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Biharmonic Submanifolds of Quaternionic Space Forms

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Abstract. In this paper, we consider biharmonic submanifolds of a quaternionic space form. We give the necessary and sufficient conditions for a submanifold to be biharmonic in a quaternionic space form, we study different particular cases for which we obtain some non-existence results and curvature estimates.

1. Introduction

Let (M^m, g) and (N^n, h) be two Riemannian manifolds. A harmonic map is a map $\varphi : (M^m, g) \to (N^n, h)$ that is a critical point of the energy functional

$$E(\varphi) = \frac{1}{2}\int_D |d\varphi|^2 v^M,$$

for any compact domain D, where v^M is the volume element [1, 6]. The Euler-Lagrange equation of $E(\varphi)$ is

$$\tau(\varphi) = \mathrm{Tr}(\nabla d\varphi) = 0,$$

where $\tau(\varphi)$ is the tension field of φ [1, 6]. The map φ is said to be biharmonic if it is a critical point of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_D |\tau(\varphi)|^2 v^M,$$

for any compact domain D. In [11], Jiang obtained the Euler-Lagrange equation of $E_2(\varphi)$. This gives us

$$\tau_2(\varphi) = \mathrm{Tr}(\nabla^\varphi \nabla^\varphi - \nabla^\varphi_\nabla)\tau(\varphi) - \mathrm{Tr}(R^N(d\varphi, \tau(\varphi))d\varphi) = 0, \tag{1.1}$$

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where $\tau_2(\varphi)$ is the bitension field of φ and R^N is the curvature tensor of N, ∇^φ denote the pull-back connection on $\varphi^{-1}TN$. Harmonic maps are always biharmonic maps by definition.

A submanifold in a Riemannian manifold is called a biharmonic submanifold if the isometric immersion defining the submanifold is a biharmonic map, and a biharmonic map is called a proper-biharmonic map if it is non-harmonic map. Also, we will call proper-biharmonic submanifolds a biharmonic submanifols which is non-harmonic [3, 7].

During the last decades, there are several results concerning the biharmonic submanifolds in space forms like real space forms [4], complex space forms [7], Sasakian space forms [8], generalized complex and Sasakian space forms [15], products of real space forms [14]. Motivated by this works, in this note, we will focus our attention on biharmonic submanifolds of quaternionic space form, we first give the necessary and sufficient condition for submanifolds to be biharmonic. Then, we apply this general result to many particular cases and obtain some non-existence results and curvature estimates.

2. Preliminaries

We recall some facts on quaternionic Kähler manifolds and their submanifolds. For a more detail we refer the reader, for example, to [2, 5, 10, 12, 13, 16].

An almost quaternionic structure on a smooth manifold N is a rank-three subbundle $\sigma \subset End(TN)$ such that a local basis $J = (J_\alpha)_{\alpha=1,2,3}$ exists of sections of σ satisfying

$$\begin{cases} J_\alpha^2 = -Id \\ J_1 J_2 = -J_2 J_1 = J_3 \end{cases} \tag{2.1}$$

where $\alpha = 1, 2, 3$. The pair (N, σ) is called an almost quaternionic manifold.

Let (N, σ) be an almost quaternionic manifold. A metric tensor field g on N is called adapted to σ if the following compatibility condition holds

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{2.2}$$

for all local basis $(J_\alpha)_{\alpha=1,2,3}$ of σ and $X, Y \in \Gamma(TN)$. The triple (N, σ, g) is said to be an almost quaternionic Hermitian manifold. It is easy to see that any almost quaternionic Hermitian manifold has dimension $4n$. An almost quaternionic Hermitian manifold (N, σ, g) is a quaternionic Kähler manifold if the Levi-Civita connection verifies

$$\nabla J_\alpha = \omega_\gamma \otimes J_\beta - \omega_\beta \otimes J_\gamma$$

for any cyclic permutation (α, β, γ) of $(1, 2, 3)$, $(\omega_\alpha)_{\alpha=1,2,3}$ being local 1-forms over the open for which $(J_\alpha)_{\alpha=1,2,3}$ is a local basis of σ.

