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Linear Approximation and Asymptotic Expansion associated to the Robin-Dirichlet Problem for a Kirchhoff-Carrier Equation with a Viscoelastic Term

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Abstract. In this paper, we consider the Robin-Dirichlet problem for a nonlinear wave equation of Kirchhoff-Carrier type with a viscoelastic term. Using the Faedo-Galerkin method and the linearization method for nonlinear terms, the existence and uniqueness of a weak solution are proved. An asymptotic expansion of high order in a small parameter of a weak solution is also discussed.

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1. Introduction

In this paper, we consider the following nonlinear Kirchhoff-Carrier wave equation with a viscoelastic term

$$
\begin{aligned}
(1.1)\qquad & u_{tt} - \lambda u_{xxt} - \frac{\partial}{\partial x}\left[\mu_1\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right)u_x\right] \\
& + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu_2\left(x,s,u(x,s),\|u(s)\|^2,\|u_x(s)\|^2\right)u_x(x,s)\right]ds \\
& = f(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2),\ 0<x<1,\ 0<t<T,
\end{aligned}
$$

associated with Robin-Dirichlet conditions

$$
(1.2)\qquad u_x(0,t) - h_0 u(0,t) = u(1,t) = 0,
$$

and initial conditions

$$
(1.3)\qquad u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x),
$$

where $\lambda > 0$, $h_0 \geq 0$ are given constants, $\tilde{u}_0$, $\tilde{u}_1$, f, g, μ_1, μ_2 are given functions satisfying conditions, which will be specified later, $\|u(t)\|^2 = \int_0^1 u^2(x,t)\,dx$, $\|u_x(t)\|^2 = \int_0^1 u_x^2(x,t)\,dx$.

When $\lambda = 0$, $g = 0$, $f = 0$, Eq. (1.1) is related to the Kirchhoff equation

$$
(1.4)\qquad \rho h u_{tt} = \left(P_0 + \frac{Eh}{2L}\int_0^L u_x^2(y,t)dy\right)u_{xx},
$$

presented by Kirchhoff in 1876 (see [9]). This equation is an extension of the classical D'Alembert wave equation which considers the effects of the changes in the length of the string during the vibrations. The parameters in (1.4) have the following meanings: u is the lateral deflection, L is the length of the string, h is the area of the cross - section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In [1], Carrier has also established the equation which models vibrations of an elastic string when changes in tension are not small

$$
(1.5)\qquad \rho u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2 dx\right)u_{xx} = 0,
$$

where $u(x,t)$ is the x - derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross - section of a string, L is the length of a string and ρ is the density of a material.

One of the early classical studies dedicated to Kirchhoff equations was given by Pohozaev [29]. After the work of Lions, for example see [13], Kirchhoff equations as well as Kirchhoff-Carrier equations of the form Eq. (1.1) received much attention (see [2, 3, 4, 6, 7, 10, 11, 18, 23, 21, 28, 27, 30, 31] and references therein). A survey of the results about the mathematical aspects of Kirchhoff model can be found in Medeiros, Limaco and Menezes [19, 20].

It is also well known that, the study of the asymptotic behavior of nonlinear equations with a viscoelastic term has attracted lots of interest of researchers (for example, see [8, 22, 26] and references therein). In [8, 22], the viscoelastic wave equation of Kirchhoff type of the form

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \int_0^t g(t-s)\,\Delta u(s)ds + hu_t = |u|^{q-1}u, \tag{1.6}$$

has been studied and the results of existence and blow up were proved. In [26], the author considered a viscoelastic plate equation with p-Laplacian, by introducing suitable energy and Lyapunov functionals, a general decay estimate for the energy was established.

The paper consists of four sections. Section 2 is devoted to some preliminaries. We begin Section 3 by establishing a sequence of approximate solutions of Prob. (1.1) - (1.3) based on the Faedo-Galerkin's method. Thanks to a priori estimates, we first prove that this sequence is bounded in an appropriate space, by using compact imbedding theorems, we next show that this sequence converges and the existence of Prob. (1.1) - (1.3) follows. By Gronwall's Lemma, the uniqueness of a weak solution is proved. In Section 4, we establish an asymptotic expansion of a weak solution $u = u_\varepsilon$ of order $N+1$ in a small parameter ε for the equation

$$\begin{aligned} & u_{tt} - \lambda u_{xxt} - \frac{\partial}{\partial x}\left[\mu_1\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right)u_x\right] \\ & \quad + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu_2\left(x,s,u(x,s),\|u(s)\|^2,\|u_x(s)\|^2\right)u_x(x,s)\right]ds \\ = \; & f(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2) + \varepsilon f_1(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2), \end{aligned} \tag{1.7}$$

$0<x<1,\ 0<t<T$, associated to (1.2), (1.3), with $\mu_1, \mu_2 \in C^{N+1}([0,1]\times[0,T^*]\times\mathbb{R}\times\mathbb{R}_+^2)$, $\mu_1(x,t,z_1,z_2,z_3) \geq \mu_* > 0$, for all $(x,t,z_1,z_2,z_3) \in [0,1]\times[0,T^*]\times\mathbb{R}\times\mathbb{R}_+^2$, $f \in C^{N+1}([0,1]\times[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+^2)$, $f_1 \in C^N([0,1]\times[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+^2)$. This result is a relative generalization of [14, 15, 16, 17, 24, 25].

2. Preliminaries

The notation we use in this paper is standard and can be found in Lion's book [12], with $\Omega = (0,1)$, $Q_T = \Omega\times(0,T)$, $T>0$ and $\|\cdot\|$ is the norm in L^2. For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. We denote $L^p(0,T;X)$, $1\leq p\leq\infty$ the Banach space of real functions $u:(0,T)\to X$ measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p}, & \text{if} \quad 1\leq p<\infty, \\ \underset{0<t<T}{ess\sup}\|u(t)\|_X, & \text{if} \quad p=\infty. \end{cases}$$

With $f \in C^k([0,1] \times [0,T^*] \times \mathbb{R}^3 \times \mathbb{R}_+^2)$, $f = f(x,t,y_1,y_2,y_3,y_4,y_5)$, $(x,t) \in [0,1] \times [0,T^*]$, $(y_1,y_2,y_3,y_4,y_5) \in \mathbb{R}^3 \times \mathbb{R}_+^2$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_{i+2} f = \frac{\partial f}{\partial y_i}$, $i = 1,...,5$ and $D^\alpha f = D_1^{\alpha_1} ... D_7^{\alpha_7} f$, $\alpha = (\alpha_1,...,\alpha_7) \in \mathbb{Z}_+^7$, $|\alpha| = \alpha_1 + ... + \alpha_7 \leq k$, $D^{(0,...,0)} f = f$.

Similarly, with $\mu \in C^k([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$, $\mu = \mu_i(x,t,z_1,z_2,z_3)$, $(x,t) \in [0,1] \times [0,T^*]$, $(z_1,z_2,z_3) \in \mathbb{R} \times \mathbb{R}_+^2$, we put $D_1 \mu = \frac{\partial \mu}{\partial x}$, $D_2 \mu = \frac{\partial \mu}{\partial y}$, $D_{j+2} \mu = \frac{\partial \mu}{\partial y_j}$, $j = 1,2,3$, and $D^\beta \mu = D_1^{\beta_1} ... D_5^{\beta_5} \mu$, $\beta = (\beta_1,..., \beta_5) \in \mathbb{Z}_+^5$, $|\beta| = \beta_1 + ... + \beta_5 \leq k$, $D^{(0,...,0)} \mu = \mu$.

On $H^1 \equiv H^1(\Omega)$, we shall use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

We set

$$V = \left\{v \in H^1(0,1) : v(1) = 0\right\}, \tag{2.2}$$

and

$$a(u,v) = \langle u_x, v_x \rangle + h_0 u(0) v(0), \text{ for all } u,\ v \in V, \tag{2.3}$$

Then, V is a closed subspace of H^1 and three norms $v \longmapsto \|v\|_{H^1}$, $v \longmapsto \|v_x\|$ and $v \longmapsto \|v\|_a = \sqrt{a(v,v)}$ are equivalent on V. On the other hand, V is continuously and densely embedded in L^2. Identifying L^2 with $\left(L^2\right)'$ (the dual of L^2), we have $V \hookrightarrow L^2 = \left(L^2\right)' \hookrightarrow V'$. We remark that the notation $\langle \cdot, \cdot \rangle$ is also used for the pairing between V and V'.

We have the following lemmas involving known properties.

Lemma 2.1. *The embedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and*

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1} \quad \textit{for all } v \in H^1. \tag{2.4}$$

Lemma 2.2. Let $h_0 \geq 0$. *Then the embedding $V \hookrightarrow C^0\left(\overline{\Omega}\right)$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \leq \|v\|_a & \textit{for all} \quad v \in V, \\ \frac{1}{\sqrt{2}} \|v\|_{H^1} \leq \|v_x\| \leq \|v\|_a \leq \sqrt{1+h_0}\, \|v\|_{H^1} & \textit{for all} \quad v \in V. \end{cases} \tag{2.5}$$

Lemma 2.3. Let $h_0 \geq 0$. *Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by* (2.3) *is continuous on $V \times V$ and coercive on V*, i.e.,

$$\begin{array}{lll} \text{(i)} & |a(u,v)| \leq (1+h_0) \|u_x\| \|v_x\|, & \textit{for all} \quad u,\ v \in V, \\ \text{(ii)} & a(v,v) \geq \|v_x\|^2, & \textit{for all} \quad v \in V. \end{array} \tag{2.6}$$

Lemma 2.4. *Let $h_0 \geq 0$. There exists the Hilbert orthonormal base $\{w_j\}$ of the space L^2 consisting of eigenfunctions w_j corresponding to eigenvalues λ_j such that*

$$\text{(2.7)} \qquad \begin{array}{ll} \text{(i)} & 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots, \ \lim_{j\to+\infty} \lambda_j = +\infty, \\ \text{(ii)} & a(w_j, v) = \lambda_j \langle w_j, v\rangle \ \textit{for all } v \in V, \quad j = 1, 2, \ldots. \end{array}$$

Furthermore, the sequence $\{w_j/\sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of V with respect to the scalar product $a(\cdot,\cdot)$. On the other hand, w_j satisfies the following boundary value problem

$$\text{(2.8)} \qquad \begin{cases} -\Delta w_j = \lambda_j w_j, \ \textit{in } \Omega, \\ w_{jx}(0) - h_0 w_j(0) = w_j(1) = 0, \ w_j \in V \cap C^\infty\left(\overline{\Omega}\right). \end{cases}$$

The proof of Lemma 2.4 can be found in [32, p.87, Theorem 7.7], with $H = L^2$ and V, $a(\cdot,\cdot)$ as defined by (2.3).

3. The Existence and Uniqueness Theorem

In this section, we consider the local existence for Problem (1.1)–(1.3), with h_0, $\lambda \in \mathbb{R}$, $h_0 \geq 0$, $\lambda > 0$. Here, it is said that u is a weak solution of Problem (1.1)–(1.3) if

$$\text{(3.1)} \qquad \begin{aligned} u \in \tilde{V}_T = &\{v \in L^\infty(0,T; V\cap H^2) : v' \in L^\infty(0,T; V \cap H^2), \\ &\qquad v'' \in L^2(0,T;V) \cap L^\infty(0,T;L^2)\}, \end{aligned}$$

and u satisfies the following variational equation

$$\text{(3.2)} \qquad \begin{aligned} &\langle u''(t), v\rangle + \lambda a(u'(t), v) + a_1[u](t; u(t), v) \\ &= \langle f[u](t), v\rangle + \int_0^t g(t-s)\, a_2[u](s; u(s), v)\, ds, \end{aligned}$$

for all $v \in V$, and a.e., $t \in (0,T)$, together with the initial conditions

$$\text{(3.3)} \qquad u(0) = \tilde{u}_0, \ u'(0) = \tilde{u}_1,$$

where, for each $\bar{u} \in \tilde{V}_T$ and $i = 1, 2$, $\{a_i[\bar{u}](t;\cdot,\cdot)\}_{0\leq t\leq T}$ is the family of symmetric bilinear forms on $V \times V$ defined by

$$\text{(3.4)} \qquad a_i[\bar{u}](t; u, v) = \langle \mu_i[\bar{u}](t) u_x, v_x\rangle + h_0 \mu_i[\bar{u}](0,t) u(0) v(0),$$

for all u, $v \in V$, $0 \leq t \leq T$, with $h_0 \geq 0$ is given constant, and

$$\text{(3.5)} \qquad \begin{aligned} \mu_i[\bar{u}](x,t) &= \mu_i\left(x, t, \bar{u}(x,t), \|\bar{u}(t)\|^2, \|\bar{u}_x(t)\|^2\right), \ i = 1, 2, \\ f[u](x,t) &= f(x, t, u(x,t), u_x(x,t), u'(x,t), \|u(t)\|^2, \|u_x(t)\|^2). \end{aligned}$$

Consider $T^* > 0$ fixed, we make the following assumptions:

(H_1) $\tilde{u}_0,\, \tilde{u}_1 \in V \cap H^2$;

(H_2) $g \in H^1(0, T^*)$;

(H_3) $\mu_1 \in C^2([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}^2_+)$,
$\mu_1(x,t,y_1,y_2,y_3) \geq \mu_* > 0,\ \forall (x,t,y_1,y_2,y_3) \in [0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}^2_+$;

(H_4) $\mu_2 \in C^1([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}^2_+)$;

(H_5) $f \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^3 \times \mathbb{R}^2_+)$.

