East Asian Math. J.
Vol. 35 (2019), No. 1, pp. 031–040
http://dx.doi.org/10.7858/eamj.2019.005

POLYNOMIAL INVARIANTS OF LONG VIRTUAL KNOTS

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Abstract. We introduce a family of polynomial invariants by using intersection index defined from a Gauss diagram of a long virtual knot, and we give some properties for long virtual knots. We extend these polynomials so that we give two-variable polynomial invariants and some example.

1. Introduction

Kauffman introduced virtual knot theory as a generalization of classical knot theory in the sense that if two classical knot diagrams are equivalent as virtual knots, then they are equivalent as classical knots [7]. Similarly, a *long virtual knot diagram* is an oriented infinitely long line in $\mathbb{R}^2$ which possibly has some encircled crossings without over/under information, called *virtual crossings*. A *long virtual knot* is the equivalence class of such a long virtual knot diagram by *generalized Reidemeister moves*, which consist of (classical) *Reidemeister moves* of type R_1, R_2 and R_3 and *virtual Reidemeister moves* of type VR_1, VR_2, VR_3 and the *semivirtual move* VR_4 as shown in Figure 1.

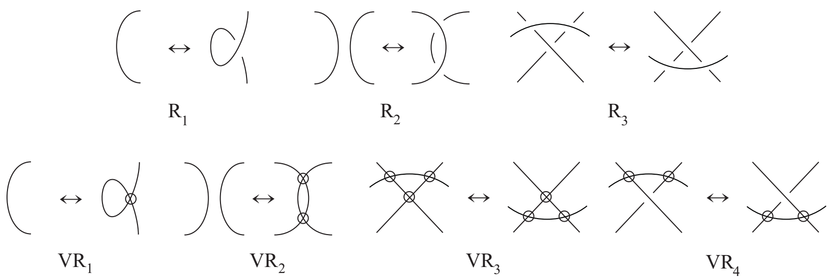

Figure 1. Generalized Reidemeister moves

Received July 18, 2018; Revised November 2, 2018; Accepted November 27, 2018.
2010 *Mathematics Subject Classification.* Primary 57M25. Secondary 57M27.
Key words and phrases. long virtual knot; intersection index; Gauss diagram; n-th polynomial.
This work was supported by a 2-Year Research Grant of Pusan National University.
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(pISSN 1226-6973, eISSN 2287-2833)

Henrich [2] introduced a polynomial invariant for virtual knots which vanishes for classical knots. Later, Im, Lee and Lee [5] extended it to an invariant of virtual links so that it is called the *index polynomial.* And Jeong [6] defined the another invariant from index polynomial by using classical crossings with zero intersection index. It is called the *zero polynomial.* Later, Im and Kim [3] introduced the *zero polynomial and nth polynomial for a Gauss diagram of a virtual knot.*

In this paper, we introduce a sequence of *n-th polynomials* of long virtual knots for each nonnegative integer n. It is organized as follows. In Section 2, we define the *zero polynomial* and *n-th polynomial* of long virtual knots and investigate some properties of these polynomials. In Section 3, we obtain two variable polynomials of long virtual knots.

2. A family of the n-th polynomials for long virtual knots

We begin this section with basic definitions and results which are needed throughout this paper.

Definition 1. A long virtual knot diagram D is a smooth immersion $f : \mathbb{R} \to \mathbb{R}^2$ such that

(1) there is a real number r so that $f(x) = (x, 0)$ for any real number x and $|x| > r$;

(2) each intersection point is double and transverse;

(3) each intersection point is endowed with classical (with a choice for underpass and overpass specified) or virtual crossing structure.

Definition 2. A long virtual knot is an equivalence class of long virtual knot diagrams modulo generalized Reidemeister moves in Figure 1.

Turaev [9] announced *pointed virtual knot diagrams* equivalent to long virtual knot diagrams, which are mapped onto virtual knot diagrams.