A submanifold M of a quaternionic Kähler manifold N is called a quaternionic submanifold (resp. totally real submanifold) if $J_\alpha(TM) \subset TM$ (resp. $J_\alpha(TM) \subset$

$TM^{\perp}$), $\alpha = 1, 2, 3$, where TM and $TM^{\perp}$ denote the tangent and normal bundle of M, respectively.

Let (N, σ, g) be a quaternionic Kähler manifold and let X be a unit vector tangent to (N, σ, g). Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $Q(X)$, is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is called a quaternionic sectional curvature. A quaternionic Kähler manifold is a quaternionic space form if its quaternionic sectional curvatures are equal to a constant, say $4c$. We denote by $N^{4n}(4c)$ the quaternionic space form of constant quaternionic sectional curvature $4c$. The Standard models of quaternionic space forms are the quaternionic projective space $\mathbb{H}P^n(4c)(c > 0)$, the quaternionic space $\mathbb{H}^n(c = 0)$ and the quaternionic hyperbolic space $\mathbb{H}H^n(4c)(c < 0)$. The Riemannian curvature tensor R of a quaternionic space form $N^{4n}(4c)$ is of the form

$$
\begin{aligned}
R(X,Y)Z \quad &= \quad c\Big\{\langle Z,Y\rangle X - \langle X,Z\rangle Y + \sum_{\alpha=1}^{3}[\langle Z,J_\alpha Y\rangle J_\alpha X - \langle Z,J_\alpha X\rangle J_\alpha Y \\
&\qquad +2\langle X,J_\alpha Y\rangle J_\alpha Z]\Big\} \qquad (2.3)
\end{aligned}
$$

for $X, Y, Z \in \Gamma(TN^{4n}(4c))$, where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on $N^{4n}(4c)$ and $(J_\alpha)_{\alpha=1,2,3}$ is a local basis of σ.

Now, let M be a submanifold of a quaternionic space form $N^{4n}(4c)$. Then for any $X \in \Gamma(TM)$, we write

$$J_\alpha X = j_\alpha X + k_\alpha X$$

where, $j_\alpha : TM \to TM$ and $k_\alpha : TM \to NM$, here NM denote the normal bundle of M. Similarly, for any $\xi \in \Gamma(NM)$, we have

$$J_\alpha \xi = l_\alpha \xi + m_\alpha \xi$$

where, $l_\alpha : NM \to TM$ and $k_\alpha : NM \to NM$. Since for any $\alpha \in \{1, 2, 3\}$, J_α satisfies (2.1) and (2.2). Then, we deduce that the operators $j_\alpha, k_\alpha, l_\alpha, m_\alpha$ satisfy the following relations

$$j_\alpha^2 X + l_\alpha k_\alpha X = -X, \tag{2.4}$$
$$m_\alpha^2 \xi + k_\alpha l_\alpha \xi = -\xi, \tag{2.5}$$
$$j_\alpha l_\alpha \xi + l_\alpha m_\alpha \xi = 0, \tag{2.6}$$
$$k_\alpha j_\alpha X + m_\alpha k_\alpha X = 0, \tag{2.7}$$
$$g(k_\alpha X, \xi) = -g(X, l_\alpha \xi), \tag{2.8}$$
$$g(j_\alpha X, Y) = -g(X, j_\alpha Y), \tag{2.9}$$
$$g(m_\alpha \xi, \eta) = -g(\xi, m_\alpha \eta). \tag{2.10}$$

for all $X, Y \in \Gamma(TM)$ and all $\xi, \eta \in \Gamma(NM)$.

3. Biharmonic Submanifolds of Quaternonic Space Forms

Let M^m be a submanifold of $N^{4n}(4c)$, $\varphi : M^m \to N^{4n}(4c)$ be the canonical inclusion. We shall denote by B, A, H, Δ and $\Delta^\perp$ the second fundamental form, the shape operator, the mean curvature vector field, the Laplacian and the Laplacian on the normal bundle of M^m in $N^{4n}(4c)$, respectively.