For each $M > 0$ given, we set the constants $K_M(f),\ \tilde{K}_M(\mu_1),\ \tilde{K}_M(\mu_2)$ as follows

$$\begin{aligned} K_M(f) &= \|f\|_{C^1(A_M)} = \|f\|_{C^0(A_M)} + \sum_{i=1}^{7} \|D_i f\|_{C^0(A_M)}, \\ \tilde{K}_M(\mu_i) &= \|\mu_i\|_{C^2(\tilde{A}_M)} = \sum_{|\beta| \leq 2} \left\|D^\beta \mu_i\right\|_{C^0(\tilde{A}_M)},\ i = 1,2, \end{aligned} \tag{3.6}$$

with

$$\begin{cases} \|f\|_{C^0(A_M)} = \sup_{(x,t,y_1,\ldots,y_5) \in A_M} |f(x,t,y_1,\ldots,y_5)|, \\ \|\mu_i\|_{C^0(\tilde{A}_M)} = \sup_{(x,t,z_1,\ldots,z_3) \in \tilde{A}_M} |\mu_i(x,t,z_1,\ldots,z_3)|,\ i = 1,2, \\ A_M = (x,t,y_1,\ldots,y_5) : \{0 \leq x \leq 1,\ 0 \leq t \leq T^*,\ |y_1|,\ |y_2|,\ |y_3| \leq \sqrt{2}M, \\ \qquad\qquad 0 \leq y_4,\ y_5 \leq M^2\}, \\ \tilde{A}_M = \left\{(x,t,z_1,\ldots,z_3) : 0 \leq x \leq 1,\ 0 \leq t \leq T^*,\ |z_1| \leq M,\ 0 \leq z_2,\ z_3 \leq M^2\right\}. \end{cases} \tag{3.7}$$

For each $T \in (0,\ T^*]$, we denote

$$V_T = \{v \in L^\infty(0,T; V \cap H^2) : v' \in L^\infty(0,T; V \cap H^2),\ v'' \in L^2(0,T;V)\}, \tag{3.8}$$

it is a Banach space with respect to the norm

$$\|v\|_{V_T} = \max\{\|v\|_{L^\infty(0,T;V\cap H^2)};\ \|v'\|_{L^\infty(0,T;V\cap H^2)};\ \|v''\|_{L^2(0,T;V)}\}. \tag{3.9}$$

For every $M > 0$, we put

$$\begin{aligned} W(M,T) &= \{v \in V_T : \|v\|_{V_T} \leq M\}, \\ W_1(M,T) &= \{v \in W(M,T) : v'' \in L^\infty(0,T;L^2)\}. \end{aligned} \tag{3.10}$$

Next, we will establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{3.11}$$

based on the associate problem (3.2), we find $u_m \in W_1(M,T)$ $(m \geq 1)$ satisfying the linear variational problem

$$\text{(3.12)} \quad \begin{cases} \langle u_m''(t), v\rangle + \lambda a(u_m'(t), v) + a_1^{(m)}(t; u_m(t), v) \\ \qquad = \int_0^t g(t-s)\, a_2^{(m)}(s; u_m(s), v)\, ds + \langle F_m(t), v\rangle, \ \forall v \in V, \\ u_m(0) = \tilde{u}_0, \ u_m'(0) = \tilde{u}_1, \end{cases}$$

where

$$\text{(3.13)} \quad \begin{cases} a_i^{(m)}(t; u, v) = a_i[u_{m-1}](t; u, v) \\ \qquad = \left\langle \mu_i^{(m)}(t)\, u_x, v_x \right\rangle + h_0 \mu_i^{(m)}(0,t)\, u(0)\, v(0), \ \forall u, v \in V; \\ F_m(x,t) = f[u_{m-1}](x,t) \\ \qquad = f(x, t, u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t), \\ \qquad\qquad \|u_{m-1}(t)\|^2, \|\nabla u_{m-1}(t)\|^2); \\ \mu_i^{(m)}(x,t) = \mu_i[u_{m-1}](x,t) \\ \qquad = \mu_i\left(x, t, u_{m-1}(x,t), \|u_{m-1}(t)\|^2, \|\nabla u_{m-1}(t)\|^2\right), \ i = 1, 2. \end{cases}$$

Then we have the following theorem.

Theorem 3.1. *Let* (H_1)–(H_5) *hold. Then there exist constants* $M, T > 0$ *such that, for* $u_0 \equiv 0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (3.11)–(3.13).

Proof. The proof consists of three steps.
Step 1. The Faedo-Galerkin approximation. Consider the basis $\{w_j\}$ for V given by Lemma 2.4. Put

$$\text{(3.14)} \quad u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j,$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of linear integrodifferential equations

$$\text{(3.15)} \quad \begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \lambda a(\dot{u}_m^{(k)}(t), w_j) + a_1^{(m)}\left(t; u_m^{(k)}(t), w_j\right) \\ \quad = \langle F_m(t), w_j\rangle + \int_0^t g(t-s)\, a_2^{(m)}\left(s; u_m^{(k)}(s), w_j\right) ds, \ 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases}$$

in which

$$\text{(3.16)} \quad \begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \text{ strongly in } V \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \tilde{u}_1 \text{ strongly in } V \cap H^2. \end{cases}$$

The system (3.15), (3.16) can be written in the form

$$\text{(3.17)}\qquad \begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda\lambda_j \dot{c}_{mj}^{(k)}(t) + \sum\limits_{i=1}^{k} a_{1ij}^{(m)}(t)\, c_{mi}^{(k)}(t) \\ \qquad\qquad = \sum\limits_{i=1}^{k} \int_0^t g(t-s)\, a_{2ij}^{(m)}(s)\, c_{mi}^{(k)}(s)ds + f_{mj}(t), \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \ \ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)}, \ 1 \leq j \leq k, \end{cases}$$

where

$$\text{(3.18)}\qquad a_{\gamma ij}^{(m)}(t) = a_{\gamma}^{(m)}(t; w_i, w_j), \ f_{mj}(t) = \langle F_m(t), w_j \rangle, \ \gamma = 1,2, \ 1 \leq i,j \leq k.$$

Note that by (3.11), using standard methods in ordinary differential equations (see [5]), the system (3.17) has a unique solution $c_{mj}^{(k)}(t)$, $1 \leq j \leq k$ on interval $[0, T_m^{(k)}] \subset [0, T]$.

Step 2. A priori estimate. First, we put

$$\text{(3.19)}\qquad S_m^{(k)}(t) = p_m^{(k)}(t) + q_m^{(k)}(t) + r_m^{(k)}(t),$$

where

$$\text{(3.20)}\qquad \begin{cases} p_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + a_1^{(m)}\left(t; u_m^{(k)}(t), u_m^{(k)}(t)\right) + 2\lambda \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|_a^2 ds, \\ q_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|_a^2 + \left\| \sqrt{\mu_1^{(m)}(t)} \Delta u_m^{(k)}(t) \right\|^2 + 2\lambda \int_0^t \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 ds, \\ r_m^{(k)}(t) = \lambda \left\| \Delta \dot{u}_m^{(k)}(t) \right\|^2 + 2\int_0^t \left\| \ddot{u}_m^{(k)}(s) \right\|_a^2 ds. \end{cases}$$

Then, it follows from (3.15), (3.19), (3.20) that

$$\begin{aligned} \text{(3.21)}\qquad & S_m^{(k)}(t) \\ = \ & S_m^{(k)}(0) + 2\left\langle J_1^{(m,k)}(0), \Delta \tilde{u}_{1k} \right\rangle + 2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s) \right\rangle ds \\ & + 2\int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s) + \ddot{u}_m^{(k)}(s)\right) ds \\ & + \int_0^t \dot{a}_1^{(m)}\left(s; u_m^{(k)}(s), u_m^{(k)}(s)\right) ds \\ & + \int_0^t ds \int_0^1 \dot{\mu}_1^{(m)}(x,s) \left| \Delta u_m^{(k)}(x,s) \right|^2 ds \\ & - 2\int_0^t \left\langle \mu_{1x}^{(m)}(s)\, u_{mx}^{(k)}(s), \Delta \dot{u}_m^{(k)}(s) \right\rangle ds \\ & + 2\int_0^t \left\langle \dot{J}_1^{(m,k)}(s), \Delta \dot{u}_m^{(k)}(s) \right\rangle ds - 2\left\langle J_1^{(m,k)}(t), \Delta \dot{u}_m^{(k)}(t) \right\rangle \end{aligned}$$

$$
\begin{aligned}
&+2\int_0^t d\tau \int_0^\tau g(\tau - s)\, a_2^{(m)}\left(s; u_m^{(k)}(s), \dot{u}_m^{(k)}(\tau)\right) ds \\
&+2\int_0^t d\tau \int_0^\tau g(\tau - s) \left\langle J_2^{(m,k)}(s), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle ds \\
&+2\int_0^t g(t - s) \left\langle J_2^{(m,k)}(s), \Delta\dot{u}_m^{(k)}(t)\right\rangle ds \\
&-2g(0)\int_0^t \left\langle J_2^{(m,k)}(\tau), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle d\tau \\
&-2\int_0^t d\tau \int_0^\tau g'(\tau - s) \left\langle J_2^{(m,k)}(s), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle ds \\
\equiv\quad & S_m^{(k)}(0) + 2\left\langle J_1^{(m,k)}(0), \Delta\tilde{u}_{1k}\right\rangle + \sum_{j=1}^{12} I_j,
\end{aligned}
$$

where

$$J_i^{(m,k)}(x,t) = \frac{\partial}{\partial x}\left(\mu_i^{(m)}(x,t)\, u_{mx}^{(k)}(x,t)\right),\ i = 1, 2. \tag{3.22}$$

Next, we need the following lemmas.

Lemma 3.2. *Put*

$$\dot{a}_i^{(m)}(t; u, v) = \left\langle \dot{\mu}_i^{(m)}(t)\, u_x, v_x\right\rangle + h_0 \dot{\mu}_i^{(m)}(0,t)\, u(0)\, v(0),\ \textit{for all } u, v \in V. \tag{3.23}$$

Then we have

$$
\begin{aligned}
&\text{(i)} && \frac{d}{dt} a_i^{(m)}\left(t; u_m^{(k)}(t), u_m^{(k)}(t)\right) = 2a_i^{(m)}\left(t; u_m^{(k)}(t), \dot{u}_m^{(k)}(t)\right) \\
&&& \qquad\qquad + \dot{a}_i^{(m)}\left(t; u_m^{(k)}(t), u_m^{(k)}(t)\right), \\
&\text{(ii)} && \left|\dot{a}_i^{(m)}(t; u, v)\right| \le \gamma_M \tilde{K}_M(\mu_i) \|u\|_a \|v\|_a, \textit{ for all } u, v \in V, \\
&\text{(iii)} && \left|a_i^{(m)}(t; u, v)\right| \le \tilde{K}_M(\mu_i) \|u\|_a \|v\|_a, \textit{ for all } u, v \in V,\ \forall t \in [0, T^*], \\
&\text{(iv)} && a_1^{(m)}(t; v, v) \ge \mu_* \|v\|_a^2, \textit{ for all } v \in V,
\end{aligned}
\tag{3.24}
$$

where $\gamma_M = 1 + M + 4M^2$, $i = 1, 2$.

Lemma 3.3. *We have*

$$
\begin{aligned}
&\text{(i)} && \left\|J_i^{(m,k)}(t)\right\| \le d_0(M) \tilde{K}_M(\mu_i) \sqrt{S_m^{(k)}(t)}, \\
&\text{(ii)} && \left\|\dot{J}_i^{(m,k)}(t)\right\| \le d_1(M) \tilde{K}_M(\mu_i) \sqrt{S_m^{(k)}(t)},
\end{aligned}
\tag{3.25}
$$

where $d_0(M) = \dfrac{2 + \sqrt{2}M}{\sqrt{\mu_*}}$, $d_1(M) = 1 + \dfrac{1}{\sqrt{\lambda}} + 2M + \dfrac{(5 + 2M)\gamma_M}{\sqrt{\mu_*}}$, $i = 1, 2$.