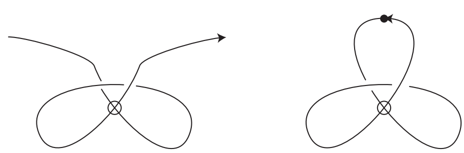

FIGURE 2. A long virtual knot diagram vs. a pointed virtual knot diagram

Two pointed virtual knot diagrams are said to be *stably homeomorphic* if there is an orientation preserving homeomorphism of regular neighborhoods of the underlying curves, sending the first diagram onto the second one and preserving the point, the orientation, and the over/under-crossing information.

Definition 3. [9] Two pointed virtual knot diagrams are stably equivalent if they can be related by a finite sequence of the following transformations:
(1) Stable homeomorphism
(2) The generalized Reidemeister moves on a knot diagram in its surface away from the point.

For the set $\mathcal{K}$ of stable equivalence classes of pointed virtual knot diagrams and the set $\mathcal{L}$ of long virtual knots, it is immediate that there is a one-to-one correspondence between $\mathcal{K}$ and $\mathcal{L}$ up to isotopy. Then we define a Gauss diagram of a long virtual knot diagram as a pointed Gauss diagram of a pointed virtual knot diagram.

A *Gauss diagram* for a virtual knot diagram consists of a counter-clockwise oriented circle S^1 together with signed, oriented m chords connecting $2m$ points on S^1. Since the preimages of the overcrossing and the undercrossing of the virtual knot diagram are connected by a chord directed from the preimage of the overcrossing which is called the tail to the preimage of the undercrossing which is called the head in a circle with an counter-clockwise orientation, we assign a sign to each chord according to the sign of the corresponding real crossing of the virtual knot diagram. For each chord c of G, we assign the signs of endpoints of the chord c so that we assign $sign(c)(-sign(c))$ to the tail(head) of c, respectively.

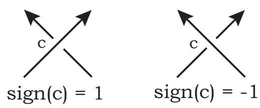

Figure 3. The sign of a crossing c

Let D be a long virtual knot diagram. Then there is a pointed virtual knot diagram $\dot{D}$ corresponding to D. We call the point of $\dot{D}$ the *infinity* in this paper. The Gauss diagram $G(\dot{D})$ of $\dot{D}$ has the point corresponding to the infinity of $\dot{D}$ and the counterclockwise orientation in Figure 4 corresponding to the orientation of $\dot{D}$. Then the Gauss diagram $G(D)$ of D can be defined as the pointed Gauss diagram $G(\dot{D})$. The point of $G(\dot{D})$ as shown in Figure 4 is also called the *infinity.*

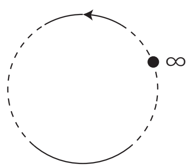

Figure 4. The infinity of the pointed Gauss diagram $G(D)$

Remark 1. In [10], Polyak proved all oriented Reidemeister moves are generated by the following four oriented Reidemeister moves I_a, I_b, II_a, III_a shown in Figure 5.

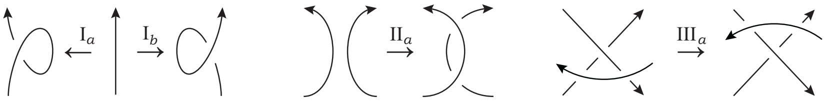

FIGURE 5. Oriented Reidemeister moves for virtual knot diagrams

Consider the corresponding Reidemeister moves I_a, I_b, II_a, III_a and $III_{a'}$ in Gauss diagrams [1] shown in Figure 6. These moves which avoid the neighborhood of the infinity are applied to pointed Gauss diagrams. Two pointed Gauss diagrams are *equivalent* if one can be transformed to the other by finitely many moves in Figure 6.