Theorem 3.1. *Let M^m be a submanifold of $N^{4n}(4c)$. Then M^m is biharmonic if and only if*

$$\begin{cases} \Delta^\perp H + \operatorname{Tr}(B(\cdot, A_H(\cdot))) - mcH + 3c\sum_{\alpha=1}^{3} k_\alpha l_\alpha H = 0, \\ \frac{m}{2}\operatorname{grad}(|H|^2) + 2\operatorname{Tr}(A_{\nabla^\perp_\cdot H}\cdot) + 3c\sum_{\alpha=1}^{3} j_\alpha l_\alpha H = 0. \end{cases} \tag{3.1}$$

Proof. Choose a local geodesic orthonormal frame $\{e_i\}_{1\leq i\leq m}$ at point p in M^m. Then calculating at p, by the use of the Gauss and Weingarten formulas, we have

$$\begin{aligned} \Delta H &= -\sum_{i=1}^{m}(\nabla^\varphi_{e_i}\nabla^\varphi_{e_i}H) = -\sum_{i=1}^{m}(\nabla^\varphi_{e_i}(-A_He_i + \nabla^\perp_{e_i}H)) \\ &= -\sum_{i=1}^{m}(-\nabla_{e_i}A_He_i - B(e_i, A_He_i) - A_{\nabla^\perp_{e_i}H}e_i + \nabla^\perp_{e_i}\nabla^\perp_{e_i}H) \\ &= \operatorname{Tr}(\nabla_\cdot A_H\cdot)\operatorname{Tr}B(\cdot, A_H\cdot) + \operatorname{Tr}(A_{\nabla^\perp_\cdot H}\cdot) + \Delta^\perp H. \end{aligned} \tag{3.2}$$

Moreover,

$$\begin{aligned} \operatorname{Tr}(\nabla_\cdot A_H\cdot) &= \sum_{i=1}^{m}\nabla_{e_i}A_H(e_i) = \sum_{i,j=1}^{m}\nabla_{e_i}(\langle A_H(e_i), e_j\rangle e_j) = \sum_{i,j=1}^{m}(e_i\langle A_H(e_i), e_j\rangle)e_j \\ &= \sum_{i,j=1}^{m}(e_i\langle B(e_j, e_i), H\rangle)e_j = \sum_{i,j=1}^{m}(e_i\langle \nabla^\varphi_{e_j}e_i, H\rangle)e_j \\ &= \sum_{i,j=1}^{m}(\langle \nabla^\varphi_{e_i}\nabla^\varphi_{e_j}e_i, H\rangle + \langle \nabla^\varphi_{e_j}e_i, \nabla^\varphi_{e_i}H\rangle)e_j \\ &= \sum_{i,j=1}^{m}(\langle \nabla^\varphi_{e_i}\nabla^\varphi_{e_j}e_i, H\rangle + \langle B(e_j, e_i), \nabla^\perp_{e_i}H\rangle)e_j \\ &= \sum_{i,j=1}^{m}(\langle \nabla^\varphi_{e_i}\nabla^\varphi_{e_j}e_i, H\rangle + \langle A_{\nabla^\perp_{e_i}H}(e_i), e_j\rangle)e_j \\ &= \sum_{i,j=1}^{m}\langle \nabla^\varphi_{e_i}\nabla^\varphi_{e_j}e_i, H\rangle e_j + \sum_{i=1}^{m}A_{\nabla^\perp_{e_i}H}(e_i). \end{aligned} \tag{3.3}$$

