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REGULARITY OF THE SCHRÖDINGER EQUATION FOR A CAUCHY-EULER TYPE OPERATOR

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Abstract. We consider the initial value problem of the Schrödinger equation for an interesting Cauchy-Euler type operator $\mathscr{R}$ on $\mathbb{C}^n$ that is an analogue of the harmonic oscillator in $\mathbb{R}^n$. We get an appropriate $L^1 - L^\infty$ dispersive estimate for the solution of the initial value problem.

1. Introduction and statement of the main result

Associated to any self-adjoint differential operator L on $\mathbb{R}^n$, one can formally define an oscillatory semigroup e^{-itL}, using the spectral theory for L. Assume that L has the spectral representation

$$Lf = \int_E \lambda dP_\lambda(f), \quad f \in L^2(\mathbb{R}^n),$$

where P_λ is a projection valued measure supported on the spectrum E of L. Then the operator e^{-itL} can be defined by

$$e^{-itL} f = \int_E e^{-it\lambda}\, dP_\lambda(f), \quad f \in L^2(\mathbb{R}^n).$$

Consider the differential operator $i\partial_t - L$ and the associated initial value problem for the Schrödinger equation for L:

$$\begin{cases} (i\partial_t - L)u & = 0 \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R} \\ u(\cdot, 0) & = f \quad \text{on} \quad \mathbb{R}^n. \end{cases}$$

Assuming $f \in L^2(\mathbb{R}^n)$, the solution u can be represented by

$$u(x,t) = e^{-itL} f(x).$$

We thus call e^{-itL}, the Schrödinger oscillatory semigroup for L.

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Let H be the most basic Schrödinger operator in $\mathbb{R}^n, n \geq 1$, the Hermite operator (or the harmonic oscillator):

$$H = -\Delta + |x|^2. \tag{1}$$

Then the Schrödinger equation for H can be written by

$$(i\partial_t - H)u = 0.$$

This is an important model in quantum mechanics (see for example [4]).

In [5], Nandakumarana and Ratnakumar considered the regularity of the following initial value problem for the Schrödinger equation for H:

$$\begin{cases} (i\partial_t - H)u & = 0 \quad \text{on} \quad \mathbb{R}^n \times \mathbb{R} \\ u(\cdot, 0) & = f \quad \text{on} \quad \mathbb{R}^n. \end{cases}$$

For $f \in L^2(\mathbb{R}^n)$ the solution to the initial value problem is given by

$$u(x,t) = e^{-itH} f(x).$$

They proved the following regularity estimate

$$\int_{-\pi}^{\pi} \|u(\cdot,t)\|^q_{L^p(\mathbb{R}^n)}\, dt \leq C_n \|f\|_2^q,$$

where $1 < q < \infty$, $2 \leq p < \Lambda$, where $\Lambda = \infty$ for $n = 1$ and $\Lambda = \frac{2n}{n-2}$ for $n \geq 2$.

Let $\mathbb{C}^n$ be the complex n-space and dV be the ordinary volume measure on $\mathbb{C}^n$. If $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ are points in $\mathbb{C}^n$, we write

$$z \cdot \overline{w} = \sum_{j=1}^n z_j \overline{w}_j, \qquad |z| = (z \cdot \overline{z})^{1/2}.$$

For any $0 < p \leq \infty$ we let $L^p_G(\mathbb{C}^n)$ denote the space of Lebesgue measurable functions f on $\mathbb{C}^n$ such that the function $f(z)e^{-\frac{1}{2}|z|^2}$ is in $L^p(\mathbb{C}^n, dV)$. When $0 < p < \infty$, it is clear that

$$L^p_G(\mathbb{C}^n) = L^p\left(\mathbb{C}^n, e^{-\frac{p}{2}|z|^2}\, dV(z)\right).$$

We define

$$\|f\|_{L^p_G} = \left[\left(\frac{p}{2\pi}\right)^n \int_{\mathbb{C}^n} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p dV(z)\right]^{\frac{1}{p}}.$$

For $p = \infty$ the norm in $L^\infty_G(\mathbb{C}^n)$ is defined by

$$\|f\|_{L^\infty_G} = \operatorname{esssup}\left\{|f(z)|e^{-\frac{1}{2}|z|^2} : z \in \mathbb{C}^n\right\}.$$

Let $F^p(\mathbb{C}^n)$ denote the space of entire functions in $L^p_G(\mathbb{C}^n)$. If $0 < p < q$, then $F^p \subset F^q$, and the inclusion is proper and continuous (see [7]). Note that F^2 is a closed subspace of the Hilbert space L^2_G (see [7]) with inner product

$$\langle f, g\rangle = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2}\, dV(z).$$

The Hermite operator H on $\mathbb{R}^n$ has the representation

$$H = \frac{1}{2}\sum_{j=1}^{n}(a_j a_j^\dagger + a_j^\dagger a_j)$$

in termes of the creation operators $a_j = -\frac{d}{dt} + x_j$ and the annihilation operator $a_j^\dagger = \frac{d}{dx_j} + x_j$, $j = 1, 2, \ldots, n$. There is an interesting operator $\mathscr{R}$ on $\mathbb{C}^n$, given by

$$\mathscr{R} = \frac{1}{2}\sum_{j=1}^{n}\left(A_j A_j^* + A_j^* A_j\right),$$

where

$$A_j = 2\frac{\partial}{\partial z_j}, \qquad A_j^* = z_j, \quad 1 \leq j \leq n.$$

Both A_j and A_j^*, as defined above, are densely defined linear operators on F^p (unbounded though). We have

$$\mathscr{R} = 2\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} + n.$$

Thus $\mathscr{R}$ is a Cauchy-Euler type operator.

