East Asian Math. J.
Vol. 35 (2019), No. 1, pp. 017–022
http://dx.doi.org/10.7858/eamj.2019.003

CHARACTERIZATION OF TRAVEL GROUPOIDS BY PARTITION SYSTEMS ON GRAPHS

Jung Rae Cho* and Jeongmi Park

Abstract. A travel groupoid is a pair $(V, *)$ of a set V and a binary operation $*$ on V satisfying two axioms. For a travel groupoid, we can associate a graph in a certain manner. For a given graph G, we say that a travel groupoid $(V, *)$ is on G if the graph associated with $(V, *)$ is equal to G. There are some results on the classification of travel groupoids which are on a given graph [1, 2, 3, 9]. In this article, we introduce the notion of vertex-indexed partition systems on a graph, and classify the travel groupoids on the graph by the those vertex-indexed partition systems.

1. Preliminaries

Definition. A *groupoid* is an algebraic system $(V, *)$, where V is a set and $*$ is a binary operation defined on V. A *travel groupoid* is a groupoid $(V, *)$ satisfying the following two axioms:

(t1) For any $u, v \in V$, $(u * v) * u = u$.
(t2) For any $u, v \in V$, if $(u * v) * v = u$ then $u = v$. □

Travel groupoid was introduced by L. Nebeský [8] in 2006 in connection with geodetic graphs [4, 5, 6] and signpost systems [7].

Proposition 1.1 ([8])**.** *Let $(V, *)$ be a travel groupoid. Then the following hold.*

(1) *For any $x \in V$, $x * x = x$.*
(2) *For any $u, v \in V$, $u * v = v$ if and only if $v * u = u$.*
(3) *For any $u, v \in V$, $u * v = u$ if and only if $u = v$.*
(4) *For any $u, v \in V$, $u * (u * v) = u * v$.* □

All graphs in this paper are simple graphs.

Let $(V, *)$ be a travel groupoid, and let G be a graph. We say that $(V, *)$ *is on* G or that G *has* $(V, *)$ if

$$V(G) = V \quad \text{and} \quad E(G) = \{\{u, v\} \mid u, v \in V, u \neq v, \text{ and } u * v = v\}.$$

Received September 6, 2018; Accepted October 20, 2018.
2010 *Mathematics Subject Classification.* 05C05, 05C12, 20N02.
Key words and phrases. travel groupoid, graph, partition system, neighborhood system .
This work was supported by a 2-year Research Grant of Pusan National University.
*Corresponding author.

©2019 The Youngnam Mathematical Society
(pISSN 1226-6973, eISSN 2287-2833)

It is then clear that every travel groupoid is on exactly one graph. However, many different travel groupoids may be on the same graph.

It follows immediately from (t1) of the definition or (4) of Proposition 1.1 that, if $(V, *)$ is a travel groupoid on a graph G, then u and $u * v$ are adjacent in G for distinct elements u and v of V.

For a vertex u of G, let $N_G(u)$ denote the neighborhood of u in G, that is, the set of vertices of G which are adjacent to u.

Lemma 1.2 ([8])**.** *Let G be a graph and let $(V, *)$ be a travel groupoid on G. For any distinct element u and v of V, we have $u * v \in N_G(u)$.* □

2. Right Translation Systems of a Groupoid

Definition. Let $(V, *)$ be a groupoid. For u and v in V, put

$$V^R_{u,v} = \{w \in V \mid u * w = v\}.$$

The system

$$\mathcal{P}^R = \{V^R_{u,v} \mid (u, v) \in V \times V\}$$

is called *right translation system* (or RTS for short) of the groupoid $(V, *)$. □

Remark. The set $V^R_{u,v}$ defined above is the inverse image of v under the left translation $L_u : V \to V$ of the groupoid $(V, *)$ by u defined by $L_u(w) = u * w$ for every w in V, i.e., $V^R_{u,v} = L_u^{-1}(v)$. We can also consider the set $V^R_{u,v}$ as the set of elements whose right translations send u to v, i.e., $V^R_{u,v} = \{w \in V \mid R_w(u) = v\}$, where the right translation $R_w : V \to V$ of $(V, *)$ by w is defined by $R_w(u) := u * w$ for every $u \in V$.

Lemma 2.1. *Let $(V, *)$ be a groupoid. Then, $(V, *)$ satisfies the condition* (t1) *if and only if the RTS $\mathcal{P}^R$ of $(V, *)$ satisfies the following property:*

(R1) *For any $u, v \in V$, if $V^R_{u,v} \neq \varnothing$, then $u \in V^R_{v,u}$.*

Proof. Let $(V, *)$ satisfies the condition (t1) and suppose $V^R_{u,v} \neq \varnothing$. Take an element $w \in V^R_{u,v}$. Then $u * w = v$. By (t1), we have $(u * w) * u = u$. Therefore, we have $v * u = u$ and so $u \in V^R_{v,u}$. Thus the property (R1) holds.