Proof. The proof of Lemmas 3.2, 3.3 are easy, hence we omit the details. □

We shall estimate the terms I_j on the right-hand side of (3.21) as follows.
Estimation of I_1. The Cauchy-Schwartz inequality leads to

$$(3.26)\qquad \begin{aligned} I_1 &= 2\int_0^t \left\langle F_m\left(s\right), \dot{u}_m^{(k)}(s)\right\rangle ds \leq 2\int_0^t \left\|F_m\left(s\right)\right\| \left\|\dot{u}_m^{(k)}(s)\right\| ds \\ &\leq TK_M^2(f) + \int_0^t S_m^{(k)}\left(s\right) ds. \end{aligned}$$

Estimation of the terms I_2 *and* I_3. We note that, by $(3.13)_2$ we have

$$(3.27)\qquad \begin{aligned} F_{mx}(x,t) &= D_1 f\left[u_{m-1}\right](x,t) + D_3 f\left[u_{m-1}\right](x,t)\nabla u_{m-1}\left(x,t\right) \\ &\quad + D_4 f\left[u_{m-1}\right](x,t)\Delta u_{m-1}\left(x,t\right) \\ &\quad + D_5 f\left[u_{m-1}\right](x,t)\nabla u'_{m-1}\left(x,t\right), \end{aligned}$$

where we use the notations

$$\begin{aligned} &D_i f\left[u_{m-1}\right](x,t) \\ &= D_i f(x,t,u_{m-1}\left(x,t\right), \nabla u_{m-1}\left(x,t\right), u'_{m-1}\left(x,t\right), \left\|u_{m-1}(t)\right\|^2, \left\|\nabla u_{m-1}(t)\right\|^2), \end{aligned}$$

$i = 1, 2, ..., 7$, so, by (3.6), (3.11) and (3.27), we obtain

$$(3.28)\qquad \begin{aligned} \left\|F_{mx}\left(t\right)\right\| &\leq (1+3M)\, K_M\left(f\right) \leq 2\gamma_M K_M\left(f\right), \\ \left\|F_m\left(t\right)\right\|_a^2 &= \left\|F_{mx}\left(t\right)\right\|^2 + h_0 F_m^2(0,t) \leq \left(4\gamma_M^2 + h_0\right) K_M^2\left(f\right). \end{aligned}$$

By Lemma 3.2, (ii), (iv) and the following inequalities

$$(3.29)\qquad 2ab \leq \beta a^2 + \frac{1}{\beta} b^2,\ \forall a, b \in \mathbb{R},\ \forall \beta > 0,$$

and

$$\begin{aligned} S_m^{(k)}\left(t\right) &\geq 2\lambda \int_0^t \left\|\dot{u}_m^{(k)}(s)\right\|_a^2 ds + 2\int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_a^2 ds \\ &\geq \min\{1, \lambda\} \int_0^t \left(\left\|\dot{u}_m^{(k)}(s)\right\|_a + \left\|\ddot{u}_m^{(k)}(s)\right\|_a\right)^2 ds, \end{aligned}$$

we shall estimate respectively the following terms I_2, I_3 on the right-hand side of (3.21) as follows

$$(3.30)\qquad \begin{aligned} I_2 &= 2\int_0^t a\left(F_m\left(s\right), \dot{u}_m^{(k)}(s) + \ddot{u}_m^{(k)}(s)\right) ds \\ &\leq 2\int_0^t \left\|F_m\left(s\right)\right\|_a \left(\left\|\dot{u}_m^{(k)}(s)\right\|_a + \left\|\ddot{u}_m^{(k)}(s)\right\|_a\right) ds \\ &\leq \frac{1}{\beta\min\{1,\lambda\}} \int_0^t \left\|F_m\left(s\right)\right\|_a^2 ds \\ &\quad + \beta\min\{1,\lambda\} \int_0^t \left(\left\|\dot{u}_m^{(k)}(s)\right\|_a + \left\|\ddot{u}_m^{(k)}(s)\right\|_a\right)^2 ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{T\left(4\gamma_M^2+h_0\right)}{\beta\min\{1,\lambda\}}K_M^2(f)+\beta S_m^{(k)}(t), \quad \forall\beta>0; \\
I_3 &= \int_0^t \dot{a}_1^{(m)}\left(s;u_m^{(k)}(s),u_m^{(k)}(s)\right)ds \leq \gamma_M\tilde{K}_M(\mu_1)\int_0^t \left\|u_m^{(k)}(s)\right\|_a^2 ds \\
&\leq \frac{\gamma_M\tilde{K}_M(\mu_1)}{\mu_*}\int_0^t S_m^{(k)}(s)\,ds.
\end{aligned}
$$

Estimation of I_4. On the other hand, by $(3.13)_3$, we have

$$
\begin{aligned}
\dot{\mu}_1^{(m)}(x,t) &= D_2\mu_1[u_{m-1}](x,t)+D_3\mu_1[u_{m-1}](x,t)\,u_{m-1}'(x,t) \\
&\quad +2D_4\mu_1[u_{m-1}](x,t)\left\langle u_{m-1}(t),u_{m-1}'(t)\right\rangle \\
&\quad +2D_5\mu_1[u_{m-1}](x,t)\left\langle \nabla u_{m-1}(t),\nabla u_{m-1}'(t)\right\rangle,
\end{aligned}
\tag{3.31}
$$

where we use the notations

$$
D_i\mu_1[u_{m-1}](x,t)=D_i\mu_1\left(x,t,u_{m-1}(x,t),\|u_{m-1}(t)\|^2,\|\nabla u_{m-1}(t)\|^2\right),
$$

$i=1,2,...,5$, it implies that

$$
\left|\dot{\mu}_1^{(m)}(x,t)\right| \leq (1+M+4M^2)\tilde{K}_M(\mu_1)=\gamma_M\tilde{K}_M(\mu_1). \tag{3.32}
$$

Hence, we deduce from (3.19), (3.20) and (3.32), that

$$
\begin{aligned}
I_4 &= \int_0^t ds\int_0^1 \dot{\mu}_1^{(m)}(x,s)\left|\Delta u_m^{(k)}(x,s)\right|^2 ds \\
&\leq \gamma_M\tilde{K}_M(\mu_1)\int_0^t \left\|\Delta u_m^{(k)}(s)\right\|^2 ds \\
&\leq \frac{\gamma_M\tilde{K}_M(\mu_1)}{\mu_*}\int_0^t S_m^{(k)}(s)ds.
\end{aligned}
\tag{3.33}
$$

Estimation of I_5. Similarly, by the following formula

$$
\mu_{1x}^{(m)}(x,t)=D_1\mu_1[u_{m-1}](x,t)+D_3\mu_1[u_{m-1}](x,t)\,\nabla u_{m-1}(x,t), \tag{3.34}
$$

and by (3.6), (3.11) and (3.34), we obtain

$$
\left|\mu_{1x}^{(m)}(x,t)\right| \leq (1+\sqrt{2}M)\tilde{K}_M(\mu_1)\leq 2\gamma_M\tilde{K}_M(\mu_1). \tag{3.35}
$$

Using the inequality (3.35) and the following inequalities

$$
\begin{aligned}
S_m^{(k)}(t) &\geq a_1^{(m)}\left(t;u_m^{(k)}(t),u_m^{(k)}(t)\right)+\lambda\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 \\
&\geq \mu_*\left\|u_{mx}^{(k)}(t)\right\|^2+\lambda\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2 \\
&\geq \min\{\mu_*,\lambda\}\left(\left\|u_{mx}^{(k)}(t)\right\|^2+\left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2\right),
\end{aligned}
$$

we shall estimate the following term I_5 as follows

$$\begin{aligned} I_5 &= -2\int_0^t \left\langle \mu_{1x}^{(m)}(s)\, u_{mx}^{(k)}(s), \Delta\dot{u}_m^{(k)}(s)\right\rangle ds \\ &\le 4\gamma_M \tilde{K}_M(\mu_1)\int_0^t \left\|u_{mx}^{(k)}(s)\right\| \left\|\Delta\dot{u}_m^{(k)}(s)\right\| ds \\ &\le \frac{2\gamma_M \tilde{K}_M(\mu_1)}{\min\{\mu_*,\lambda\}}\int_0^t S_m^{(k)}(s)ds. \end{aligned} \tag{3.36}$$

Estimation of I_6. By Lemma 3.3, (ii) and the inequality $S_m^{(k)}(t) \ge \lambda \left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2$, we have

$$\begin{aligned} I_6 &= 2\int_0^t \left\langle \dot{J}_1^{(m,k)}(s), \Delta\dot{u}_m^{(k)}(s)\right\rangle ds \\ &\le 2\int_0^t \left\|\dot{J}_1^{(m,k)}(s)\right\| \left\|\Delta\dot{u}_m^{(k)}(s)\right\| ds \\ &\le \frac{2}{\sqrt{\lambda}} d_1(M)\tilde{K}_M(\mu_1)\int_0^t S_m^{(k)}(s)\, ds. \end{aligned} \tag{3.37}$$

Estimation of I_7. By Lemma 3.3, (ii), we have

$$\begin{aligned} \left\|J_1^{(m,k)}(t)\right\|^2 &= \left(\left\|J_1^{(m,k)}(0) + \int_0^t \dot{J}_1^{(m,k)}(s)\, ds\right\|\right)^2 \\ &\le 2\left\|J_1^{(m,k)}(0)\right\|^2 + 2T\int_0^t \left\|\dot{J}_1^{(m,k)}(s)\right\|^2 ds \\ &\le 2\left\|J_1^{(m,k)}(0)\right\|^2 + 2Td_1^2(M)\tilde{K}_M^2(\mu_1)\int_0^t S_m^{(k)}(s)\, ds. \end{aligned} \tag{3.38}$$

We deduce from the inequalities (3.38) and $S_m^{(k)}(t) \ge \lambda \left\|\Delta\dot{u}_m^{(k)}(t)\right\|^2$ that

$$\begin{aligned} I_7 &= -2\left\langle J_1^{(m,k)}(t), \Delta\dot{u}_m^{(k)}(t)\right\rangle \le \frac{2}{\sqrt{\lambda}}\left\|J_1^{(m,k)}(t)\right\| \sqrt{S_m^{(k)}(t)} \\ &\le \frac{1}{\beta\lambda}\left\|J_1^{(m,k)}(t)\right\|^2 + \beta S_m^{(k)}(t) \\ &\le \frac{2}{\beta\lambda}\left\|J_1^{(m,k)}(0)\right\|^2 + \frac{2}{\beta\lambda}Td_1^2(M)\tilde{K}_M^2(\mu_1)\int_0^t S_m^{(k)}(s)\, ds \\ &\quad + \beta S_m^{(k)}(t), \end{aligned} \tag{3.39}$$

for all $\beta > 0$.

Estimation of I_8. By Lemma 3.3, (ii) and the inequalities

$$
\begin{aligned}
S_m^{(k)}(t) &\geq a_1^{(m)}\left(t; u_m^{(k)}(t), u_m^{(k)}(t)\right) + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 \\
&\geq \mu_* \left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 \geq 2\sqrt{\mu_*}\left\|u_m^{(k)}(t)\right\|_a \left\|\dot{u}_m^{(k)}(t)\right\|_a,
\end{aligned}
$$

we have

$$
\begin{aligned}
I_8 &= 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, a_2^{(m)}\left(s; u_m^{(k)}(s), \dot{u}_m^{(k)}(\tau)\right) ds \\
&\leq 2\tilde{K}_M(\mu_2) \int_0^t d\tau \int_0^\tau |g(\tau - s)| \sqrt{\frac{S_m^{(k)}(s)}{\mu_*}} \sqrt{S_m^{(k)}(\tau)} ds \\
&\leq \frac{2}{\sqrt{\mu_*}} \tilde{K}_M(\mu_2) \int_0^t d\tau \int_0^\tau |g(\tau - s)| \sqrt{S_m^{(k)}(s)} \sqrt{S_m^{(k)}(\tau)} ds \\
&\leq \frac{2}{\sqrt{\mu_*}} \tilde{K}_M(\mu_2) \sqrt{T}\, \|g\|_{L^2(0,T^*)} \int_0^t S_m^{(k)}(s)\, ds.
\end{aligned} \tag{3.40}
$$