Further, using (2.3) and (2.4) we have

$$\begin{aligned}
\sum_{i,j=1}^{m}\langle\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_j}e_i,H\rangle e_j &= \sum_{i,j=1}^{m}\langle R^{N^{4n}(4c)}(e_i,e_j)e_i+\nabla^{\varphi}_{e_j}\nabla^{\varphi}_{e_i}e_i+\nabla^{\varphi}_{[e_i,e_j]}e_i,H\rangle e_j\\
&= c\sum_{i,j=1}^{m}\sum_{\alpha=1}^{3}\{\langle\langle e_i,e_j\rangle e_i-\langle e_i,e_i\rangle e_j+\langle e_i,J_\alpha e_j\rangle J_\alpha e_i\\
&\quad -\langle e_i,J_\alpha e_i\rangle J_\alpha e_j+2\langle e_i,J_\alpha e_j\rangle J_\alpha e_i,H\rangle e_j\}\\
&\quad +\sum_{i,j=1}^{m}\langle\nabla^{\varphi}_{e_j}B(e_i,e_i)+\nabla^{\varphi}_{e_j}\nabla_{e_i}e_i,H\rangle e_j\\
&= 3c\sum_{i,j=1}^{m}\sum_{\alpha=1}^{3}\langle J_\alpha(\langle J_\alpha e_j,e_i\rangle e_i),H\rangle e_j+m\sum_{j=1}^{m}\langle\nabla^{\varphi}_{e_j}H,H\rangle e_j\\
&\quad +\sum_{i,j=1}^{m}\langle\nabla_{e_j}\nabla_{e_i}e_i+B(e_j,\nabla_{e_i}e_i),H\rangle e_j\\
&= 3c\sum_{i,j=1}^{m}\sum_{\alpha=1}^{3}\langle j_\alpha^2 e_j,H\rangle e_j+\frac{m}{2}\sum_{j=1}^{m}e_j(|H|^2)e_j\\
&= 3c\sum_{i,j=1}^{m}\sum_{\alpha=1}^{3}\langle -e_j-l_\alpha k_\alpha e_j,H\rangle e_j+\frac{m}{2}\operatorname{grad}(|H|^2)\\
&= \frac{m}{2}\operatorname{grad}(|H|^2). \qquad (3.4)
\end{aligned}$$

Reporting (3.4) into (3.3), we find

$$\operatorname{Tr}(\nabla.A_H\cdot)=\frac{m}{2}\operatorname{grad}(|H|^2)+\sum_{i=1}^{m}A_{\nabla^{\perp}_{e_i}H}(e_i). \qquad (3.5)$$

Replacing (3.5) into (3.2), we get the following formula

$$\Delta H=\frac{m}{2}\operatorname{grad}(|H|^2)+\operatorname{Tr}(B(\cdot,A_H(\cdot)))+2\operatorname{Tr}(A_{\nabla^{\perp}_{\cdot}H}\cdot)+\Delta^{\perp}H. \qquad (3.6)$$

Furthermore, we have

$$\tau(\varphi)=\operatorname{Tr}(\nabla d\varphi)=mH. \qquad (3.7)$$

From (1.1) and (3.7), we find

$$\begin{aligned}
\tau_2(\varphi) &= \operatorname{Tr}(\nabla^{\varphi}\nabla^{\varphi}-\nabla^{\varphi}_{\nabla})\tau(\varphi)-\operatorname{Tr}(R^{N^{4n}(4c)}(d\varphi,\tau(\varphi))d\varphi)\\
&= \sum_{i=1}^{m}(\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i}-\nabla^{\varphi}_{\nabla_{e_i}e_i})mH-\sum_{i=1}^{m}R^{N^{4n}(4c)}(d\varphi(e_i),mH)d\varphi(e_i)\\
&= -m\{\Delta H+\sum_{i=1}^{m}R^{N^{4n}(4c)}(d\varphi(e_i),H)d\varphi(e_i)\} \qquad (3.8)
\end{aligned}$$

By (2.3), we get

$$\begin{aligned}\sum_{i=1}^{m} R^{N^{4n}(4c)}(d\varphi(e_i), H)d\varphi(e_i) &= -mcH + 3c\sum_{i=1}^{m}\sum_{\alpha=1}^{3}\langle l_\alpha H, e_i\rangle J_\alpha e_i \\ &= -mcH + 3c\sum_{\alpha=1}^{3} J_\alpha l_\alpha H \\ &= -mcH + 3c\sum_{\alpha=1}^{3} j_\alpha l_\alpha H + 3c\sum_{\alpha=1}^{3} k_\alpha l_\alpha H. \end{aligned} \tag{3.9}$$