Now, suppose that the RTS of $(V, *)$ satisfies the property (R1). Take any u, v in V and let $u * v = x$. Then $v \in V^R_{u,x}$ and so $V^R_{u,x} \neq \varnothing$. By (R1), $u \in V^R_{x,u}$ and so $x * u = u$. Since $u * v = x$, we have $(u * v) * u = u$. Thus $(V, *)$ satisfies the condition (t1). □

Lemma 2.2. *Let $(V, *)$ be a groupoid. Then, $(V, *)$ satisfies the condition* (t2) *if and only if the RTS $\mathcal{P}^R$ of $(V, *)$ satisfies the following property:*

(R2) *For any $u, v \in V$ with $u \neq v$, $V^R_{u,v} \cap V^R_{v,u} = \varnothing$.*

Proof. Let $(V, *)$ satisfies the condition (t2) and assume $V^R_{u,v} \cap V^R_{v,u} \neq \varnothing$ for some $u, v \in V$ with $u \neq v$. Take an element $x \in V^R_{u,v} \cap V^R_{v,u}$. Then $u * x = v$ and $v * x = u$. Therefore, we have $(u * x) * x = u$, which is a contradiction to the property (t2). Therefor $V^R_{u,v} \cap V^R_{v,u} = \varnothing$ and the property (R2) holds.

Now suppose that the RTS of $(V,*)$ satisfies the property (R2). Assume that there exist $u, v \in V$ with $u \neq v$ such that $(u * v) * v = u$. Put $x = u * v$. Then $v \in V^R_{u,x}$. Since $(u * v) * v = u$, we have $x * v = u$ and so $v \in V^R_{x,u}$. Therefore $v \in V^R_{u,x} \cap V^R_{x,u}$, which is a contradiction to the property (R2). Therefore there exist no pair $u, v \in V$ with $u \neq v$ such that $(u*v)*v = u$, that is, $(V,*)$ satisfies the condition (t2). □

The following gives a characterization of travel groupoids in terms of the RTSs of groupoids.

Theorem 2.3. *Let $(V,*)$ be a groupoid. Then, $(V,*)$ is a travel groupoid if and only if the right translation system of $(V,*)$ satisfies the properties* (R1) *and* (R2)*.*

Proof. The theorem follows from Lemmas 2.1 and 2.2. □

3. Vertex-indexed Partition Systems

Now we introduce the notion of vertex-indexed partition systems on a graph and characterize travel groupoids on the graph by those systems.

For a vertex u in a graph G, let $N_G[u]$ denote the closed neighborhood of u in G, i.e., $N_G[u] := \{u\} \cup N_G(u)$.

Definition. Let $G = (V, E)$ be a graph. A *vertex-indexed partition system* (or VPS for short) on G is a system

$$\mathcal{P} = \{V_{u,v} \subseteq V \mid (u,v) \in V \times V\}$$

satisfying the following conditions:

(i) For any $u \in V$, $\{V_{u,v} \mid v \in N_G[u]\}$ is a partition of V.
(ii) For any $u \in V$, $V_{u,u} = \{u\}$.
(iii) For any $u, v \in V$ with $u \neq v$, $v \in V_{u,v} \iff \{u,v\} \in E$.
(iv) For any $u, v \in V$ with $u \neq v$, $V_{u,v} = \varnothing \iff \{u,v\} \notin E$.
(v) For any $u, v \in V$ with $u \neq v$, $V_{u,v} \cap V_{v,u} = \varnothing$. □

Remark. If $\mathcal{P} = \{V_{u,v} \mid (u,v) \in V \times V\}$ is a VPS on a graph G, then it follows from the conditions (i) and (iv) that, for any $u, v \in V$, there exists a unique vertex w such that $v \in V_{u,w}$.