Remark 1. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{z_1^k}{\sqrt{2}^k (k+1)\sqrt{k!}}.$$

Then $f \in F^2$, but $\mathscr{R} f \notin F^2$.

The remark above tells us that $\mathcal{D}om(\mathscr{R}) \subsetneq F^2$. Thus $\mathscr{R}$ is an unbounded operator on F^2.

The Segal-Bargmann transform $\mathcal{B}$ is defined by

$$\mathcal{B}f(z) = \frac{1}{\pi^{n/4}} \int_{\mathbb{R}^n} f(x) e^{x\cdot z - \frac{1}{2}|x|^2 - \frac{1}{4} z\cdot z}\, dV(x),$$

where $dV(x)$ is the volume measure on $\mathbb{R}^n$. It is well-known that the Segal-Bargmann transform is a unitary isomorphism between $L^2(\mathbb{R}^n)$ and $F^2(\mathbb{C}^n)$ ([1], [7]). Moreover, we know that

$$\mathcal{B}H = \mathscr{R}\mathcal{B} \quad \text{on} \quad L^2(\mathbb{R}^n).$$

Motivated by these relations, we consider the initial value problem:

$$\begin{cases} (i\partial_t - \mathscr{R})u &= 0 \quad \text{on} \quad \mathbb{C}^n \times (0, \infty) \\ u(\cdot, 0) &= f \quad \text{on} \quad \mathbb{C}^n. \end{cases} \tag{2}$$

We get an appropriate $L^1 - L^\infty$ dispersive estimate for the solution of the initial value problem as following.

Theorem 1.1. *For $f \in F^1(\mathbb{C}^n)$, the solution $u(z,t) = e^{-it\mathscr{R}} f(z)$ of the initial value problem (2) satisfies the following regularity estimate*

$$\sup_{0<t<\infty} \|u(\cdot,t)\|_{F^q} \le \|f\|_{F^p},$$

where $1 \le p \le 2$, $2 \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

2. Proof of Theorem 1.1

We define

$$e_\alpha(z) = \frac{z^\alpha}{\|z^\alpha\|_{F^2}} = \frac{z^\alpha}{\sqrt{\alpha!}}.$$

Then $\{e_\alpha : \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for F^2. We know that $\mathscr{R}$ is a positive, self-adjoint operator on $\mathcal{D}om(\mathscr{R})$ with the discrete spectrum $\sigma(\mathscr{R}) = \{2|\alpha| + n : \alpha \in \mathbb{N}_0^n\}$ [2]. For $f \in F^2$ let

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha e_\alpha(z)$$

be the orthonormal decomposition of f. Associated with the operator $\mathscr{R}$ is a semigroup $\{B_t\}_{t\ge 0}$ defined by the expansion

$$B_t f(z) = \sum_{\alpha \in \mathbb{N}_0^n} e^{-i(2|\alpha|+n)t} c_\alpha e_\alpha(z).$$

It is easy to see that $B_t f(z)$ converges in F^2 for every fixed $t \ge 0$ whenever $f \in F^2$. Moreover, $B_t f(z) \to f(z)$ in F^2 as $t \to 0^+$ by the dominated convergence theorem since $|e^{-i(2|\alpha|+n)t} - 1| \le 2$. Thus $u(z,t) = B_t f(z)$ is the solution of the initial value problem:

$$\begin{cases} (i\partial_t - \mathscr{R})u &= 0 \quad \text{on} \quad \mathbb{C}^n \times (0,\infty) \\ u(\cdot,0) &= f \quad \text{on} \quad \mathbb{C}^n. \end{cases}$$

We know that $\{B_t\}_{t\ge 0}$ is a strongly continuous semigroup. Moreover, $-i\mathscr{R}$ is the infinitesimal generator of $\{B_t\}_{t\ge 0}$ [2]. That is,

$$\lim_{t\to 0^+} \frac{B_t f - f}{t} = -i\mathscr{R} f.$$

Thus, we have (see [3])

$$B_t = e^{-it\mathscr{R}}.$$

It is well-known ([1], [7]) that for $f \in F^2$ we have the reproducing formula such that