Substituting (3.6) and (3.9) into (3.8), we obtain

$$\begin{aligned}\tau_2(\varphi) &= -m\{\frac{m}{2}\operatorname{grad}(|H|^2) + \operatorname{Tr}(B(\cdot, A_H(\cdot))) + 2\operatorname{Tr}(A_{\nabla^\perp_\cdot H}\cdot) + \Delta^\perp H - mcH \\ &\quad +3c\sum_{\alpha=1}^{3} j_\alpha l_\alpha H + 3c\sum_{\alpha=1}^{3} k_\alpha l_\alpha H\}. \end{aligned} \tag{3.10}$$

Obviously $j_\alpha l_\alpha H$ is tangent and $k_\alpha l_\alpha H$ is normal for all $\alpha \in \{1,2,3\}$, comparing the tangential and the normal parts, we get (3.1) and this completes the proof. □

Corollary 3.2. *Let $N^{4n}(4c)$ be a quaternionic space form and M^{4n-1} a real hypersurface of $N^{4n}(4c)$. Then M^{4n-1} is biharmonic if and only if*

$$\begin{cases}\Delta^\perp H + \operatorname{Tr}(B(\cdot, A_H(\cdot))) - 4c(n+2)H = 0, \\ \frac{4n-1}{2}\operatorname{grad}(|H|^2) + 2\operatorname{Tr}(A_{\nabla^\perp_\cdot H}\cdot) = 0.\end{cases} \tag{3.11}$$

Proof. As M^{4n-1} is a hypersurface, then J_α for all $\alpha \in \{1,2,3\}$ maps normal vectors on tangent vectors, that is, $m_\alpha = 0$ for all $\alpha \in \{1,2,3\}$. Hence, by relation (2.5), we have $k_\alpha l_\alpha H = -H$ and by relation (2.6), $j_\alpha l_\alpha H = 0$, which gives the result by Theorem 3.1. □

Corollary 3.3. *Let $N^{4n}(4c)$ be a quaternionic space form and M^m a totally real submanifold of $N^{4n}(4c)$. Then M^m is biharmonic if and only if*

$$\begin{cases}\Delta^\perp H + \operatorname{Tr}(B(\cdot, A_H(\cdot))) - c(m+9)H = 0, \\ \frac{m}{2}\operatorname{grad}(|H|^2) + 2\operatorname{Tr}(A_{\nabla^\perp_\cdot H}\cdot) = 0.\end{cases} \tag{3.12}$$

Proof. As M^m is a totally real submanifold, then $j_\alpha = 0$ for all $\alpha \in \{1,2,3\}$ and by the use of (2.6) we deduce that $m_\alpha = 0$ for all $\alpha \in \{1,2,3\}$. Moreover, by relation (2.5), we have $k_\alpha l_\alpha H = -H$, which gives the proof by Theorem 3.1. □

Remark 3.4. In [5], Chen proved that every quaternionic submanifold of a quaternionic Kähler manifold is totally geodesic. Then we deduce that every quaternionic submanifold of $N^{4n}(4c)$ is biharmonic, and there exists no proper-biharmonic quaternionic submanifold in $N^{4n}(4c)$.

Corollary 3.5. *Let $N^{4n}(4c)$ be a quaternionic space form and M^m a totally real submanifold of $N^{4n}(4c)$ with parallel mean curvature vector field. Then it is biharmonic if and only if*

$$\operatorname{Tr}(B(\cdot, A_H(\cdot))) = c(m+9)H.$$

Proof. Since M^m has parallel mean curvature, we obtain immediately the result from the Corollary 3.3. □

Proposition 3.6. *Let M^{4n-1} be a real hypersurface of $N^{4n}(4c)$ with non-zero constant mean curvature. Then M^{4n-1} is proper-biharmonic if and only if*

$$|B|^2 = 4c(n+2), \tag{3.13}$$

or equivalently, M^{4n-1} is proper-biharmonic if and only if the scalar curvature of M^{4n-1} satisfies

$$\operatorname{Scal}^{M^{4n-1}} = c\Big\{(4n-1)(4n+7) - 4n - 17\Big\} + (4n-1)^2|H|^2.$$

Proof. Assume that M^{4n-1} is a real hypersurface of $N^{4n}(4c)$ with non-zero constant mean curvature. Then, by Corollary 3.2, the first equation of (3.11) becomes

$$\operatorname{Tr}(B(\cdot, A_H(\cdot))) = 4c(n+2)H. \tag{3.14}$$

As, for hypersurfaces, we have $A_H = HA$, then we can write

$$\operatorname{Tr}(B(\cdot, A_H(\cdot))) = H\operatorname{Tr}(B(\cdot, A(\cdot))) = H|B|^2. \tag{3.15}$$

Reporting (3.15) into (3.14), we get the identity (3.13).