Estimation of I_9, I_{10}, I_{11}, I_{12}. By Lemma 3.3, (ii) and the inequalities

$$
\begin{aligned}
S_m^{(k)}(t) &\geq a_1^{(m)}\left(t; u_m^{(k)}(t), u_m^{(k)}(t)\right) + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 \geq \mu_* \left\|u_m^{(k)}(t)\right\|_a^2 + \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 \\
&\geq 2\sqrt{\mu_*}\left\|u_m^{(k)}(t)\right\|_a \left\|\dot{u}_m^{(k)}(t)\right\|_a, \\
S_m^{(k)}(t) &\geq \lambda \left\|\Delta \dot{u}_m^{(k)}(t)\right\|^2,
\end{aligned}
$$

it is not difficult to estimate the following terms

$$
\begin{aligned}
I_9 &= 2\int_0^t d\tau \int_0^\tau g(\tau - s) \left\langle J_2^{(m,k)}(s), \Delta \dot{u}_m^{(k)}(\tau)\right\rangle ds \\
&\leq 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \left\|J_2^{(m,k)}(s)\right\| \left\|\Delta \dot{u}_m^{(k)}(\tau)\right\| ds \\
&\leq \frac{2}{\sqrt{\lambda}} d_0(M) \tilde{K}_M(\mu_2) \int_0^t d\tau \int_0^\tau |g(\tau - s)| \sqrt{S_m^{(k)}(s)} \sqrt{S_m^{(k)}(\tau)} ds \\
&\leq \frac{2}{\sqrt{\lambda}} d_0(M) \tilde{K}_M(\mu_2) \sqrt{T}\, \|g\|_{L^2(0,T^*)} \int_0^t S_m^{(k)}(s)\, ds;
\end{aligned} \tag{3.41}
$$

$$
\begin{aligned}
I_{10} &= 2\int_0^t g(t - s) \left\langle J_2^{(m,k)}(s), \Delta \dot{u}_m^{(k)}(t)\right\rangle ds \\
&\leq \frac{2}{\sqrt{\lambda}} d_0(M) \tilde{K}_M(\mu_2) \int_0^t |g(t - s)| \sqrt{S_m^{(k)}(s)} \sqrt{S_m^{(k)}(t)} ds \\
&\leq \frac{1}{\beta\lambda} d_0^2(M) \tilde{K}_M^2(\mu_2) \|g\|_{L^2(0,T^*)}^2 \int_0^t S_m^{(k)}(s)\, ds + \beta S_m^{(k)}(t),\ \forall \beta > 0;
\end{aligned}
$$

$$\begin{aligned}
I_{11} &= -2g(0)\int_0^t \left\langle J_2^{(m,k)}(\tau), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle d\tau \\
&\le 2\left|g(0)\right| \int_0^t \left\| J_2^{(m,k)}(\tau)\right\| \left\|\Delta\dot{u}_m^{(k)}(\tau)\right\| d\tau \\
&\le \frac{2}{\sqrt{\lambda}}\left|g(0)\right| d_0(M)\tilde{K}_M(\mu_2)\int_0^t S_m^{(k)}(\tau)\, d\tau; \\
I_{12} &= -2\int_0^t d\tau \int_0^\tau g'(\tau-s)\left\langle J_2^{(m,k)}(s), \Delta\dot{u}_m^{(k)}(\tau)\right\rangle ds \\
&\le 2\int_0^t d\tau \int_0^\tau \left|g'(\tau-s)\right| \left\| J_2^{(m,k)}(s)\right\| \left\|\Delta\dot{u}_m^{(k)}(\tau)\right\| ds \\
&\le \frac{2}{\sqrt{\lambda}} d_0(M)\tilde{K}_M(\mu_2)\int_0^t d\tau \int_0^\tau \left|g'(\tau-s)\right| \sqrt{S_m^{(k)}(s)}\sqrt{S_m^{(k)}(\tau)} ds \\
&\le \frac{2}{\sqrt{\lambda}} d_0(M)\tilde{K}_M(\mu_2)\sqrt{T}\left\|g'\right\|_{L^2(0,T^*)}\int_0^t S_m^{(k)}(s)\, ds.
\end{aligned}$$

Choosing $\beta = \frac{1}{6}$, it follows from (3.21), (3.26), (3.30), (3.33), (3.36), (3.37), (3.39)-(3.41), that

(3.42) $$S_m^{(k)}(t) \le \bar{S}_{m,k} + D_M^{(1)} T K_M^2(f) + D_M^{(2)}(T)\int_0^t S_m^{(k)}(s)\, ds,$$

where

(3.43) $$\begin{aligned}
\bar{S}_{m,k} &= 2S_m^{(k)}(0) + 4\left\langle J_1^{(m,k)}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{24}{\lambda}\left\| J_1^{(m,k)}(0)\right\|^2, \\
D_M^{(1)} &= 2 + \frac{12\left(4\gamma_M^2 + h_0\right)}{\min\{1,\lambda\}},
\end{aligned}$$

$$\begin{aligned}
D_M^{(2)}(T) &= 2 + 4\left(\frac{1}{\mu_*} + \frac{1}{\min\{\mu_*,\lambda\}}\right)\gamma_M \tilde{K}_M(\mu_1) \\
&\quad + \frac{4}{\sqrt{\lambda}}\left(\left|g(0)\right| + \sqrt{T}\left\|g'\right\|_{L^2(0,T^*)} + \sqrt{T}\left\|g\right\|_{L^2(0,T^*)}\right) d_0(M)\tilde{K}_M(\mu_2) \\
&\quad + 4\left(\frac{1}{\sqrt{\lambda}} + \frac{6}{\lambda} T d_1(M)\tilde{K}_M(\mu_1)\right) d_1(M)\tilde{K}_M(\mu_1) \\
&\quad + 4\left(\frac{\sqrt{T}}{\sqrt{\mu_*}} + \frac{3}{\lambda} d_0^2(M)\tilde{K}_M(\mu_2)\left\|g\right\|_{L^2(0,T^*)}\right)\tilde{K}_M(\mu_2)\left\|g\right\|_{L^2(0,T^*)}.
\end{aligned}$$

Estimation of $\bar{S}_{m,k}$.

Notice that the formula $J_1^{(m,k)}(0) = \frac{\partial}{\partial x}\left[\mu_1\left(x, 0, \tilde{u}_0(x), \left\|\tilde{u}_0\right\|^2, \left\|\tilde{u}_{0x}\right\|^2\right)\tilde{u}_{0kx}\right]$ independent of m. By means of the convergences in (3.16), we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$\begin{aligned}
(3.44) \qquad \bar{S}_{m,k} &= 2S_m^{(k)}(0) + 4\left\langle J_1^{(m,k)}(0), \Delta\tilde{u}_{1k}\right\rangle + \frac{24}{\lambda}\left\|J_1^{(m,k)}(0)\right\|^2 \\
&= 2a_1^{(m)}(0;\tilde{u}_{0k},\tilde{u}_{0k}) + 2\left\|\sqrt{\mu_1^{(m)}(0)}\Delta\tilde{u}_{0k}\right\|^2 \\
&\quad +2\|\tilde{u}_{1k}\|^2 + 2\|\tilde{u}_{1k}\|_a^2 + 2\lambda\|\Delta\tilde{u}_{1k}\|^2 \\
&\quad +4\left\langle \frac{\partial}{\partial x}\left(\mu_1^{(m)}(\cdot,0)\,\tilde{u}_{0kx}\right), \Delta\tilde{u}_{1k}\right\rangle \\
&\quad +\frac{24}{\lambda}\left\|\frac{\partial}{\partial x}\left(\mu_1^{(m)}(\cdot,0)\,\tilde{u}_{0kx}\right)\right\|^2 \\
&\leq \frac{1}{2}M^2 \quad \text{for all } m,k\in\mathbb{N}.
\end{aligned}$$

Hence, from $(3.43)_{2,3}$, we can choose $T \in (0, T^*]$, such that

$$(3.45) \qquad \left(\frac{M^2}{2} + D_M^{(1)} T K_M^2(f)\right) e^{TD_M^{(2)}(T)} \leq M^2,$$

and

$$(3.46) \qquad k_T = \left(1 + \frac{1}{\sqrt{2\lambda}} + \frac{1}{\sqrt{\mu_*}}\right)\sqrt{T\bar{D}_1(M)}\exp(T\bar{D}_2(M)) < 1,$$

where

$$\begin{aligned}
(3.47) \quad \bar{D}_1(M) &= 8(1+2M)^2K_M^2(f) \\
&\quad + \frac{6(1+4M)^2M^2}{\lambda}\left(\tilde{K}_M^2(\mu_1) + \tilde{K}_M^2(\mu_2)\|g\|_{L^1(0,T^*)}^2\right), \\
\bar{D}_2(M) &= 1 + \frac{\gamma_M\tilde{K}_M(\mu_1)}{\mu_*} + \frac{3\tilde{K}_M^2(\mu_2)\,T^*\|g\|_{L^2(0,T^*)}^2}{\lambda\mu_*}.
\end{aligned}$$

Finally, it follows from (3.42), (3.44) and (3.45), that

$$(3.48) \qquad S_m^{(k)}(t) \leq M^2 e^{-TD_M^{(2)}(T)} + D_M^{(2)}(T)\int_0^t S_m^{(k)}(s)\,ds.$$

By using Gronwall's Lemma, we deduce from (3.48) that

$$(3.49) \qquad S_m^{(k)}(t) \leq M^2 e^{-TD_M^{(2)}(T)} e^{tD_M^{(2)}(T)} \leq M^2,$$

for all $t \in [0, T]$, for all m and $k \in \mathbb{N}$.

Therefore, we have

$$(3.50) \qquad u_m^{(k)} \in W(M,T), \text{ for all } m \text{ and } k.$$

Step 3. Limiting process. From (3.50), we deduce the existence of a subsequence of $\{u_m^{(k)}\}$ still so denoted, such that

$$\text{(3.51)} \qquad \begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weak*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weak*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^2(0,T;V) \text{ weak}, \\ u_m \in W(M,T). \end{cases}$$

Passing to limit in (3.15), we have u_m satisfying (3.12) in $L^2(0,T)$.
The proof of Theorem 3.1 is complete. □

We note that $W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2) \cap L^2(0,T;V)\}$ is a Banach space with respect to the norm (see Lions [12]).

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2)} + \|v'\|_{L^2(0,T;V)}.$$

We use the result given in Theorem 3.1 to prove the existence and uniqueness of a weak solution of Prob. (1.1) - (1.3). Hence, we get the main result in this section as follows.

Theorem 3.4. *Let* $(H_1) - (H_5)$ *hold. Then*

(i) *Prob.* (1.1) - (1.3) *has a unique weak solution* $u \in W_1(M,T)$, *where* $M > 0$ *and* $T > 0$ *are chosen constants as in Theorem 3.1.*

Furthermore,

(ii) *The recurrent sequence* $\{u_m\}$ *defined by* (3.11) - (3.13) *converges to the solution* u *of Prob.* (1.1) - (1.3) *strongly in* $W_1(T)$.

And we have the estimate

$$\text{(3.52)} \qquad \|u_m - u\|_{W_1(T)} \leq C_T k_T^m, \text{ for all } m \in \mathbb{N},$$

where the constant $k_T \in [0,1)$ *is defined as in* (3.46) *and* C_T *is a constant depending only on* T, h_0, f, g, μ_1, μ_2, $\tilde{u}_0$, $\tilde{u}_1$ *and* k_T.

Proof. (a) *Existence of the solution.*

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

(3.53)

$$\begin{cases} \langle w_m''(t), v\rangle + \lambda a(w_m'(t), v) + a_1^{(m+1)}(t; w_m(t), v) \\ \qquad = \langle F_{m+1}(t) - F_m(t), v\rangle + \int_0^t g(t-s)\, a_2^{(m+1)}(s; w_m(s), v)\, ds \\ \qquad - \bar{a}_1^{(m)}(t; u_m(t), v) + \int_0^t g(t-s)\, \bar{a}_2^{(m)}(s; u_m(s), v)\, ds, \ \forall v \in V, \\ w_m(0) = w_m'(0) = 0, \end{cases}$$

where

$$\begin{aligned}
(3.54) \qquad \bar{a}_i^{(m)}(t;u,\phi) &= a_i^{(m+1)}(t;u,\phi) - a_i^{(m)}(t;u,\phi) \\
&= \left\langle \left(\mu_i^{(m+1)}(t) - \mu_i^{(m)}(t)\right) u_x, \phi_x \right\rangle \\
&\quad + h_0 \left(\mu_i^{(m+1)}(0,t) - \mu_i^{(m)}(0,t)\right) u(0)\,\phi(0),
\end{aligned}$$

$\forall u,\ \phi \in V$, with $i = 1, 2$.

Note that

$$(3.55) \qquad \left|\bar{a}_i^{(m)}(t;u,\phi)\right| \le \left\|\mu_i^{(m+1)}(t) - \mu_i^{(m)}(t)\right\|_{C^0(\bar{\Omega})} \|u\|_a \|\phi\|_a,\ \forall u,\ \phi \in V,$$

$$\frac{d}{dt} a_1^{(m+1)}(t; w_m(t), w_m(t)) = 2a_1^{(m+1)}\left(t; w_m(t), w_m'(t)\right) + \dot{a}_1^{(m+1)}(t; w_m(t), w_m(t)).$$

Taking $w = w_m'(t)$ in $(3.53)_1$, after integrating in t, we get

$$\begin{aligned}
(3.56) \qquad Z_m(t) &= \int_0^t \dot{a}_1^{(m+1)}(s; w_m(s), w_m(s))ds \\
&\quad + 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle ds \\
&\quad + 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, a_2^{(m+1)}(s; w_m(s), w_m'(\tau))\, ds \\
&\quad - 2\int_0^t \bar{a}_1^{(m)}(s; u_m(s), w_m'(s))\, ds \\
&\quad + 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, \bar{a}_2^{(m)}(s; u_m(s), w_m'(\tau))\, ds \\
&\equiv J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned}$$

where

$$\begin{aligned}
(3.57)\ Z_m(t) &= \|w_m'(t)\|^2 + a_1^{(m+1)}(t; w_m(t), w_m(t)) + 2\lambda \int_0^t \|w_m'(s)\|_a^2 ds \\
&\ge \|w_m'(t)\|^2 + \mu_* \|w_m(t)\|_a^2 + 2\lambda \int_0^t \|w_m'(s)\|_a^2 ds,
\end{aligned}$$

and the integrals on the right - hand side of (3.56) are estimated as follows.