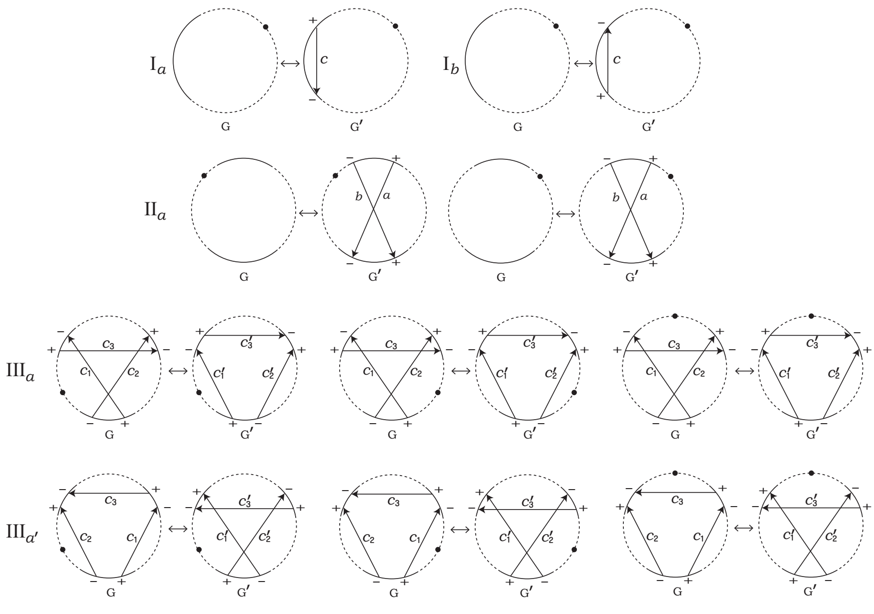

FIGURE 6. Reidemeister moves I_a, I_b, II_a, III_a and $III_{a'}$ for pointed Gauss diagrams

From the above one-to-one correspondence, the polynomial invariant defined for pointed Gauss diagrams is also an invariant for long virtual knot diagrams. Then we focus on the polynomial invariants for pointed Gauss diagrams.

Let G be a pointed Gauss diagram and let c be a chord of G. From the computation of intersection indices for long virtual knots [4], we introduce the lead side of the chord c on G. In Figure 7, the chord c divides the circle of G into two arcs. The *lead side of the chord* c is the arc containing the infinity. We denote the collection of endpoints except those of c on the lead side of a chord c by $L(c)$. Then *the intersection index of* c is the sum of signs of endpoints in $L(c)$. It is denoted by $ind(c)$.

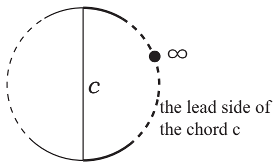

FIGURE 7. The lead side of the chord c

Let G be a pointed Gauss diagram and let $C(G)$ be the set of chords of G. Define a subset of $C(G)$ for each non-negative integer n as

$$C_n(G) = \{c \in C(G) | ind(c) = kn \text{ for some integer } k\}.$$

Definition 4. Let G be a pointed Gauss diagram and let c be a chord of $C_n(G)$. For a non-negative integer n, the *d_n-function for* c denoted by $d_n(c)$ is defined as the sum of signs of endpoints in $L(c)$ whose chords belong to $C_n(G)$. We define the n-th polynomial $Z^n_G \in \mathbb{Z}[t^{\pm 1}]$ for G as

$$Z^n_G(t) = \sum_{c \in C_n(G)} sign(c)(t^{d_n(c)} - 1).$$

Remark 2. $Z^1_G(t)$ is the original index polynomial for a pointed Gauss diagram G.

Then we have the following main result.

Theorem 2.1. *If G and G' are equivalent pointed Gauss diagrams, then $Z^n_G(t) = Z^n_{G'}(t)$ for each non-negative integer n.*

Proof. By mimicking the proof of Theorem 3.3 [3], we can get the result.

Suppose that the number of chords in G is less than or equal to the number of chords in G'. First, we consider the case that G' is obtained from G by applying a single I_a-move. Let c be a new chord of G'. As we see in Figure 6, we have $ind(c) = 0$ and the chord related to c does not meet with any other chord of $Z_n(G')$. Therefore, we obtain $d_n(c) = 0$ and

$$Z^n_{G'}(t) = Z^n_G(t) + sign(c)(t^{d_n(c)} - 1) = Z^n_G(t).$$

For an I_b-move, its result is the same as before.