For the second equivalence, by the use of the Gauss equation, we find

$$\begin{aligned}\operatorname{Scal}^{M^{4n-1}} &= \sum_{i,j=1}^{4n-1}\langle R^{N^{4n}(4c)}(e_i,e_j)e_j,e_i\rangle + \sum_{i,j=1}^{4n-1}\langle B(e_i,e_i), B(e_j,e_j)\rangle \\ &\quad -\sum_{i,j=1}^{4n-1}\langle B(e_j,e_i), B(e_j,e_i)\rangle,\end{aligned} \tag{3.16}$$

where $\{e_i\}_{1\le i\le 4n-1}$ is a local orthonormal frame of M^{4n-1}. Therefore

$$\operatorname{Scal}^{M^{4n-1}} = \sum_{i,j=1}^{4n-1}\langle R^{N^{4n}(4c)}(e_i,e_j)e_j,e_i\rangle + (4n-1)^2|H|^2 - |B|^2. \tag{3.17}$$

Using (2.3) we have

$$\begin{aligned}\sum_{i,j=1}^{4n-1}\langle R^{N^{4n}(4c)}(e_i,e_j)e_j,e_i\rangle &= c\Big\{(4n-1)^2 - (4n-1) + 9(4n-1) - 9\Big\} \\ &= c\Big\{(4n-1)(4n+7) - 9\Big\}.\end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.17), we get

$$\mathrm{Scal}^{M^{4n-1}} = c\Big\{(4n-1)(4n+7)-9\Big\} + (4n-1)^2|H|^2 - |B|^2. \tag{3.19}$$

Hence, we deduce that M is proper-biharmonic if and only if $|B|^2 = 4c(n+2)$, that is, if and only if

$$\mathrm{Scal}^{M^{4n-1}} = c\Big\{(4n-1)(4n+7)-4n-17\Big\} + (4n-1)^2|H|^2. \qquad \Box$$

Remark 3.7. The first equivalence of the previous proposition has been proven in [9] for the quaternionic projective space.

Corollary 3.8. *There exists no biharmonic real hypersurface with constant mean curvature in a quaternionic space form $N^{4n}(4c)$ of negative scalar curvature.*

Proof. The assertion follows immediately from the first equivalence of Proposition 3.6. $\Box$

Example 3.9. The geodesic sphere $S^{4n-1}(u)$ of radius u $(0 < u < \frac{\pi}{2})$ in the quaternionic Euclidian space $\mathbb{R}^{4n}(=\mathbb{H}^n)$ is curvature adapted hypersurface of the quaternionic projective space $\mathbb{H}P^n(4)$, i.e., $J_\alpha\xi$ is a direction of the principal curvature for all $\alpha = 1,2,3$, where ξ is the unit normal vector field along S^{4n-1} (see [2],[9]). Furthemore it is proper-biharmonic hypersurface in $\mathbb{H}P^n(4)$ if and only if $(\cot u)^2 = \frac{2n+7\pm 2\sqrt{n^2+4n+13}}{4n-1}$. Indeed, the principal curvatures of $S^{4n-1}(u)$ are given as follows (see [2],[9]),

$$\begin{cases} \lambda_1 = \cot u \text{ (with multiplicity } m_1 = 4(n-1)), \\ \lambda_2 = 2\cot(2u) \text{ (with multiplicity } m_2 = 3). \end{cases} \tag{3.20}$$

The mean curvature H and the square of the second fundamental form $|B|^2$ of $S^{4n-1}(u)$ are given by (see [9])

$$\begin{aligned} H &= \frac{1}{4n-1}\{4(n-1)\lambda_1 + 3\lambda_2\}, \\ &= \frac{1}{4n-1}\{4(n-1)\cot u + 6\cot(2u)\}, \\ &= \frac{4(n-1)}{4n-1}t + \frac{3}{4n-1}\big(t-\frac{1}{t}\big), \end{aligned}$$

where $t = \cot u$.

