\rangle,$$

i.e.,

$$\int_\Omega [(-(|u'|^{2m-2}u')' - \lambda_1^m|u|^{2m-2}u)\phi_1]dx = 0$$
$$= \int_\Omega [((b-\lambda_1^m)|u|^{2m-2}u^+ - (a-\lambda_1^m)|u|^{2m-2}u^-)\phi_1]dx. \tag{3.3}$$

Since $b-\lambda_1^m > 0$ and $-(a-\lambda_1^m) > 0$, the only possibility to hold (3.3) is that $u = 0$.

Lemma 3.1. *(A priori bound) Assume that $m \in N$, $m < \infty$, $-\infty < a < \lambda_1^m, \cdots, \lambda_n^m < b < \lambda_{n+1}^m$, $s \in R$. Then there exist $s_1 < 0$, $s_2 > 0$ and a constant $C > 0$ depending only on a, b and s such that for any any s with $s_1 \le s \le s_2$, any solution u of (1.1) satisfies $\|u\|_{W^{1,2m}(\Omega)} < C$.*

Proof. Suppose that the lemma is false. Then there exists a sequence $(u_n)_n$, $(a_n)_n$, $(b_n)_n$ and $(t_n)_n$ such that $-\infty < a_n < \lambda_1^m, \cdots, \lambda_n^m < b_n < \lambda_{n+1}^m$, $s_1 \le t_n \le s_2$, $\|u_n\|_{W^{1,2m}(\Omega)} = \rho_n \to \infty$ and

$$-(|u_n'|^{2m-2}u_n')' = b_n|u_n|^{p-2}u_n^+ - a_n|u_n|^{2m-2}u_n^- + t_n\phi_1^{2m-1} \qquad \text{in } \Omega \tag{3.4}$$

or equivalently

$$u_n = (-L_{2m})^{-1}(b_n|u_n|^{2m-2}u_n^+ - a_n|u_n|^{2m-2}u_n^- + t_n\phi_1^{2m-1}) \qquad \text{in } \Omega.$$

Let us set $w_n = \frac{u_n}{\|u_n\|_{W^{1,2m}(\Omega)}}$. Then $(w_n)_n$ is bounded, so there exists a subsequence, up to a subsequence $(w_n)_n$ such that $(w_n)_n \to w$ weakly for some w in $W^{1,2m}(\Omega)$. Dividing (3.4) by $\|u_n\|^{2m-1}_{W^{1,2m}(\Omega)}$, we have

$$\frac{-(|u_n'|^{2m-2}u_n')'}{\|u_n\|^{2m-1}_{W^{1,2m}(\Omega)}} = b_n\frac{|u_n|^{2m-2}u_n^+}{\|u_n\|^{2m-1}_{W^{1,2m}(\Omega)}} - a_n\frac{|u_n|^{2m-2}u_n^-}{\|u_n\|^{2m-1}_{W^{1,2m}(\Omega)}} + \frac{t_n\phi_1^{2m-1}}{\|u_n\|^{2m-1}_{W^{1,2m}(\Omega)}} \qquad \text{in } \Omega, \tag{3.5}$$

i.e.,

$$w_n = (-L_{2m})^{-1}(b_n|w_n|^{2m-2}w_n^+ - a_n|w_n|^{2m-2}w_n^- + \frac{t_n\phi_1^{2m-1}}{\|u_n\|^{2m-1}_{W^{1,2m}(\Omega)}}) \qquad \text{in } \Omega.$$