Second, we consider the case that G' is obtained from G by applying a single II_a-move. Let a and b be the two new chords of G'. Since a and b have the same indices, either $a, b \in C_n(G')$ or $a, b \notin C_n(G')$. If $a, b \notin C_n(G')$, then $Z^n_G(t) = Z^n_{G'}(t)$. Otherwise, endpoints in $L(a)$ whose chords belong to

$C_n(G')$ are the same as the ones in $L(b)$ whose chords belong to $C_n(G')$. Thus, $d_n(a) = d_n(b)$. Since the sign of a is different from the sign of b,

$$Z^n_{G'}(t) = Z^n_G(t) + sign(a)(t^{d_n(a)} - 1) + sign(b)(t^{d_n(b)} - 1) = Z^n_G(t).$$

Finally we consider the case that G' is obtained from G by applying a single III_a-move in Figure 6.

Let c_1, c_2 and c_3 be chords of G in the process of III_a-move. We denote the corresponding three chords of G' by c'_1, c'_2 and c'_3. It is known that the indices of c_i and c'_i are the same for $i = 1, 2, 3$. On both sides of the III_a-move, we obtain $ind(c_1) - ind(c_2) + ind(c_3) = 0$ $(ind(c'_1) - ind(c'_2) + ind(c'_3) = 0)$, respectively [1]. Thus, if these three chords of G are not in $C_n(G)$, then the result follows immediately. Otherwise, either only one chord belongs to $C_n(G)$ or all of them belong to $C_n(G)$. In both cases, we can check that d_n-values of crossings are not changed and the conclusion follows.

Similarly, for the case of an $III_{a'}$-move, it is the same proof as III_a-move.

Since the intersection index is not changed from virtual Reidemeister moves and semivirtual move, the result is true. □

Definition 5. Let D be a long virtual knot diagram and $G(D)$ be a corresponding pointed Gauss diagram. The nth polynomial of D is defined by

$$Z^n_D(t) := Z^n_{G(D)}(t).$$

Example 2.2. *Let D be a long virtual knot diagram and $G(D)$ be the corresponding Gauss diagram in Figure 8. By the computation, we get $i(a) = -1$, $i(b) = 0$, $i(c) = -1$ and $i(d) = 0$ for each chord of $G(D)$. Then, $Z^1_{G(D)}(t)$ is zero, but for chords $\{b, d\}$ with zero indices, we have $d_0(b) = d_0(d) = 1$.*

Therefore, the zero polynomial is $Z^0_{G(D)}(t) = 2(t - 1)$, D is non-trivial.

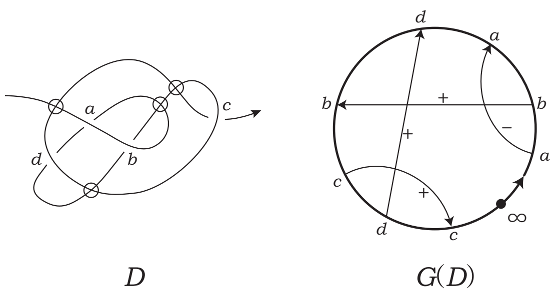

FIGURE 8. The nontrivial example in $Z^n_D(t)$

For a pointed Gauss diagram G, $-G$ is the pointed Gauss diagram with the clockwise oriented circle S^1 which is called the *inverse of* G, while keeping the orientation and the sign of each chord. If G and $-G$ represent the same pointed Gauss diagrams, then G is said to be *invertible*.

For a pointed Gauss diagram G, we denote by G^* the pointed Gauss diagram obtained by changing both the orientation and the sign of all chords of G while keeping the orientation of the circle S^1 of G. G^* is called the *mirror image of* G. If G and G^* represent the same pointed Gauss diagrams, then the Gauss diagram G is called *amphicheiral*.