Definition. Let $\mathcal{P} = \{V_{u,v} \mid (u,v) \in V \times V\}$ be a VPS on a graph G. For $u, v \in V$, let $f_u(v)$ denote the unique vertex w such that $v \in V_{u,w}$. Define a binary operation $*$ on V by $u*v := f_u(v)$ for all u and v in V, and we call $(V,*)$ the *groupoid associated with* $\mathcal{P}$. □

Lemma 3.1. *Let G be a graph and $\mathcal{P}$ be a VPS on G. If $(V,*)$ is the groupoid associated with $\mathcal{P}$, then the RTS $\mathcal{P}^R$ of $(V,*)$ is the same as the VPS $\mathcal{P}$ on G. That is, $\mathcal{P}^R = \mathcal{P}$.*

Proof. It is trivial by definitions of $(V,*)$ and $\mathcal{P}^R$. □

Lemma 3.2. *Let G be a graph and $\mathcal{P}$ a VPS on G. Then, the groupoid associated with $\mathcal{P}$ is a travel groupoid on G.*

Proof. By Lemma 3.1 $\mathcal{P}^R = \mathcal{P}$. So we will use the notation $V_{u,v}$ for $V^R_{u,v}$ as well.

Let $u,\ v \in V$. If $u = v$ then $u \in V_{u,u}$ by (ii). Suppose $u \neq v$ and $V_{u,v} \neq \varnothing$. Then $\{u, v\} \in E$ by (iv), which yields $u \in V_{v,u}$ by (iii). Thus, $\mathcal{P}$ satisfies the property (R1). Moreover, the condition (v) is the same as the property (R2). Therefore, by Theorem 2.3, $(V, *)$ is a travel groupoid on G.

Now, we show that $(V, *)$ is on the graph G. Suppose $(V, *)$ is on the graph $G_{(V,*)}$. It suffices to show that $G_{(V,*)} = G$. Take any edge $\{u, v\}$ in $G_{(V,*)}$. Then, we have $u * v = v$. Therefore $v \in V_{u,v}$. Thus $\{u, v\}$ is an edge in G by (iii), and so $E(G_{(V,*)}) \subseteq E(G)$. Now, take any edge $\{u, v\}$ in G. Then, we have $v \in V_{u,v}$. Therefore $u * v = v$. Thus $\{u, v\}$ is an edge in $G_{(V,*)}$ by the definition of $G_{(V,*)}$, and so $E(G) \subseteq E(G_{(V,*)})$. Hence we have $G_{(V,*)} = G$. □

Lemma 3.3. *Let G be a graph, and let $(V, *)$ be a travel groupoid on G. Then, the RTS $\{V^R_{u,v} \mid (u, v) \in V \times V\}$ of $(V, *)$ is a VPS on G.*

Proof. Fix any $u \in V$ and let $w \in V$. Put $v = u * w$ then $v \in N_G[u]$ and $w \in V^R_{u,v}$. Thus $w \in \bigcup_{v\in N_G[u]} V^R_{u,v}$. Therefore, $\bigcup_{v\in N_G[u]} V^R_{u,v} = V$. Assume that $V^R_{u,x} \cap V^R_{u,y} \neq \varnothing$ for some $x, y \in N_G[u]$ with $x \neq y$. Take $z \in V^R_{u,x} \cap V^R_{u,y}$. Then $u*z = x$ and $u*z = y$. Therefore we have $x = y$, which is a contradiction to the assumption that $x \neq y$. Thus $V^R_{u,x} \cap V^R_{u,y} = \varnothing$ for any $x, y \in N_G[u]$ with $x \neq y$. Therefore $\{V^R_{u,v} \mid v \in N_G[u]\}$ is a partition of V. Thus the condition (i) holds.

By Proposition 1.1(3), $u * v = u$ if and only if $u = v$. Therefore $V^R_{u,u} = \{u\}$ and thus the condition (ii) holds.

If u and v are adjacent in G, then we have $u * v = v$, and so $v \in V^R_{u,v}$. If $v \in V^R_{u,v}$, then we have $u * v = v$, and so $\{u, v\}$ is an edge of G. Thus the condition (iii) holds.

Since $u * w$ is a neighbor of u in G for any $w \in V \setminus \{u\}$, if u and v are not adjacent in G with $u \neq v$, then there is no w such that $u * w = v$, and so $V^R_{u,v} = \varnothing$. If $V^R_{u,v} = \varnothing$, then we have $u * v \neq v$, and so u and v are not adjacent in G. Thus the condition (iv) holds.

Since $(V, *)$ satisfies the condition (t2), it follows from Lemma 2.2 that the condition (v) holds.

Hence the RTS of $(V, *)$ is a VPS on G. □

The following theorem yields a characterization of travel groupoids on a graph in terms of vertex-indexed partition systems on the graph.

Theorem 3.4. *Let G be a graph. Then, there exists one-to-one correspondence between the set of all travel groupoids on G and the set of all VPSs on G.*

Proof. Let $V := V(G)$. Let $\mathrm{TG}(G)$ denote the set of all travel groupoids on G and let $\mathrm{VPS}(G)$ denote the set of all VPSs on G.