Since by Lemma 2.1, the embedding $W^{1,2m}(\Omega) \hookrightarrow L^{2m}(\Omega)$ is compact, and when $m < \infty$, $(-L_{2m})^{-1}$ is compact operator, $(w_n)_n \to w$ strongly in $W^{1,2m}(\Omega)$. Moreover $(a_n)_n$ and $(b_n)_n$ satisfying $-\infty < a_n < \lambda_1^m, \cdots, \lambda_n^m < b_n < \lambda_{n+1}^m$ converge strongly to some a and b with $-\infty < a < \lambda_1^m, \cdots, \lambda_n^m < b < \lambda_{n+1}^m$. Moreover $(t_n)_n$ with $s_1 \le t_n \le s_2$ also converge strongly to some s with $s_1 \le s \le s_2$. Limiting (3.5) as $n \to \infty$, we have

$$-(|w'|^{2m-2}w')' = b|w|^{2m-2}w^+ - a|w|^{2m-2}w^-. \tag{3.6}$$

By (ii) of Theorem 1.1, (3.6) has only trivial solution, which is absurd because $\|w\|_{W^{1,2m}(\Omega)} = 1$. Thus the lemma is proved. □

We shall consider the Leray-Schauder degree on a large ball

Lemma 3.2. *Assume that $m \in N$, $m < \infty$, $-\infty < a < \lambda_1^m, \cdots, \lambda_n^m < b < \lambda_{n+1}^m$. Then there exist a constant $R > 0$ depending on a, b, s, and $s_1 < 0$ and $s_2 > 0$ such that for any s with $s_1 \le s \le s_2$, the Leray-Schauder degree*

$$d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}), B_R(0), 0) = 0,$$

where $-L_{2m}u = -(|u'|^{2m-2}u')'$.

Proof. Let us consider the homotopy

$$F(x,u) = u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}). \tag{3.7}$$

By (i) of Theorem 1.1, for any $s > 0$, (1.1) has no solution. Thus there exist $s_2 > 0$ and a large $R > 0$ such that (3.7) has no zero in $B_R(0)$ for any $s \ge s_2$, and by the a priori bound in Lemma 3.1, there exists $s_1 < 0$ such that for any s with $s_1 \le s \le s_2$, all solutions of

$$u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}) = 0$$

satisfy $\|u\|_{W^{1,2m}(\Omega)} \le R$ and (3.7) has no zero on ∂B_R for any $s_1 \le s \le s_2$. Since

$$d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s_2\phi_1^{2m-1}), B_R(0), 0) = 0,$$

by homotopy arguments, for any $s_1 \le s \le s_2$, we have

$$\begin{aligned} &d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}), B_R(0), 0) \\ &= d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1} + \lambda(s_2 - s)\phi_1^{2m-1}), B_R(0), 0) \\ &= d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s_2\phi_1^{2m-1}), B_R(0), 0) = 0 \end{aligned}$$

for any $0 \le \lambda \le 1$. Thus the lemma is proved. □

Lemma 3.3. *Let K be a compact set in $L^{2m}(\Omega)$. Let $\xi > 0$ a.e. Then there exists a modulus of continuity $\alpha : R \to R$ depending only on K and ξ such that*

$$\||(|\tau| - \frac{\xi}{\eta})^+\|_{L^{2m}(\Omega)} \le \alpha(\eta) \qquad \textit{for all } \ \tau \in K.$$

Proof. For any $\tau \in K$, Let $\tau_n = (|\tau| - \frac{\xi}{\eta})^+$. Since $0 \le \tau_n \le |\tau|$ and $\tau_n(x) \to 0$ as $\eta \to 0$ a.e., it follows that $\|\tau_n\|_{L^{2m}(\Omega)} \to 0$ for all $\tau \in K$. We claim that for given $\epsilon > 0$, there exists $\delta > 0$ such that if $\tau \in K$, then $\|\tau_n\|_{L^{2m}(\Omega)} \le 2\epsilon$ for all $\eta \in [0, \delta]$. Choose $\{\tau_i,\ i = 1, \cdots, N\}$ as an ϵ net for K. Choose δ so that $\|(\tau_i)_\delta\|_{L^{2m}(\Omega)} < \epsilon$ for $i = 1, \cdots, N$. Then for any $\tau \in K$, there exists τ_k, α, $\|\alpha\|_{L^{2m}(\Omega)} < \epsilon$ that $\tau = \tau_K + \alpha$. Since $(u+v)^+ \le u^+ + v^+$, we have $\|\tau_\delta\|_{L^{2m}(\Omega)} \le (\tau_K)_\delta + |\alpha|$ and therefore $\|\tau_\eta\|_{L^{2m}(\Omega)} \le \|\tau_\delta\|_{L^{2m}(\Omega)} + \|\alpha\|_{L^{2m}(\Omega)} \le 2\epsilon$ □