Proposition 2.3. *Let G be a pointed Gauss diagram and $-G$ be a inverse of G. Then $Z^n_{-G}(t) = Z^n_G(t)$ for each non-negative integer n.*

Proof. Since $-G$ is obtained from G by reversing the orientation, for a chord c of G and the corresponding chord c' of $-G$, $ind(c) = ind(c')$ as Figure 9. Thus, we get $d_n(c) = d_n(c')$ for each $c \in C_n(G)$ and the corresponding chord $c' \in C_n(-G)$. As a consequence, we have $Z^n_{-G}(t) = Z^n_G(t)$ for each non-negative integer n.

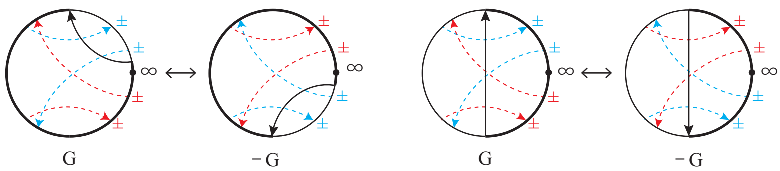

FIGURE 9

□

Proposition 2.4. *Let G be a pointed Gauss diagram and G^* be a mirror image of G. Then $Z^n_{G^*}(t) = -Z^n_G(t^{-1})$ for each non-negative integer n.*

Proof. Following the proof of Proposition 2.3, for a chord c of G and the corresponding chord c' of G^*, $ind(c) = -ind(c')$ but $sign(c) = -sign(c')$ as Figure 10. Then, we have $Z^n_{G^*}(t) = -Z^n_G(t^{-1})$ for each non-negative integer n.

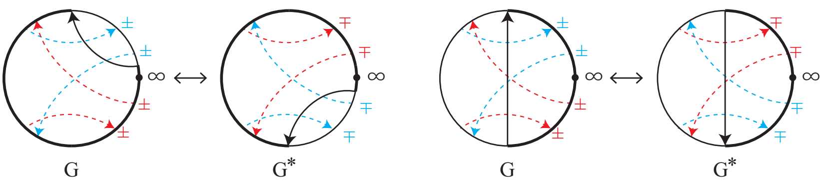

FIGURE 10

□

Remark 3. Let D be a long virtual knot diagram, $-D$ be a inverse of D where $-D$ is obtained from D by changing the orientation of D, and D^* be a mirror image of D. Then $Z^n_D(t) = Z^n_{-D}(t) = -Z^n_{D^*}(t^{-1})$.

Example 2.5. *In Example 2.2, the long virtual knot diagram D is cheiral by Proposition 2.4.*

3. A sequence of two variable polynomials for long virtual knots

In this section, we obtain a sequence of the two-variable polynomials by using early overcrossings and early undercrossings for a long virtual knot diagram.

Definition 6. [8] Let D be a long virtual knot diagram and let c be a classical crossing of D. Then c is called an *early overcrossing (early undercrossing) of D* if the over arc (under arc) of c appears earlier than the under arc (over arc) of c along the orientation of D, respectively.

Then in a pointed Gauss diagram G, we define the *early overchord* c (*early underchord*) if the over (under) information of c appears earlier than the under (over respectively) information along the counterclockwise orientation from the infinity of G. We denote the set of early overchords of G (early overcrossings of D) by $O(G)$ $(O(D))$ and the set of early underchords of G (early undercrossings of D) by $U(G)$ $(U(D))$ for a pointed Gauss diagram G (a long virtual knot diagram D). Then we consider the 2-variable n-th polynomial for pointed Gauss diagrams (long virtual knot diagrams), respectively.

Definition 7. Let G be a pointed Gauss diagram. Then we define a two-variable n-th polynomial for each non-negative integer n as

$$Z_G^n(t_1,t_2)=\sum_{c\in C_n(G)\cap O(G)} sign(c)(t_1^{d_n(c)}-1)+\sum_{c\in C_n(G)\cap U(G)} sign(c)(t_2^{d_n(c)}-1).$$

For a long virtual knot diagram D and a corresponding pointed Gauss diagram $G(D)$, a two-variable n-th polynomial of D is defined by $Z_D^n(t_1,t_2):=Z_{G(D)}^n(t_1,t_2)$ for non-negative integer n.

We have the following result.