$$\begin{aligned} |B|^2 &= 4(n-1)\lambda_1^2 + 3\lambda_2^2 \\ &= 4(n-1)t^2 + 3\big(t-\frac{1}{t}\big)^2 \\ &= (4n-1)t^2 + \frac{3}{t^2} - 6. \end{aligned}$$

On the other hand, using (3.19) we derive

$$\mathrm{Scal}^{S^{4n-1}(u)} = (4n-1)(4n+7) - 9 + (4n-1)^2|H|^2 - |B|^2.$$

Then, by using the second equivalence of the Proposition 3.6, we have that $S^{4n-1}(u)$ is proper biharmonic if and only if

$$(4n-1)t^2 + \frac{3}{t^2} - 2(2n+7) = 0 \Leftrightarrow t^2 = \frac{2n+7 \pm 2\sqrt{n^2+4n+13}}{4n-1},$$

which has always solutions. We set for example $t = \sqrt{3}$ for $n = 2$, then the geodesic sphere $S^7(\frac{\pi}{6})$ is proper-biharmonic hypersurface in $\mathbb{H}P^2(4)$.

In the next proposition we give an estimate of the mean curvature for a biharmonic totally real submanifold of $\mathbb{H}P^n(4)$.

Proposition 3.10. *Let M^m a totally real submanifold of $\mathbb{H}P^n(4)$ with non-zero constant mean curvature.*

(1) *If M^m is proper-biharmonic, then $0 < |H|^2 \leq \frac{m+9}{m}$.*
(2) *If $|H|^2 = \frac{m+9}{m}$, then M^m is proper-biharmonic if and only if it is pseudo-umbilical and $\nabla^\perp H = 0$.*

Proof. We assume that M^m is a biharmonic totally real submanifold of $\mathbb{H}P^n(4)$ with non-zero constant mean curvature. By the first equation of (3.12), we have

$$\Delta^\perp H + \mathrm{Tr}(B(\cdot, A_H(\cdot))) - (m+9)H = 0. \tag{3.21}$$

Then taking the scalar product of (3.21) with H, we find

$$\langle \Delta^\perp H, H\rangle + \langle \mathrm{Tr}(B(\cdot, A_H(\cdot))), H\rangle - (m+9)\langle H, H\rangle = 0. \tag{3.22}$$

Since $|H|$ is a constant, we get

$$\langle \Delta^\perp H, H\rangle = (m+9)|H|^2 - |A_H|^2. \tag{3.23}$$

Using the Bochner formula, we have

$$|\nabla^\perp H|^2 + |A_H|^2 = (m+9)|H|^2. \tag{3.24}$$

By using Cauchy-Schwarz inequality, i.e., $|A_H|^2 \geq m|H|^4$ in the above equation, we obtain

$$(m+9)|H|^2 \geq m|H|^4 + |\nabla^\perp H|^2 \geq m|H|^4. \tag{3.25}$$

So, we can write

$$0 < |H|^2 \leq \frac{m+9}{m},$$

because $|H|$ is a non-zero constant. This gives the proof of **1)**.

If $|H|^2 = \frac{m+9}{m}$ and M^m is biharmonic. From (3.25), we derive $\nabla^\perp H = 0$ and from (3.24), we have

$$|A_H|^2 = \frac{(m+9)^2}{m}. \tag{3.26}$$

That is, M^m is pseudo-umbilical.

Conversely, if $|H|^2 = \frac{m+9}{m}$ and M^m is pseudo-umbilical with $\nabla^\perp H = 0$, then we get immediately

$$\Delta^\perp H + \operatorname{Tr}(B(\cdot, A_H(\cdot))) - (m+9)H = 0,$$

and

$$\frac{m}{2}\operatorname{grad}(|H|^2) + 2\operatorname{Tr}(A_{\nabla^\perp_\cdot H}\cdot) = 0.$$

Therefore, by Corollary 3.3, M^m is proper-biharmonic. This completes the proof. □

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