Lemma 3.4. *Assume that $m \in N$, $m < \infty$, $-\infty < a < \lambda_1^m, \cdots, \lambda_n^m < b < \lambda_{n+1}^m$. Then there exist a small constant ϵ and $s_1 < 0$ such that for any s with $s_1 \le s < 0$, the Leray-Schauder degree*

$$d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}), B_{\epsilon|s|}(u_0), 0) = (-1)^n,$$

where $u_0 = (\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 > 0$ is a positive solution of (1.1).

Proof. Let us set $M = (-L_{2m} - bg_{2m})^{-1}$. Then (1.1) can be rewritten as

$$(-L_p - bg_{2m})(u) = b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u + s\phi_1^{2m-1}$$

or equivalently

$$u = M(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u + s\phi_1^{2m-1}). \tag{3.8}$$

The operator M is compact on $L^{2m}(\Omega)$, and the set $K = M(\bar{B})$, where $\bar{B}$ is the closed unit ball in $L^{2m}(\Omega)$. Then K is a compact set. Let us set $\gamma = \min\{b - \lambda_n^m, \lambda_{n+1}^m - b\}$. We can observe that if $m < \infty$, then $\|M\|_{L^{2m}(\Omega)} \le \|\frac{1}{\gamma^{\frac{1}{2m-1}}} g_{2m}^{-1}\|_{L^{2m}(\Omega)}$. Let α be the modulus continuity of Lemma 3.3 corresponding to K and $\xi = M\phi_1^{2m-1} = (\frac{1}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1$ and choose $\epsilon > 0$ so that

$$\alpha(\epsilon^{\frac{1}{2m-1}}((b-a)^{\frac{1}{2m-1}} + \gamma^{\frac{1}{2m-1}}) \le \frac{\gamma^{\frac{1}{2m-1}}}{4(b-a)^{\frac{1}{2m-1}}((b-a)^{\frac{1}{2m-1}} + \gamma^{\frac{1}{2m-1}})}. \tag{3.9}$$

We have

$$\|b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u\|_{L^{2m}(\Omega)} \le (b-a)\||u|^{2m-2}u^-\|_{L^{2m}(\Omega)}. \tag{3.10}$$

It follows from that

$$\|M\big(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u\big)\|_{L^{2m}(\Omega)} \le \frac{(b-a)^{\frac{1}{2m-1}}}{\gamma^{\frac{1}{2m-1}}}\|u^-\|_{L^{2m}(\Omega)}. \tag{3.11}$$

For $u = (\frac{|s|}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 + (|s|\epsilon v)^{\frac{1}{2m-1}}$ with $v \in \bar{B}$,

$$\begin{aligned}\|u^-\|_{L^{2m}(\Omega)} =& \|((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 + (|s|\epsilon)^{\frac{1}{2m-1}} v^{\frac{1}{2m-1}})^-\|_{L^{2m}(\Omega)} \\ \le& \|((|s|\epsilon v)^{\frac{1}{2m-1}})^-\|_{L^{2m}(\Omega)} \le (|s|\epsilon)^{\frac{1}{2m-1}}\end{aligned}$$

since $(\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 > 0$. Then $T(u) = M(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u + s\phi^{2m-1})$ can be rewritten as

$$T(u) = (\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 + (|s|\epsilon)^{\frac{1}{2m-1}}((b-a)^{\frac{1}{2m-1}} + \gamma^{\frac{1}{2m-1}})w^{\frac{1}{2m-1}}, \qquad w \in K.$$