Theorem 3.1. *For two equivalent pointed Gauss diagrams G and G', $Z_G^n(t_1,t_2)=Z_{G'}^n(t_1,t_2)$.*

Proof. Suppose that G' is obtained from G by applying a single Polyak's Reidemeister move [10] and the number of chords of G is less than or equal to the number of chords of G'.

For the I_a or I_b-move, we have $a\in O(G')$ or $b\in U(G')$. Then we follow the proof as the one of Theorem 2.1.

For an II_a-move, let a and b be new chords of G'. If a and b both are not in $C_n(G')$, the result is obtained. Otherwise, either $a,b\in O(G')$ or $a,b\in U(G')$ in Figure 6. If $a,b\in C_n(G')\cap O(G')$, we get $sign(a)=-sign(b)$ and $d_n(a)=d_n(b)$. Therefore,

$$Z_{G'}^n(t_1,t_2)=Z_G^n(t_1,t_2)+sign(a)(t_1^{d_n(a)}-1)+sign(b)(t_1^{d_n(b)}-1)=Z_G^n(t_1,t_2).$$

If $a, b \in C_n(G') \cap U(G')$,

$$Z^n_{G'}(t_1, t_2) = Z^n_G(t_1, t_2) + sign(a)(t_2^{d_n(a)} - 1) + sign(b)(t_2^{d_n(b)} - 1) = Z^n_G(t_1, t_2).$$

For an III_a-move, let c_1, c_2 and c_3 be chords of G in the process of the III_a-move and c'_1, c'_2 and c'_3 be the corresponding chords of G'. It c_i is in $O(G)$ $(U(G))$ for any $i = 1, 2, 3$, then c'_i is in $O(G)$ $(U(G)$ respectively). The the conclusion follows immediately.

Similarly, for the case of an $III_{a'}$-move, it is the same proof as III_a-move.

Since the intersection index is not changed from virtual Reidemeister moves and semivirtual move, the result is true. $\square$

The two-variable polynomials have some properties for the inverses and mirror images of pointed Gauss diagrams.

Proposition 3.2. *Let G be a pointed Gauss diagram and $-G$ be a inverse of G. Then $Z^n_{-G}(t_1, t_2) = Z^n_G(t_2, t_1)$ for each non-negative integer n.*

Proof. Since $-G$ has the reversed orientation from G, the set of early overchords and the set of early underchords of G are interchanged. The conclusion follows. $\square$

Proposition 3.3. *Let G be a pointed Gauss diagram and G^* be a mirror image of G. Then $Z^n_{G^*}(t_1, t_2) = -Z^n_G(t_2^{-1}, t_1^{-1})$ for each non-negative integer n.*

Proof. Following the proof of Proposition 2.4 and 3.2, the result is obtained. $\square$

Finally we give an example of a non-trivial long virtual knot which can be recognized by a two variable polynomial although it cannot be found by any n-th one variable polynomial for any non-negative integer n.

Example 3.4. *Let D be a long virtual knot diagram in Figure 11.*

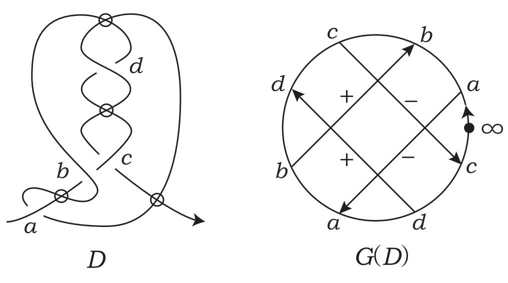

FIGURE 11. The nontrivial example in $Z^n_D(t_1, t_2)$

Since $ind(a) = 2$, $ind(b) = 2$, $ind(c) = -2$, and $ind(d) = -2$, we get $Z^n_G(t) = 0$ for any non-negative integer n. But since we have $a, c \in O(D)$ and $b, d \in U(D)$, two variable 2-nd polynomial $Z^2_D(t_1, t_2)$ is $-t_1^2 - t_1^{-2} + t_2^2 + t_2^{-2}$.

Therefore, D is non-trivial and non-invertible.

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