First integral J_1. By Lemma 3.2, (ii) and (3.57), we have

$$(3.58) \qquad |J_1| \le \int_0^t \left|\dot{a}_1^{(m+1)}(s; w_m(s), w_m(s))\right| ds \le \frac{\gamma_M \tilde{K}_M(\mu_1)}{\mu_*} \int_0^t Z_m(s)ds.$$

Second integral J_2. By the following inequality

$$\begin{aligned}
(3.59) \qquad & \|F_{m+1}(t) - F_m(t)\| \\
&\le 2(1+2M)K_M(f)\left[\|\nabla w_{m-1}(t)\| + \left\|w_{m-1}'(t)\right\|\right] \\
&\le 2(1+2M)K_M(f)\, \|w_{m-1}\|_{W_1(T)},
\end{aligned}$$

we obtain

$$
\begin{aligned}
(3.60) \qquad |J_2| &\leq 2\int_0^t \|F_{m+1}(s) - F_m(s)\| \, \|w_m'(s)\| \, ds \\
&\leq 4(1+2M)K_M(f) \|w_{m-1}\|_{W_1(T)} \int_0^t \|w_m'(s)\| \, ds \\
&\leq 4T(1+2M)^2 K_M^2(f) \|w_{m-1}\|_{W_1(T)}^2 + \int_0^t Z_m(s) ds.
\end{aligned}
$$

Third integral J_3. By Lemma 3.2, (iii) and (3.57), we have

$$
\begin{aligned}
(3.61) \quad |J_3| &\leq 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \left| a_2^{(m+1)}(s; w_m(s), w_m'(\tau)) \right| ds \\
&\leq 2\tilde{K}_M(\mu_2) \int_0^t d\tau \int_0^\tau |g(\tau - s)| \, \|w_m(s)\|_a \left\|w_m'(\tau)\right\|_a ds \\
&\leq \frac{2}{\sqrt{\mu_*}} \tilde{K}_M(\mu_2) \sqrt{T^*} \|g\|_{L^2(0,T^*)} \left(\int_0^t \left\|w_m'(\tau)\right\|_a^2 d\tau \right)^{1/2} \left(\int_0^t Z_m(s) ds \right)^{1/2} \\
&\leq 2\beta\lambda \int_0^t \left\|w_m'(\tau)\right\|_a^2 d\tau + \frac{1}{2\beta\lambda\mu_*} \tilde{K}_M^2(\mu_2) T^* \|g\|_{L^2(0,T^*)}^2 \int_0^t Z_m(s) ds \\
&\leq \beta Z_m(t) + \frac{1}{2\beta\lambda\mu_*} \tilde{K}_M^2(\mu_2) T^* \|g\|_{L^2(0,T^*)}^2 \int_0^t Z_m(s) ds, \ \forall \beta > 0.
\end{aligned}
$$

Fourth integral J_4. By $(3.55)_1$ and the following inequality

$$
\begin{aligned}
(3.62) \quad \left\| \mu_i^{(m+1)}(s) - \mu_i^{(m)}(s) \right\|_{C^0(\bar{\Omega})} &\leq (1+4M)\tilde{K}_M(\mu_i) \|\nabla w_{m-1}(s)\| \\
&\leq (1+4M)\tilde{K}_M(\mu_i) \|w_{m-1}\|_{W_1(T)}, \ i = 1, 2,
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
(3.63) \quad J_4 &= -2\int_0^t \bar{a}_1^{(m)}(s; u_m(s), w_m'(s)) \, ds \\
&\leq 2\int_0^t \left\| \mu_1^{(m+1)}(s) - \mu_1^{(m)}(s) \right\|_{C^0(\bar{\Omega})} \|u_m(s)\|_a \|w_m'(s)\|_a \, ds \\
&\leq 2(1+4M)\tilde{K}_M(\mu_1) \|w_{m-1}\|_{W_1(T)} \int_0^t \|u_m(s)\|_a \|w_m'(s)\|_a \, ds \\
&\leq 2(1+4M)M\tilde{K}_M(\mu_1) \|w_{m-1}\|_{W_1(T)} \int_0^t \|w_m'(s)\|_a \, ds \\
&\leq 2\beta\lambda \int_0^t \|w_m'(s)\|_a^2 ds + \frac{1}{2\beta\lambda} T(1+4M)^2 M^2 \tilde{K}_M^2(\mu_1) \|w_{m-1}\|_{W_1(T)}^2 \\
&\leq \beta Z_m(t) + \frac{1}{2\beta\lambda} T(1+4M)^2 M^2 \tilde{K}_M^2(\mu_1) \|w_{m-1}\|_{W_1(T)}^2, \ \forall \beta > 0.
\end{aligned}
$$

Fifth term J_5. Similarly to (3.63), we have

(3.64)
$$\begin{aligned}
J_5 &= 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, \bar{a}_2^{(m)}\left(s; u_m(s), w_m'(\tau)\right) ds \\
&\le 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \left\| \mu_2^{(m+1)}(s) - \mu_2^{(m)}(s) \right\|_{C^0(\bar{\Omega})} \|u_m(s)\|_a \left\| w_m'(\tau) \right\|_a ds \\
&\le 2(1+4M)M\tilde{K}_M(\mu_2) \|w_{m-1}\|_{W_1(T)} \int_0^t d\tau \int_0^\tau |g(\tau - s)| \left\| w_m'(\tau) \right\|_a ds \\
&\le 2(1+4M)M\tilde{K}_M(\mu_2) \|w_{m-1}\|_{W_1(T)} \|g\|_{L^1(0,T^*)} \int_0^t \left\| w_m'(\tau) \right\|_a d\tau \\
&\le 2\beta\lambda \int_0^t \left\| w_m'(s) \right\|_a^2 ds + \frac{1}{2\beta\lambda} T(1+4M)^2 M^2 \tilde{K}_M^2(\mu_2) \|g\|_{L^1(0,T^*)}^2 \|w_{m-1}\|_{W_1(T)}^2 \\
&\le \beta Z_m(t) + \frac{1}{2\beta\lambda} T(1+4M)^2 M^2 \tilde{K}_M^2(\mu_2) \|g\|_{L^1(0,T^*)}^2 \|w_{m-1}\|_{W_1(T)}^2, \ \forall \beta > 0.
\end{aligned}$$

Choosing $\beta = \dfrac{1}{6}$, it follows from (3.56), (3.58), (3.60), (3.61), (3.63), (3.64) that

$$Z_m(t) \le T\bar{D}_1(M) \|w_{m-1}\|_{W_1(T)}^2 + 2\bar{D}_2(M) \int_0^t Z_m(s) ds, \tag{3.65}$$

where $\bar{D}_1(M)$ and $\bar{D}_2(M)$ are the constants as in (3.47).

Using Gronwall's Lemma, we deduce from (3.65) that

$$\|w_m\|_{W_1(T)} \le k_T \|w_{m-1}\|_{W_1(T)}, \ \forall m \in \mathbb{N}, \tag{3.66}$$

where $k_T \in (0,1)$ is defined as in (3.46), which implies that

$$\|u_m - u_{m+p}\|_{W_1(T)} \le \|u_0 - u_1\|_{W_1(T)} (1-k_T)^{-1} k_T^m, \quad \forall m, p \in \mathbb{N}. \tag{3.67}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{3.68}$$

Note that $u_m \in W(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases}
u_{m_j} \to u & \text{in} \quad L^\infty(0,T; V \cap H^2) \text{ weak*}, \\
u_{m_j}' \to u' & \text{in} \quad L^\infty(0,T; V \cap H^2) \text{ weak*}, \\
u_{m_j}'' \to u'' & \text{in} \quad L^2(0,T; V) \text{ weak}, \\
u \in W(M,T). &
\end{cases} \tag{3.69}$$

We note that

$$\|F_m - f[u]\|_{L^\infty(0,T;L^2)} \le 2(1+2M)K_M(f) \|u_{m-1} - u\|_{W_1(T)}. \tag{3.70}$$

Hence, we deduce from (3.68) and (3.70) that

$$F_m \to f[u] \text{ strongly in } L^\infty(0,T;L^2). \tag{3.71}$$

We also note that

$$\begin{aligned} &\left\|\mu_i^{(m)}(t) - \mu_i[u](t)\right\|_{C^0(\bar{\Omega})} \\ &\le (1+4M)\tilde{K}_M(\mu_i)\left\|u_{m-1} - u\right\|_{W_1(T)}, \ a.e.\ t \in (0,T),\ i = 1,2. \end{aligned} \tag{3.72}$$

On the other hand, for all $v \in V$, we have

$$\begin{aligned} &\left|a_1^{(m)}(t; u_m(t), v) - a_1[u](t; u(t), v)\right| \\ &\le \left|a_1^{(m)}(t; u_m(t), v) - a_1[u](t; u_m(t), v)\right| + \left|a_1[u](t; u_m(t) - u(t), v)\right| \\ &\le \left\|\mu_1^{(m)}(t) - \mu_1[u](t)\right\|_{C^0(\bar{\Omega})} \left\|u_m(t)\right\|_a \left\|v\right\|_a \\ &\quad + \tilde{K}_M(\mu_1)\left\|u_m(t) - u(t)\right\|_a \left\|v\right\|_a \\ &\le \tilde{K}_M(\mu_1)\left[M(1+4M)\left\|u_{m-1} - u\right\|_{W_1(T)} + \left\|u_m - u\right\|_{W_1(T)}\right]\left\|v\right\|_a . \end{aligned} \tag{3.73}$$

Hence

$$\begin{aligned} &\left|\int_0^T a_1^{(m)}(t; u_m(t), v)\phi(t)dt - \int_0^T a_1[u](t; u(t), v)\phi(t)dt\right| \\ &\le \tilde{K}_M(\mu_1)\Big[M(1+4M)\left\|u_{m-1} - u\right\|_{W_1(T)} \\ &\qquad + \left\|u_m - u\right\|_{W_1(T)}\Big]\left\|v\right\|_a \left\|\phi\right\|_{L^1(0,T)} \to 0, \forall v \in V, \forall \phi \in L^1(0,T). \end{aligned} \tag{3.74}$$

Similarly

$$\begin{aligned} &\left|\int_0^T \left(\int_0^t g(t-s)\, a_2^{(m)}(s; u_m(s), v)\, ds\right)\phi(t)dt \right. \\ &\left. - \int_0^T \left(\int_0^t g(t-s)\, a_2[u](s; u(s), v)\, ds\right)\phi(t)dt\right| \\ &\le \tilde{K}_M(\mu_2)\left\|g\right\|_{L^1(0,T)}\Big[M(1+4M)\left\|u_{m-1} - u\right\|_{W_1(T)} \\ &\qquad + \left\|u_m - u\right\|_{W_1(T)}\Big]\left\|v\right\|_a \left\|\phi\right\|_{L^1(0,T)} \to 0, \end{aligned} \tag{3.75}$$

$\forall v \in V$, $\forall \phi \in L^1(0,T)$.

Finally, passing to limit in (3.12), (3.13) as $m = m_j \to \infty$, it implies from (3.68), (3.69), (3.71), (3.74) and (3.75) that there exists $u \in W(M,T)$ satisfying the equation

$$
\begin{aligned}
(3.76)\qquad & \langle u''(t), w\rangle + \lambda a(u'(t), w) + a_1[u](t; u(t), w) \\
= & \int_0^t g\left(t-s\right) a_2[u]\left(s; u\left(s\right), w\right) ds + \left\langle f[u](t), w\right\rangle,
\end{aligned}
$$

for all $w \in V$ and the initial conditions

$$
(3.77)\qquad u(0) = \tilde{u}_0,\ \ u'(0) = \tilde{u}_1.
$$

On the other hand, we have from (H_2), (H_3), (H_6), and $(3.69)_{1,2,3}$, that

$$
\begin{aligned}
(3.78)\qquad u_{tt} &= \lambda u_{xxt} + \mu_1[u]u_{xx} + \frac{\partial}{\partial x}\left(\mu_1[u]\right) u_x \\
&\quad - \int_0^t g\left(t-s\right) \mu_2[u]\left(x,s\right) u_{xx}\left(x,s\right) ds \\
&\quad - \int_0^t g\left(t-s\right) \frac{\partial}{\partial x}\left(\mu_2[u]\right)\left(x,s\right) u_x\left(x,s\right) ds + f[u] \\
&\equiv \tilde{F} \in L^{\infty}(0,T;L^2).
\end{aligned}
$$

Thus $u \in W_1(M,T)$. The existence result follows.