If u is a solution of (3.8), then $u = Tu$ and by Lemma 3.3,

$$\|u^-\|_{L^{2m}(\Omega)} = \|((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 + ((|s|\epsilon)^{\frac{1}{2m-1}}((b-a)^{\frac{1}{2m-1}} + \gamma^{\frac{1}{2m-1}})w^{\frac{1}{2m-1}})_-\|_{L^{2m}(\Omega)}$$

$$\leq ((|s|\epsilon)^{\frac{1}{2m-1}}((b-a)^{\frac{1}{2m-1}} + \gamma^{\frac{1}{2m-1}})\alpha(\epsilon^{\frac{1}{2m-1}}((b-a)^{\frac{1}{2m-1}} + \gamma^{\frac{1}{2m-1}}) < \frac{\gamma^{\frac{1}{2m-1}}(|s|\epsilon)^{\frac{1}{2m-1}}}{4(b-a)^{\frac{1}{2m-1}}}. \tag{3.12}$$

Combining (3.11) with (3.12), we have

$$\|M(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u)\|_{L^{2m}(\Omega)}$$

$$\leq \frac{(b-a)^{\frac{1}{2m-1}}}{\gamma^{\frac{1}{2m-1}}}\|u^-\|_{L^{2m}(\Omega)} \leq \frac{1}{4}(|s|\epsilon)^{\frac{1}{2m-1}} \leq \frac{1}{4}|s|\epsilon.$$

Thus we have shown that any solution $u \in (\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 + |s|\epsilon\bar{B}$ of (3.8) belong to $(\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1 + \frac{1}{4}|s|\epsilon\bar{B}$. This estimate holds if we replace $b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u$ by $\lambda(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u)$ with $0 \leq \lambda \leq 1$. Thus the equation

$$u = (-L_{2m})^{-1}(s\phi_1^{2m-1} + b|u|^{2m-2}u + \lambda(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u))$$

has no solution on the boundary of the ball $B_{\epsilon|s|}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1)$ for $0 \leq \lambda \leq 1$. By the homotopy invariance degree,

$$d_{LS}(u - (-L_{2m})^{-1}(s\phi_1^{2m-1} + b|u|^{2m-2}u + \lambda(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u),$$
$$B_{\epsilon|s|}(\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1, 0)$$

is defined for $0 \leq \lambda \leq 1$ and is independent of λ. For $\lambda = 0$,

$$d_{LS}(u - (-L_{2m})^{-1}(s\phi_1^{2m-1} + b|u|^{2m-2}u, B_{\epsilon|s|}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1), 0) = (-1)^n.$$

since $u = (\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1$ is the unique solution of the equation and since there are n eigenvalues $\lambda_1^m, \cdots, \lambda_n^m$ of $-L_{2m}$ to the left of b and thus the operator $I - b(-L_{2m})^{-1}$ has n negative eigenvalues, while all the rest are positive. When $\lambda = 1$, we have

$$d_{LS}((u - (-L_{2m})^{-1}(s\phi_1^{2m-1} + b|u|^{2m-2}u^+ + 1(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- - b|u|^{2m-2}u),$$
$$B_{\epsilon|s|}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1), 0)$$
$$= d_{LS}(s\phi_1^{2m-1} + b|u|^{2m-2}u^+ - a|u|^{2m-2}u^-, B_{\epsilon|s|}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1), 0).$$

Thus by the homotopy invariance of degree, we have

$$d_{LS}(s\phi_1^{2m-1} + b|u|^{2m-2}u^+ - a|u|^{2m-2}u^-, B_{\epsilon|s|}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1), 0)$$

$$= d_{LS}(u - (-L_{2m})^{-1}(s\phi_1^{2m-1} + b|u|^{2m-2}u, B_{\epsilon|s|}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1), 0) = (-1)^n.$$

Thus the lemma is proved. □

Lemma 3.5. *Assume that $m \in N$, $m < \infty$, $-\infty < a < \lambda_1^m, \cdots \lambda_n^m < b < \lambda_{n+1}^m$ and $s_1 < 0$. Then there exist a constant $\epsilon > 0$ depending on a, b, s such that for any s with $s_1 \le s < 0$, the Leray-Schauder degree*

$$d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}), B_{\epsilon|s|}(u_1), 0) = 1,$$

where $u_1 = -(\frac{s}{a-\lambda_1^m})^{\frac{1}{2m-1}}\phi_1 < 0$ is a negative solution of (1.1).