$$f(z) = \int_{\mathbb{C}^n} f(w) K(z,w) e^{-|z|^2}\, dV(w),$$

where $K(z,w)$ is the reproducing kernel defined by

$$K(z,w) = \sum_\alpha e_\alpha(z) \overline{e_\alpha(w)}.$$

By the spectral theory,

$$\begin{aligned}
u(z,t) &= e^{-it\mathscr{R}} f(z) \\
&= e^{-it\mathscr{R}} \int_{\mathbb{C}^n} f(w) \sum_{\alpha} e_\alpha(z)\overline{e_\alpha(w)} e^{-|w|^2} dV(w) \\
&= e^{-it\mathscr{R}} \left(\sum_{\alpha} e_\alpha(z) \right) \int_{\mathbb{C}^n} f(w)\overline{e_\alpha(w)} e^{-|w|^2} dV(w) \\
&= \sum_{\alpha} e^{-it(2|\alpha|+n)} e_\alpha(z) \int_{\mathbb{C}^n} f(w)\overline{e_\alpha(w)} e^{-|w|^2} dV(w) \\
&= \int_{\mathbb{C}^n} f(w) \sum_{\alpha} e^{-it(2|\alpha|+n)} e_\alpha(z)\overline{e_\alpha(w)} e^{-|w|^2} dV(w) \\
&= \int_{\mathbb{C}^n} f(w) K_t(z,w) e^{-|w|^2} dV(w).
\end{aligned}$$

Interchanging the order of summation and integration is justified by the dominated convergence theorem since

$$\sum_{\alpha} |e_\alpha(z)| \int_{\mathbb{C}^n} |f(w)||e_\alpha(w)| e^{-|w|^2} dV(w) \le \sum_{\alpha} \frac{|z^\alpha|}{\sqrt{\alpha!}} \|f\|_{F^2}$$

and the power series on the right side of the inequality above is convergent for every $z \in \mathbb{C}^n$.

Note that

$$\begin{aligned}
K_t(z,w) = &\sum_{\alpha} e^{-it(2|\alpha|+n)} e_\alpha(z)\overline{e_\alpha(w)} \\
=& e^{-int} \sum_{\alpha} e^{-2it|\alpha|} \frac{z^\alpha \bar{w}^\alpha}{\alpha!} \\
=& e^{-int} \exp(e^{-2it} z \cdot \bar{w}).
\end{aligned}$$

Hence

$$|K_t(z,w)| = \exp[\mathrm{Re}(e^{-2it} z \cdot \bar{w})] \le e^{|e^{-2it} z \cdot \bar{w}|} = e^{|z \cdot \bar{w}|}.$$

We first prove that the $B_t = e^{-it\mathscr{R}}$ maps L^1 to L^∞ and L^2 to L^2, respectively, and combine them with Riez-Thorin interpolation to drive the desired result.

Now we can calculate that

$$\begin{aligned}\|u(\cdot,t)\|_{F^\infty} &= \sup_{z\in\mathbb{C}^n} |u(z,t)|e^{-\frac{1}{2}|z|^2} \\ &\le \sup_{z\in\mathbb{C}^n}\left[\int_{\mathbb{C}^n} |f(w)||K_t(z,w)|e^{-|w|^2-\frac{1}{2}|z|^2}dV(w)\right] \\ &\le \sup_{z\in\mathbb{C}^n}\left[\int_{\mathbb{C}^n} |f(w)|e^{-|w|^2-\frac{1}{2}|z|^2+|z\cdot\bar{w}|}dV(w)\right] \\ &\le \left[\int_{\mathbb{C}^n} |f(w)|e^{-\frac{1}{2}|w|^2}dV(w)\right] = \|f\|_{F^1},\end{aligned}$$

where we used the following relation in third inequailty:

$$-|w|^2 - \frac{1}{2}|z|^2 + |z\cdot\bar{w}| \le -|w|^2 - \frac{1}{2}|z|^2 + |z||w| \le -\frac{1}{2}|w|^2.$$

On the other hand, for $f \in F^2$, we have a holomorphic expansion of $f(z) = \sum c_\alpha e_\alpha(z)$. Then

$$\begin{aligned}u(z,t) &= e^{-it\mathscr{R}}f(z) \\ &= e^{-int}\sum_\alpha e^{-2it|\alpha|}c_\alpha e_\alpha(z).\end{aligned}$$

So we have

$$\begin{aligned}\|u(\cdot,t)\|^2_{F^2} &= \langle u(\cdot,t), u(\cdot,t)\rangle \\ &= \left\langle e^{-int}\sum_\alpha e^{-2it|\alpha|}c_\alpha e_\alpha, e^{-int}\sum_\beta e^{-2it|\beta|}c_\beta e_\beta\right\rangle \\ &= \sum_{\alpha,\beta} c_\alpha\overline{c_\beta}e^{-2it(|\alpha|-|\beta|)}\langle e_\alpha, e_\beta\rangle \\ &= \sum_\alpha |c_\alpha|^2 = \|f\|^2_{F^2}.\end{aligned}$$

Hence by Riesz-Thorin interpolation theorem [6], for $p \in [1,2]$ we have

$$\|u(\cdot,t)\|_{F^q} \le \|f\|_{F^p},$$

where (p,q) is a conjugate pair.

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