We define a map $\Phi : \mathrm{VPS}(G) \to \mathrm{TG}(G)$ as follows: For $\mathcal{P} \in \mathrm{VPS}(G)$, let $\Phi(\mathcal{P})$ be the groupoid $(V, *)_{\mathcal{P}}$ associated with $\mathcal{P}$. By Lemma 3.2, $(V, *)_{\mathcal{P}}$ is a travel groupoid on G.

We define a map $\Psi : \mathrm{TG}(G) \to \mathrm{VPS}(G)$ as follows: For $(V, *) \in \mathrm{TG}(G)$, let $\Psi((V, *))$ be the RTS $\mathcal{P}^R$ of $(V, *)$. By Lemma 3.3, $\Psi((V, *))$ is a vertex-indexed partition system on G.

Then, we can check that $\Psi(\Phi(\mathcal{P})) = \mathcal{P}$ holds for any $\mathcal{P} \in \mathrm{VPS}(G)$ and that $\Phi(\Psi((V, *))) = (V, *)$ holds for any $(V, *) \in \mathrm{TG}(G)$. Hence the map Φ is a one-to-one correspondence between the sets $\mathrm{VPS}(G)$ and $\mathrm{TG}(G)$. $\square$

Example 3.5. Let $G = (V, E)$ be the graph in Figure 1 defined by $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\}$, and $(V, *)$ be the groupoid defined by the multiplication table in Table 1. It can be checked that $(V, *)$ is a travel groupoid on the graph G.

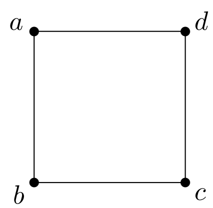

Figure 1

$*$	a	b	c	d
a	a	b	d	d
b	a	b	c	a
c	b	b	c	d
d	a	c	c	d

Table 1

Let $\mathcal{P} = \{V_{u,v} \mid (u, v) \in V \times V\}$ be the system of subsets of V defined by

$$\begin{array}{llll} V_{a,a} = \{a\}, & V_{a,b} = \{b\}, & V_{a,c} = \varnothing, & V_{a,d} = \{c, d\}, \\ V_{b,a} = \{a, d\}, & V_{b,b} = \{b\}, & V_{b,c} = \{c\}, & V_{b,d} = \varnothing, \\ V_{c,a} = \varnothing, & V_{c,b} = \{a, b\}, & V_{c,c} = \{c\}, & V_{c,d} = \{d\}, \\ V_{d,a} = \{a\}, & V_{d,b} = \varnothing, & V_{d,c} = \{b, c\}, & V_{d,d} = \{d\}. \end{array}$$

Then $\mathcal{P}$ is a VPS on G. Now one can check that the RTS of $(V, *)$ is $\mathcal{P}$ and the groupoid associated with $\mathcal{P}$ is $(V, *)$. $\square$

References

[1] J. R. Cho, J. Park, and Y. Sano: Travel groupoids on infinite graphs, *Czechoslovak Mathematical Journal* **64** (2014) 763–766.

[2] J. R. Cho, J. Park, and Y. Sano: The non-confusing travel groupoids on a finite connected graph, *Lecture Notes in Computer Science* **8845** (2014) 14–17.

[3] J. R. Cho, J. Park, and Y. Sano: T-neighbor systems and travel groupoids on a graph, *Graphs and Combinatorics* **33** (2017) 1521–1529.

[4] L. Nebeský: An algebraic characterization of geodetic graphs, *Czechoslovak Mathematical Journal* **48(123)** (1998) 701–710.

[5] L. Nebeský: A tree as a finite nonempty set with a binary operation, *Mathematica Bohemica* **125** (2000) 455–458.

[6] L. Nebeský: New proof of a characterization of geodetic graphs, *Czechoslovak Mathematical Journal* **52(127)** (2002) 33–39.

[7] L. Nebeský: On signpost systems and connected graphs, *Czechoslovak Mathematical Journal* **55(130)** (2005) 283–293.

[8] L. Nebeský: Travel groupoids, *Czechoslovak Mathematical Journal* **56(131)** (2006) 659–675.

[9] D. K. Matsumoto and A. Mizusawa: A construction on smooth travel groupoids on finite graphs, *Graphs and Combinatorics* **32** (2016) 1117–1124.

Jung Rae Cho
Department of Mathematics
Pusan National University
Busan, 609-735, Korea
E-mail address: `jungcho@pusan.ac.kr`

Jeongmi Park
Faculty of Engineering
Information and Systems
University of Tsukuba
Tsukuba, Ibaraki, 305-8573, Japan
E-mail address: `park.jeongmi.ft@u.tsukuba.ac.jp`