Proof. We can prove this lemma by the almost identical proof to that of Lemma 3.4. □

Proof of (iii) of Theorem 1.1

By Lemma 3.4 and Lemma 3.5, there is a solution $(\frac{s}{\lambda_1^m-b})^{\frac{1}{2m-1}}\phi_1 > 0$ in $B_{|s|\epsilon}((\frac{s}{\lambda_1^m-b})^{\frac{1}{2m-1}}\phi_1)$ and a solution $-(\frac{s}{a-\lambda_1^m})^{\frac{1}{2m-1}}\phi_1 < 0$ in $B_{|s|\epsilon}(-(\frac{s}{a-\lambda_1^m})^{\frac{1}{2m-1}}\phi_1)$. We may assume that $|s|\epsilon < (\frac{s}{\lambda_1^m-b})^{\frac{1}{2m-1}}$ and $|s|\epsilon < (\frac{s}{a-\lambda_1^m})^{\frac{1}{2m-1}}$. Then these two balls $B_{|s|\epsilon}((\frac{s}{\lambda_1^m-b})^{\frac{1}{2m-1}}\phi_1)$ and $B_{|s|\epsilon}(-(\frac{s}{a-\lambda_1^m})^{\frac{1}{2m-1}}\phi_1)$ are disjoint. Then there is a large ball B_R centred at origin and containing $B_{|s|\epsilon}((\frac{s}{\lambda_1^m-b})^{\frac{1}{2m-1}}\phi_1)$ and $B_{|s|\epsilon}(-(\frac{s}{a-\lambda_1^m})^{\frac{1}{2m-1}}\phi_1)$. Since

$$d_{LS}(u - (-L_{2m})^{-1}(b|u|^{2m-2}u^+ - a|u|^{2m-2}u^- + s\phi_1^{2m-1}), B_R(0), 0) = 0,$$

$$d_{LS}(u-(-L_{2m})^{-1}(b|u|^{2m-2}u^+-a|u|^{2m-2}u^-+s\phi_1^{2m-1}), B_{|s|\epsilon}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1), 0) = (-1)^n$$

and

$$d_{LS}(u-(-L_{2m})^{-1}(b|u|^{2m-2}u^+-a|u|^{2m-2}u^-+s\phi_1^{2m-1}), B_{|s|\epsilon}(-(\frac{s}{a - \lambda_1^m})^{\frac{1}{2m-1}}\phi_1), 0) = 1,$$

we have

$$d_{LS}(u-(-L_{2m})^{-1}(b|u|^{2m-2}u^+-a|u|^{2m-2}u^-+s\phi_1^{2m-1}), B_R(0)\setminus(B_{|s|\epsilon}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1)$$

$$\cup B_{|s|\epsilon}(-(\frac{s}{a - \lambda_1^m})^{\frac{1}{2m-1}}\phi_1), 0)$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ -2 & \text{if } n \text{ is even.} \end{cases}$$

Thus if n is odd, then we can not assure that there exists a third solution in

$$B_R(0)\setminus\big(B_{|s|\epsilon}((\frac{s}{\lambda_1^m - b})^{\frac{1}{2m-1}}\phi_1) \cup B_{|s|\epsilon}(-(\frac{s}{a - \lambda_1^m})^{\frac{1}{2m-1}}\phi_1)\big),$$

and if n is even, then we can assure that there exists a third solution in $B_R(0)\backslash\big(B_{|s|\epsilon}((\frac{s}{\lambda_1^m-b})^{\frac{1}{2m-1}}\phi_1)$
$\cup B_{|s|\epsilon}(-(\frac{s}{a-\lambda_1^m})^{\frac{1}{2m-1}}\phi_1)\big)$. Thus (iii) of Theorem 1.1. is proved.

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