(b) *Uniqueness of the solution.*

Let $u_1,\ u_2 \in W_1(M,T)$ be two weak solutions of Prob. (1.1) - (1.3). Then $u = u_1 - u_2$ satisfies the variational problem

$$
(3.79)\qquad \begin{cases}
\langle u''(t), w\rangle + \lambda a(u'(t), w) + a_1[u_1](t; u(t), w) \\
= \langle f[u_1](t) - f[u_2](t), w\rangle + \int_0^t g\left(t-s\right) a_2[u_1]\left(s; u\left(s\right), w\right) ds \\
-\left\langle \left(\mu_1[u_1](t) - \mu_1[u_2](t)\right) u_{2x}(t), w_x\right\rangle \\
-h_0\left(\mu_1[u_1](0,t) - \mu_1[u_2](0,t)\right) u_2(0,t) w(0) \\
+\int_0^t g\left(t-s\right)\left[\left\langle \left(\mu_2[u_1](s) - \mu_2[u_2](s)\ \right) u_{2x}(s), w_x\right\rangle\right. \\
\left. +h_0\left(\mu_2[u_1](0,s) - \mu_2[u_2](0,s)\ \right) u_2\left(0,s\right) w(0)\right] ds,\ \forall w \in V, \\
u(0) = u'(0) = 0,
\end{cases}
$$

where

$$
\begin{aligned}
(3.80)\quad a_j[u_i](t; v, w) &= \langle \mu_j[u_i](t) v_x, w_x\rangle \\
&\quad + h_0 \mu_j[u_i](0,t) v(0) w(0),\ \ v,\ \ w \in V, \\
\mu_j[u_i](x,t) &= \mu_j\left(x, t, u_i\left(x,t\right), \left\|u_i(t)\right\|^2, \left\|u_{ix}(t)\right\|^2\right),\ \ i,j = 1,2.
\end{aligned}
$$

We take $w = u'$ in $(3.79)_1$ and integrate in t to get

$$
\begin{aligned}
Z(t) = &\int_0^t \dot{a}_1[u_1](s; u(s), u(s)) ds \\
&+ 2\int_0^t \langle f[u_1](s) - f[u_2](s), u'(s)\rangle \, ds \\
&+ 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, a_2[u_1]\,(s; u\,(s)\,, u'(\tau))\, ds \\
&- 2\int_0^t [\langle(\mu_1[u_1](s) - \mu_1[u_2](s)\ )\, u_{2x}(s), u'_x(s)\rangle \\
&+h_0\,(\mu_1[u_1](0,s) - \mu_1[u_2](0,s)\ )\, u_2\,(0,s)\, u'(0,s)]\, ds \\
&+ 2\int_0^t d\tau \int_0^\tau g(\tau - s)\, [\langle(\mu_2[u_1](s) - \mu_2[u_2](s)\ )\, u_{2x}(s), u'_x(\tau)\rangle \\
&+h_0\,(\mu_2[u_1](0,s) - \mu_2[u_2](0,s)\ )\, u_2\,(0,s)\, u'(0,\tau)]\, ds,
\end{aligned} \tag{3.81}
$$

where

$$
Z(t) = \|u'(t)\|^2 + a_1[u_1](t; u(t), u(t)) + 2\lambda \int_0^t \|u'(s)\|_a^2 \, ds. \tag{3.82}
$$

Put

$$
\begin{aligned}
\tilde{Z}_M = &\frac{2\gamma_M \tilde{K}_M(\mu_1)}{\mu_*} + 8\left(1 + \frac{1}{\sqrt{\mu_*}}\right)(1+M)K_M(f) \\
&+ \frac{6}{\lambda\mu_*}\Big[\tilde{K}_M^2(\mu_2)\, T\, \|g\|^2_{L^2(0,T^*)} \\
&\qquad +(1+4M)^2 M^2 \left(\tilde{K}_M^2(\mu_1) + \tilde{K}_M^2(\mu_2) T \|g\|^2_{L^2(0,T^*)}\right)\Big],
\end{aligned}
$$

then it follows from (3.81) that

$$
Z(t) \le \tilde{Z}_M \int_0^t Z(s) ds.
$$

By Gronwall's Lemma, we deduce $Z(t) = 0$, i.e., $u_1 \equiv u_2$. Theorem 3.4 is proved completely. □

4. Asymptotic Expansion of the Solution with respect to a Small Parameter

In this section, let $(H_1) - (H_5)$ hold. We also make the following assumptions:

(H_6) $f_1 \in C^1([0,1] \times [0, T^*] \times \mathbb{R}^3 \times \mathbb{R}^2_+)$.

We consider the following perturbed problem, where ε is a small parameter, with $|\varepsilon| < 1$:

$$(P_\varepsilon)\begin{cases} u_{tt} - \lambda u_{xxt} - \dfrac{\partial}{\partial x}\left[\mu_1[u](x,t)u_x\right] + \int_0^t g\,(t-s)\,\dfrac{\partial}{\partial x}\left[\mu_2[u](x,s)u_x\,(x,s)\right] ds \\ \qquad\quad = F_\varepsilon[u](x,t),\ 0<x<1,\ 0<t<T, \\ u_x(0,t) - h_0 u(0,t) = u(1,t) = 0, \\ u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \\ \mu_i[u](x,t) = \mu_i\left(x,t,u(x,t),\|u(t)\|^2,\|u_x(t)\|^2\right),\ i=1,2, \\ F_\varepsilon[u](x,t) = f[u](x,t) + \varepsilon f_1[u](x,t), \\ f[u](x,t) = f(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2). \end{cases}$$

By the assumptions $(H_1)-(H_6)$ and theorem 3.4, Prob. (P_ε) has a unique weak solution u depending on $\varepsilon : u = u_\varepsilon$. When $\varepsilon = 0$, (P_ε) is denoted by (P_0). We shall study the asymptotic expansion of the solution u_ε of Prob. (P_ε) with respect to a small parameter ε.

We use the following notations. For a multi-index $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{Z}_+^N$, and $x = (x_1, ..., x_N) \in \mathbb{R}^N$, we put

$$\begin{cases} |\alpha| = \alpha_1 + ... + \alpha_N,\ \alpha! = \alpha_1!...\alpha_N!, \\ x^\alpha = x_1^{\alpha_1}...x_N^{\alpha_N}, \\ \alpha,\ \beta \in \mathbb{Z}_+^N,\ \alpha \le \beta \Longleftrightarrow \alpha_i \le \beta_i \quad \forall i = 1, ..., N. \end{cases}$$

First, we shall need the following lemma.

Lemma 4.1. *Let m, $N \in \mathbb{N}$, $x = (x_1, ..., x_N) \in \mathbb{R}^N$, and $\varepsilon \in \mathbb{R}$. Then*

$$\left(\sum_{i=1}^N x_i \varepsilon^i\right)^m = \sum_{k=m}^{mN} P_N^{[m]}[x]_k \varepsilon^k, \tag{4.1}$$

where the coefficients $P_N^{[m]}[x]_k$, $m \le k \le mN$ depending on $x = (x_1, ..., x_N)$ are defined by the formula

$$P_N^{[m]}[x]_k = \begin{cases} x_k, & 1 \le k \le N,\ m = 1, \\ \sum\limits_{\alpha \in A_k^{[m]}(N)} \dfrac{m!}{\alpha!} x^\alpha, & m \le k \le mN,\ m \ge 2, \end{cases} \tag{4.2}$$

with $A_k^{[m]}(N) = \left\{\alpha \in \mathbb{Z}_+^N : |\alpha| = m,\ \sum\limits_{i=1}^N i\alpha_i = k\right\}$.

Proof. The proof of this lemma is easy, hence we omit the details. □

Now, we assume that

(H_7) μ_1, $\mu_2 \in C^{N+1}([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2)$,
$\mu_1(x,t,z_1,z_2,z_3) \ge \mu_* > 0$, for all $(x,t,z_1,z_2,z_3) \in [0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^2$;

(H_8) $f \in C^{N+1}([0,1]\times[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+^2)$, $f_1 \in C^{N}([0,1]\times[0,T^*]\times\mathbb{R}^3\times\mathbb{R}_+^2)$.

We also use the notations

$$\begin{aligned} f[u](x,t) &= f(x,t,u,u_x,u_t,\|u(t)\|^2,\|u_x(t)\|^2), \\ \mu_i[u](x,t) &= \mu_i\left(x,t,u,\|u(t)\|^2,\|u_x(t)\|^2\right),\ i=1,2. \end{aligned}$$

Let u_0 be a unique weak solution of problem (P_0) (as in Theorem 3.4) corresponding to $\varepsilon = 0$, *i.e.*,

$$(P_0)\begin{cases} u_0'' - \lambda\Delta u_0' - \dfrac{\partial}{\partial x}\left[\mu_1[u_0](x,t)u_{0x}\right] \\ \quad + \int_0^t g(t-s)\dfrac{\partial}{\partial x}\left[\mu_2[u_0](x,s)u_{0x}(x,s)\right]ds = f[u_0],\ 0<x<1,\ 0<t<T, \\ u_{0x}(0,t) - h_0u_0(0,t) = u_0(1,t) = 0, \\ u_0(x,0) = \tilde{u}_0(x),\ u_0'(x,0) = \tilde{u}_1(x), \\ u_0 \in W_1(M,T). \end{cases}$$

Considering the sequence of weak solutions u_r, $1 \le r \le N$, of the following problems:

$$(\tilde{P}_r)\begin{cases} u_r'' - \lambda\Delta u_r' - \dfrac{\partial}{\partial x}\left[\mu_1[u_0](x,t)u_{rx}\right] \\ \quad + \int_0^t g(t-s)\dfrac{\partial}{\partial x}\left[\mu_2[u_0](x,s)u_{rx}(x,s)\right]ds = F_r,\ 0<x<1,\ 0<t<T, \\ u_{rx}(0,t) - h_0u_r(0,t) = u_r(1,t) = 0, \\ u_r(x,0) = u_r'(x,0) = 0, \\ u_r \in W_1(M,T), \end{cases}$$

where F_r, $1 \le r \le N$, are defined by the recurrence formulas

$$F_r = \begin{cases} f[u_0], & r=0, \\ \pi_r[N,f] + \pi_{r-1}[N-1,f_1] \\ \quad + \sum\limits_{i=1}^{r}\dfrac{\partial}{\partial x}\left(\rho_i[\mu_1]\nabla u_{r-i} - \int_0^t g(t-s)\rho_i[\mu_2]\nabla u_{r-i}(s)ds\right), & 1 \le r \le N, \end{cases} \tag{4.3}$$

with $\pi_r[N,f] = \pi_r[N,f;u_0,u_1,...,u_r]$, $\rho_r[\mu_i] = \rho_r[\mu_i,N,u_0,\bar{u},\sigma^{(1)},\sigma^{(2)}]$, $0 \le r \le N$, defined by the formulas:
(a) *Formula* $\pi_r[N,f] = \pi_r[N,f;u_0,\bar{u},\sigma^{(1)},\sigma^{(2)}]$:

$$\pi_r[N,f] = \begin{cases} f[u_0], & r=0, \\ \sum\limits_{1\le|m|\le r}\frac{1}{m!}D^mf[u_0]\tilde{\Phi}_r[m,N,\bar{u},\sigma^{(1)},\sigma^{(2)}], & 1 \le r \le N, \end{cases} \tag{4.4}$$

in which

$$\tilde{\Phi}_r[m,N,\bar{u},\sigma^{(1)},\sigma^{(2)}] \tag{4.5}$$

$$= \sum_{(k_1,...,k_5)\in \bar{A}_r(m,N)} P_N^{[m_1]}[\bar{u}]_{k_1} P_N^{[m_2]}[\nabla \bar{u}]_{k_2} P_N^{[m_3]}[\bar{u}']_{k_3} P_{2N}^{[m_4]}[\sigma^{(1)}]_{k_4} P_{2N}^{[m_5]}[\sigma^{(2)}]_{k_5},$$

and $\bar{A}_r(m,N) = \{(k_1,...,k_5) \in \mathbb{Z}_+^5 : k_1 + ... + k_5 = r,\ m_i \le k_i \le m_i N,\ i = 1,...,3;\ m_j \le k_j \le 2m_j N,\ j = 4,5\}$, $m = (m_1,...,m_5) \in \mathbb{Z}_+^5$, $|m| = m_1 + ... + m_5$, $m! = m_1!...m_5!$, $D^m f = D_3^{m_1} D_4^{m_2} D_5^{m_3} D_6^{m_4} D_7^{m_5} f$, and $\sigma^{(1)} = (\sigma_1^{(1)},...,\sigma_{2N}^{(1)})$, $\sigma^{(2)} = (\sigma_1^{(2)},...,\sigma_{2N}^{(2)})$, are defined by

$$(4.6)\qquad \sigma_i^{(1)} = \begin{cases} 2\langle u_0, u_1\rangle, & i = 1, \\ 2\langle u_0, u_i\rangle + \sum\limits_{j=1}^{i} \langle u_j, u_{i-j}\rangle, & 2 \le i \le N, \\ \sum\limits_{j=1}^{i} \langle u_j, u_{i-j}\rangle, & N+1 \le i \le 2N, \end{cases}$$

$$\sigma_i^{(2)} = \begin{cases} 2\langle \nabla u_0, \nabla u_1\rangle, & i = 1, \\ 2\langle \nabla u_0, \nabla u_i\rangle + \sum\limits_{j=1}^{i} \langle \nabla u_j, \nabla u_{i-j}\rangle, & 2 \le i \le N, \\ \sum\limits_{j=1}^{i} \langle \nabla u_j, \nabla u_{i-j}\rangle, & N+1 \le i \le 2N; \end{cases}$$

(b) *Formula* $\rho_r[\mu_i] = \rho_r[\mu_i, N, u_0, \bar{u}, \sigma^{(1)}, \sigma^{(2)}]$:

$$(4.7)\qquad \rho_r[\mu_i] = \begin{cases} \mu_i[u_0], & r = 0, \\ \sum\limits_{1\le|\gamma|\le r} \frac{1}{\gamma!} D^\gamma \mu_i[u_0] \Phi_r[N, \bar{u}, \sigma^{(1)}, \sigma^{(2)}, \gamma], & 1 \le r \le N, \end{cases}$$

where

$$(4.8)\quad \Phi_r[N, \bar{u}, \sigma^{(1)}, \sigma^{(2)}, \gamma] = \sum_{(k_1,k_2,k_3)\in \tilde{A}_r(\gamma,N)} P_N^{[\gamma_1]}[\bar{u}]_{k_1} P_{2N}^{[\gamma_2]}[\sigma^{(1)}]_{k_2} P_{2N}^{[\gamma_3]}[\sigma^{(2)}]_{k_3},$$

with $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}_+^3$, $1 \le |\gamma| = \gamma_1 + ... + \gamma_3 \le r$, $\gamma! = \gamma_1!...\gamma_3!$, $D^\gamma \mu_i = D_3^{\gamma_1} D_4^{\gamma_2} D_5^{\gamma_3} \mu_i$, $\tilde{A}_r(\gamma, N) = \{(k_1,k_2,k_3) \in \mathbb{Z}_+^3 : |\gamma| = r,\ \gamma_1 \le k_1 \le \gamma_1 N,\ \gamma_2 \le k_2 \le 2\gamma_2 N,\ \gamma_3 \le k_3 \le 2\gamma_3 N\}$. Then, we have the following lemma.

Lemma 4.2. *Let* $\pi_r[N,f] = \pi_r[N, f; u_0, \bar{u}, \sigma^{(1)}, \sigma^{(2)}]$, $\rho_r[\mu_i] = \rho_r[\mu_i, N, u_0, \bar{u}, \sigma^{(1)}, \sigma^{(2)}]$, $0 \le r \le N$, *be the functions defined by formulas* (4.4) *and* (4.7). *Let* $h = \sum\limits_{r=0}^{N} u_r \varepsilon^r$. *Then we have*

$$(4.9)\qquad f[h] = \sum_{r=0}^{N} \pi_r[N,f]\varepsilon^r + |\varepsilon|^{N+1}\, \bar{R}_N^{(1)}[f,\varepsilon],$$

$$(4.10)\qquad \mu_i[h] = \sum_{r=0}^{N} \rho_r[\mu_i]\varepsilon^r + |\varepsilon|^{N+1}\, \widetilde{R}_N^{(1)}[\mu_i,\varepsilon],$$

with $\left\|\widetilde{R}_N^{(1)}[\mu_i,\varepsilon]\right\|_{L^\infty(0,T;L^2)}+\left\|\bar{R}_N^{(1)}[f,\varepsilon]\right\|_{L^\infty(0,T;L^2)}\leq C$, *where* C *is a constant depending only on* N, T, f, μ_1, μ_2, u_r, $0\leq r\leq N$.

Proof. (i) In the case of $N=1$, the proof of (4.9) is easy, hence we omit the details. We only prove the case of $N\geq 2$. Let $h=u_0+\sum_{i=1}^N u_i\varepsilon^i\equiv u_0+h_1$. We rewrite as below

$$
\begin{aligned}
f[h] &= f\left(x,t,h(x,t),\nabla h(x,t),h'(x,t),\|h(t)\|^2,\|\nabla h(t)\|^2\right)\\
&= f(x,t,u_0+h_1,\nabla u_0+\nabla h_1,u_0'+h_1',\|u_0+h_1\|^2,\|\nabla u_0+\nabla h_1\|^2)\\
&= f(x,t,u_0+h_1,\nabla u_0+\nabla h_1,u_0'+h_1',\|u_0\|^2+\xi_4,\|\nabla u_0\|^2+\xi_5),
\end{aligned}
\tag{4.11}
$$

where $\xi_4=\|u_0+h_1\|^2-\|u_0\|^2$, $\xi_5=\|\nabla u_0+\nabla h_1\|^2-\|\nabla u_0\|^2$.

By using Taylor's expansion of the function $f[u_0+h_1]$ around the point $[u_0]=(x,t,u_0,\nabla u_0,u_0',\|u_0\|^2,\|\nabla u_0\|^2)$ up to order $N+1$, we obtain

$$
\begin{aligned}
f[u_0+h_1] =& f[u_0]+D_3f[u_0]h_1+D_4f[u_0]\nabla h_1\\
&+D_5f[u_0]h_1'+D_6f[u_0]\xi_4+D_7f[u_0]\xi_5\\
&+\sum_{\substack{2\leq|m|\leq N\\ m=(m_1,\ldots,m_5)\in\mathbb{Z}_+^5}}\frac{1}{m!}D^mf[u_0]h_1^{m_1}(\nabla h_1)^{m_2}(h_1')^{m_3}\xi_4^{m_4}\xi_5^{m_5}\\
&+R_N^{(1)}[f,h_1,\xi_4,\xi_5],
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
&R_N^{(1)}[f,h_1,\xi_4,\xi_5] \\
&= \sum_{\substack{|m|=N+1\\ m\in\mathbb{Z}_+^5}}\frac{N+1}{m!}\left(\int_0^1(1-\theta)^ND^mf(\theta)d\theta\right)h_1^{m_1}(\nabla h_1)^{m_2}(h_1')^{m_3}\xi_4^{m_4}\xi_5^{m_5}\\
&= |\varepsilon|^{N+1}\bar{R}_N^{(1)}[f,h_1,\xi_4,\xi_5],
\end{aligned}
\tag{4.13}
$$

$$
D^mf(\theta) = D^mf(x,t,u_0+\theta h_1,\nabla u_0+\theta\nabla h_1,u_0'+\theta h_1',\|u_0\|^2+\theta\xi_4,\|\nabla u_0\|^2+\theta\xi_5).
$$

By the formula (4.1), it follows that

$$
\begin{aligned}
h_1^{m_1} &= \left(\sum_{i=1}^N u_i\varepsilon^i\right)^{m_1}=\sum_{k=m_1}^{m_1N}P_N^{[m_1]}[\bar{u}]_k\varepsilon^k,\\
(\nabla h_1)^{m_2} &= \left(\sum_{i=1}^N \nabla u_i\varepsilon^i\right)^{m_2}=\sum_{k=m_2}^{m_2N}P_N^{[m_2]}[\nabla\bar{u}]_k\varepsilon^k,\\
(h_1')^{m_3} &= \left(\sum_{i=1}^N u_i'\varepsilon^i\right)^{m_3}=\sum_{k=m_3}^{m_3N}P_N^{[m_3]}[\bar{u}']_k\varepsilon^k,
\end{aligned}
\tag{4.14}
$$

where $\bar{u} = (u_1, ..., u_N)$, $\nabla \bar{u} = (\nabla u_1, ..., \nabla u_N)$, $\bar{u}' = (u_1', ..., u_N')$.

On the other hand,

$$\xi_4 = \|u_0 + h_1\|^2 - \|u_0\|^2 = 2\langle u_0, h_1\rangle + \|h_1\|^2 \equiv \sum_{i=1}^{2N} \sigma_i^{(1)} \varepsilon^i, \tag{4.15}$$

with $\sigma_i^{(1)}$, $1 \le i \le 2N$ are defined by $(4.6)_1$.

By the formula (4.1), it follows from (4.15) that

$$\xi_4^{m_4} = \left(\sum_{i=1}^{2N} \sigma_i^{(1)} \varepsilon^i \right)^{m_4} = \sum_{k=m_4}^{2m_4 N} P_{2N}^{[m_4]}[\sigma^{(1)}]_k \varepsilon^k, \tag{4.16}$$

where $\sigma^{(1)} = (\sigma_1^{(1)}, ..., \sigma_{2N}^{(1)})$.

Similarly, we have

$$\xi_5^{m_5} = \left(\sum_{i=1}^{2N} \sigma_i^{(2)} \varepsilon^i \right)^{m_5} = \sum_{k=m_5}^{2m_5 N} P_{2N}^{[m_5]}[\sigma^{(2)}]_k \varepsilon^k, \tag{4.17}$$

where $\sigma^{(2)} = (\sigma_1^{(2)}, ..., \sigma_{2N}^{(2)})$, are defined by $(4.6)_2$.

Therefore, it follows from (4.14), (4.16), (4.17) that

$$\begin{aligned} h_1^{m_1} (\nabla h_1)^{m_2} (h_1')^{m_3} \xi_4^{m_4} \xi_5^{m_5} \quad &= \quad \sum_{r=|m|}^{N} \tilde{\Phi}_r[m, N, \bar{u}, \sigma^{(1)}, \sigma^{(2)}] \varepsilon^r \\ &\quad + |\varepsilon|^{N+1} \bar{R}_N^{(2)}[m, N, \bar{u}, \sigma^{(1)}, \sigma^{(2)}], \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} &\tilde{\Phi}_r[m, N, \bar{u}, \sigma^{(1)}, \sigma^{(2)}] \\ &= \sum_{(k_1, ..., k_5) \in \bar{A}_r(m,N)} P_N^{[m_1]}[\bar{u}]_{k_1} P_N^{[m_2]}[\nabla \bar{u}]_{k_2} P_N^{[m_3]}[\bar{u}']_{k_3} P_{2N}^{[m_4]}[\sigma^{(1)}]_{k_4} P_{2N}^{[m_5]}[\sigma^{(2)}]_{k_5}, \end{aligned} \tag{4.19}$$

with $\bar{A}_r(m, N)$ as in (4.5) and

$$|\varepsilon|^{N+1} \bar{R}_N^{(2)}[m, N, \bar{u}, \sigma^{(1)}, \sigma^{(2)}] = \sum_{r=N+1}^{(|m|+m_4+m_5)N} \tilde{\Phi}_r[m, N, \bar{u}, \sigma^{(1)}, \sigma^{(2)}] \varepsilon^r.$$

Hence, we deduce from (4.12), (4.18) that

$$
\begin{aligned}
(4.20)\qquad & f[u_0+h_1] \\
&= f[u_0] + \sum_{\substack{1\le |m|\le N\\ m=(m_1,\dots,m_5)\in\mathbb{Z}_+^5}} \frac{1}{m!} D^m f[u_0] \sum_{r=|m|}^{N} \tilde{\Phi}_r[m,N,\bar{u},\sigma^{(1)},\sigma^{(2)}]\varepsilon^r \\
&\quad + |\varepsilon|^{N+1}\, \bar{R}_N^{(1)}[f,\varepsilon] \\
&= f[u_0] + \sum_{r=1}^{N}\left(\sum_{1\le|m|\le r} \frac{1}{m!} D^m f[u_0] \tilde{\Phi}_r[m,N,\bar{u},\sigma^{(1)},\sigma^{(2)}]\right)\varepsilon^r \\
&\quad + |\varepsilon|^{N+1}\, \bar{R}_N^{(1)}[f,\varepsilon] \\
&= \sum_{r=0}^{N} \pi_r[N,f]\varepsilon^r + |\varepsilon|^{N+1}\, \bar{R}_N^{(1)}[f,\varepsilon],
\end{aligned}
$$

where

$$
\begin{aligned}
(4.21)\qquad \bar{R}_N^{(1)}[f,\varepsilon] &= \sum_{\substack{2\le |m|\le N\\ m=(m_1,\dots,m_5)\in\mathbb{Z}_+^5}} \frac{1}{m!} D^m f[u_0] \bar{R}_N^{(2)}[m,N,\bar{u},\sigma^{(1)},\sigma^{(2)}] \\
&\quad + \bar{R}_N^{(1)}[f,h_1,\xi_4,\xi_5],
\end{aligned}
$$

with $\pi_r[N,f]$, $0\le r\le N$ are defined by (4.4), and $\left\|\bar{R}_N^{(1)}[f,\varepsilon]\right\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on N, T, f, u_r, $r=0,1,\dots,N$.

Hence, the formula (4.9) is proved.

(ii) In the case of $\mu_i[h]=\mu_i\left(x,t,h(x,t),\|h(t)\|^2,\|\nabla h(t)\|^2\right)$. Applying the formulas (4.4) - (4.6) and (4.9) with $f=f(x,t,z_1,z_2,z_3)$, $D_j f=0$, $j=4,5$, $D_3 f = D_3\mu_i = \dfrac{\partial \mu_i}{\partial z_1}$, $D_6 f = D_4\mu_i = \dfrac{\partial \mu_i}{\partial z_2}$, $D_7 f = D_5\mu_i = \dfrac{\partial \mu_i}{\partial z_3}$ and $\pi_r[N,f]=\rho_r[\mu_i]$, $0\le r\le N$, we obtain formulas (4.6) - (4.8) and the formula (4.10) is proved.

This completes the proof of the lemma 4.2. $\square$

Remark 4.1. Lemma 4.2 is a generalization of a formula contained in [15] (formula (4.38), p. 262) and it is useful to obtain the following Lemma 4.3. These Lemmas are the key to obtain the asymptotic expansion of the weak solution $u=u_\varepsilon$ of order $N+1$ in a small parameter ε.

Let $u = u_\varepsilon \in W_1(M,T)$ be a unique weak solution of the problem (P_ε). Then $v = u - \sum_{r=0}^{N} u_r \varepsilon^r \equiv u - h = u - u_0 - h_1$ satisfies the problem

$$
(4.22)\quad \begin{cases}
v'' - \lambda \Delta v' - \dfrac{\partial}{\partial x}\left[\mu_1[v+h] v_x\right] + \int_0^t g\,(t-s)\,\dfrac{\partial}{\partial x}\left[\mu_2[v+h] v_x\,(x,s)\right] ds \\
= f[v+h] - f[h] + \varepsilon\left(f_1[v+h] - f_1[h]\right) \\
+ \dfrac{\partial}{\partial x}\left[\left(\mu_1[v+h] - \mu_1[h]\right) h_x\right] \\
- \int_0^t g\,(t-s)\,\dfrac{\partial}{\partial x}\left[\left(\mu_2[v+h] - \mu_2[h]\right) h_x\,(x,s)\right] ds \\
+ E_\varepsilon(x,t),\ 0 < x < 1,\ 0 < t < T, \\
v_x(0,t) - h_0 v(0,t) = v(1,t) = 0, \\
v(x,0) = v'(x,0) = 0,
\end{cases}
$$

where

$$
(4.23)\quad \begin{aligned}
E_\varepsilon(x,t) = & f[h] - f[u_0] + \varepsilon f_1[h] + \frac{\partial}{\partial x}\left[\left(\mu_1[h] - \mu_1[u_0]\right) h_x\right] \\
& - \int_0^t g\,(t-s)\,\frac{\partial}{\partial x}\left[\left(\mu_2[h] - \mu_2[u_0]\right) h_x\,(x,s)\right] ds - \sum_{r=1}^{N} F_r \varepsilon^r.
\end{aligned}
$$

Lemma 4.3. *Under the assumptions (H_1), (H_2), (H_7) and (H_8), there exists a constant $\bar{C}_*$ such that*

$$
(4.24)\quad \|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le \bar{C}_* \left|\varepsilon\right|^{N+1},
$$

where $\bar{C}_*$ is a constant depending only on N, T, f, f_1, μ_1, μ_2, u_r, $0 \le r \le N$.

Proof. In the case of $N = 1$, the proof of Lemma 4.3 is easy. The details are omitted. We only consider $N \ge 2$.

By using formulas (4.9), (4.10) for the functions $f_1[h]$, $\mu_1[h]$ and $\mu_2[h]$, we obtain

$$
(4.25)\quad \begin{aligned}
f_1[h] &= \sum_{r=0}^{N-1} \pi_r[N-1, f_1]\varepsilon^r + \left|\varepsilon\right|^N \bar{R}_{N-1}^{(1)}[f_1, \varepsilon], \\
\mu_i[h] &= \sum_{r=0}^{N} \rho_r[\mu_i]\varepsilon^r + \left|\varepsilon\right|^{N+1} \widetilde{R}_N^{(1)}[\mu_i, \varepsilon],\ \ i = 1,2.
\end{aligned}
$$

By $(4.25)_1$, we rewrite $\varepsilon f_1[h]$ as follows

$$
(4.26)\quad \varepsilon f_1[h] = \sum_{r=1}^{N} \pi_{r-1}[N-1, f_1]\varepsilon^r + \varepsilon\left|\varepsilon\right|^N \bar{R}_{N-1}^{(1)}[f_1, \varepsilon].
$$

First, we deduce from (4.9) and (4.26), that

$$
\begin{aligned}
(4.27) \qquad f[h] - f[u_0] + \varepsilon f_1[h] &= \sum_{r=1}^{N} \left(\pi_r[N,f] + \pi_{r-1}[N-1,f_1] \right) \varepsilon^r \\
&\quad + |\varepsilon|^{N+1} \bar{R}_N^{(1)}[f,f_1,\varepsilon],
\end{aligned}
$$

where $\bar{R}_N^{(1)}[f,f_1,\varepsilon] = \bar{R}_N^{(1)}[f,\varepsilon] + \frac{\varepsilon}{|\varepsilon|}\bar{R}_{N-1}^{(1)}[f_1,\varepsilon]$ is bounded in $L^\infty(0,T;L^2)$ by a constant depending only on N, T, f, f_1, u_r, $0 \le r \le N$.

On the other hand, we deduce from (4.10) and $(4.25)_2$ that

$$
\begin{aligned}
(4.28) \qquad \frac{\partial}{\partial x}\left[\left(\mu_1[h] - \mu_1[u_0] \right) h_x \right] &= \frac{\partial}{\partial x}\left[\left(\sum_{r=1}^{N} \rho_r[\mu_1]\varepsilon^r + |\varepsilon|^{N+1} \widetilde{R}_N^{(1)}[\mu_1,\varepsilon] \right) h_x \right] \\
&= \sum_{r=1}^{N}\sum_{i=1}^{r} \frac{\partial}{\partial x}\left(\rho_i[\mu_1]\nabla u_{r-i} \right)\varepsilon^r + |\varepsilon|^{N+1} \widetilde{R}_N^{(2)}[\mu_1,\varepsilon],
\end{aligned}
$$

where

$$
(4.29) \qquad \widetilde{R}_N^{(2)}[\mu_1,\varepsilon] = \frac{\partial}{\partial x}\left[\widetilde{R}_N^{(1)}[\mu_1,\varepsilon]h_x + \frac{1}{|\varepsilon|^{N+1}} \sum_{r=N+1}^{2N}\sum_{i=1}^{r} \rho_i[\mu_1]\nabla u_{r-i}\varepsilon^r \right].
$$

Similarly

$$
(4.30) \quad \frac{\partial}{\partial x}\left[\left(\mu_2[h] - \mu_2[u_0] \right) h_x \right] = \sum_{r=1}^{N}\sum_{i=1}^{r} \frac{\partial}{\partial x}\left(\rho_i[\mu_2]\nabla u_{r-i} \right)\varepsilon^r + |\varepsilon|^{N+1} \widetilde{R}_N^{(2)}[\mu_2,\varepsilon],
$$

where

$$
(4.31) \qquad \widetilde{R}_N^{(2)}[\mu_2,\varepsilon] = \frac{\partial}{\partial x}\left[\widetilde{R}_N^{(1)}[\mu_2,\varepsilon]h_x + \frac{1}{|\varepsilon|^{N+1}} \sum_{r=N+1}^{2N}\sum_{i=1}^{r} \rho_i[\mu_2]\nabla u_{r-i}\varepsilon^r \right].
$$

Combining (4.3), (4.4), (4.7), (4.23), (4.27), (4.28) and (4.30), we then obtain

$$
(4.32) \quad E_\varepsilon(x,t) = |\varepsilon|^{N+1} \left(\bar{R}_N^{(1)}[f,f_1,\varepsilon] + \widetilde{R}_N^{(2)}[\mu_1,\varepsilon] - \int_0^t g(t-s)\, \widetilde{R}_N^{(2)}[\mu_2,\varepsilon] ds \right).
$$

By the functions $u_r \in W_1(M,T)$, $0 \le r \le N$, we obtain from (4.27), (4.29), (4.31) and (4.32) that

$$
(4.33) \qquad \|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le \bar{C}_* |\varepsilon|^{N+1},
$$

where $\bar{C}_*$ is a constant depending only on N, T, f, f_1, μ_1, μ_2, u_r, $0 \le r \le N$.

This completes thee proof of lemma 4.3. □

Now, we estimate $v = u - \sum\limits_{r=0}^{N} u_r \varepsilon^r$.

By multiplying the two sides of (4.22) by v', we verify without difficulty that

$$
\begin{aligned}
(4.34) \qquad S(t) = {} & 2\int_0^t \langle E_\varepsilon(s), v'(s)\rangle ds + \int_0^t \dot{a}_1[v+h](s, v(s), v(s)) ds \\
& -2\int_0^t \left[a_1[v+h](s, h(s), v'(s)) - a_1[h](s, h(s), v'(s))\right] ds \\
& +2\int_0^t d\tau \int_0^\tau g\left(\tau - s\right) a_2[v+h](s, v(s), v'(\tau)) ds \\
& +2\int_0^t \langle f[v+h] - f[h] + \varepsilon\left(f_1[v+h] - f_1[h]\right), v'(s)\rangle ds \\
& +2\int_0^t d\tau \int_0^\tau g\left(\tau - s\right)\left[a_2[v+h](s, h(s), v'(\tau)) - a_2[h](s, h(s), v'(\tau))\right] ds \\
& \equiv \sum_{i=1}^{6} S_i,
\end{aligned}
$$

where

$$
\begin{aligned}
(4.35) \qquad S(t) &= \|v'(t)\|^2 + a_1[v+h](t, v(t), v(t)) + 2\lambda \int_0^t \|v'(s)\|_a^2 \, ds \\
&\geq \|v'(t)\|^2 + \mu_* \|v(t)\|_a^2 + 2\lambda \int_0^t \|v'(s)\|_a^2 \, ds.
\end{aligned}
$$

Put $M_* = (N+2)M$, it is not difficult to prove that the following inequalities hold

$$
\begin{aligned}
(4.36) \qquad S_1 &= 2\int_0^t \langle E_\varepsilon(s), v'(s)\rangle ds \leq T\bar{C}_*^2 \left|\varepsilon\right|^{2N+2} + \int_0^t S(s) ds; \\
S_2 &= \int_0^t \dot{a}_1[v+h](s, v(s), v(s)) ds \leq \gamma_{M_*} \tilde{K}_{M_*}\left(\mu_1\right) \int_0^t \|v(s)\|_a^2 \, ds \\
&\leq \frac{1}{\mu_*} \gamma_{M_*} \tilde{K}_{M_*}\left(\mu_1\right) \int_0^t S(s) ds;
\end{aligned}
$$

$$
\begin{aligned}
S_3 &= -2\int_0^t \left[a_1[v+h](s, h(s), v'(s)) - a_1[h](s, h(s), v'(s))\right] ds \\
&\leq 2(1+4M_*)M_* \tilde{K}_{M_*}\left(\mu_1\right) \int_0^t \|v(s)\|_a \|v'(s)\|_a \, ds \\
&\leq \frac{1}{6} S(t) + \frac{3}{\lambda \mu_*}(1+4M_*)^2 M_*^2 \tilde{K}_{M_*}^2\left(\mu_1\right) \int_0^t S(s) ds;
\end{aligned}
$$

$$\begin{aligned}
S_4 &= 2\int_0^t d\tau \int_0^\tau g(\tau-s)\, a_2[v+h](s,v(s),v'(\tau))ds \\
&\le 2\tilde{K}_{M_*}(\mu_2)\int_0^t d\tau \int_0^\tau |g(\tau-s)|\, \|v(s)\|_a \|v'(\tau)\|_a\, ds \\
&\le \frac{1}{6}S(t) + \frac{3}{\lambda\mu_*}T\tilde{K}^2_{M_*}(\mu_2)\, \|g\|^2_{L^2(0,T^*)}\int_0^t S(s)ds; \\
S_5 &= 2\int_0^t \langle f[v+h]-f[h]+\varepsilon\left(f_1[v+h]-f_1[h]\right), v'(s)\rangle ds \\
&\le 8\left(1+\frac{1}{\sqrt{\mu_*}}\right)(1+2M_*)\left(K_{M_*}(f)+K_{M_*}(f_1)\right)\int_0^t S(s)ds; \\
S_6 &= 2\int_0^t d\tau \int_0^\tau g(\tau-s)\left[a_2[v+h](s,h(s),v'(\tau)) - a_2[h](s,h(s),v'(\tau))\right]ds \\
&\le 2\int_0^t d\tau \int_0^\tau |g(\tau-s)|\, \|\mu_2[v+h](s)-\mu_2[h](s)\|_{C^0([0,1])}\|h(s)\|_a \|v'(\tau)\|_a\, ds \\
&\le 2(1+4M_*)M_*\tilde{K}_{M_*}(\mu_2)\int_0^t d\tau \int_0^\tau |g(\tau-s)|\, \|v(s)\|_a\|v'(\tau)\|_a\, ds \\
&\le \frac{1}{6}S(t) + \frac{3}{\lambda\mu_*}T(1+4M_*)^2M_*^2\tilde{K}^2_{M_*}(\mu_2)\, \|g\|^2_{L^2(0,T^*)}\int_0^t S(s)ds.
\end{aligned}$$

Combining (4.34) and (4.36), we then obtain

$$S(t) \le 2T\bar{C}_*^2\, |\varepsilon|^{2N+2} + 2\tilde{D}_T(M)\int_0^t S(s)ds, \tag{4.37}$$

where

$$\begin{aligned}
\tilde{D}_T(M) =& 1 + \frac{1}{\mu_*}\gamma_{M_*}\tilde{K}_{M_*}(\mu_1) + 8\left(1+\frac{1}{\sqrt{\mu_*}}\right)(1+2M_*)\left(K_{M_*}(f)+K_{M_*}(f_1)\right) \\
&+ \frac{3}{\lambda\mu_*}\Big[(1+4M_*)^2M_*^2\tilde{K}^2_{M_*}(\mu_1) \\
&\qquad + \left(1+(1+4M_*)^2M_*^2\right)T\,\|g\|^2_{L^2(0,T^*)}\tilde{K}^2_{M_*}(\mu_2)\Big].
\end{aligned}$$

By Gronwall's lemma, we obtain from (4.37) that

$$S(t) \le 2T\bar{C}_*^2\, |\varepsilon|^{2N+2}\exp\left(2T\tilde{D}_T(M)\right).$$

Hence

$$\|v\|_{W_1(T)} \le \left(1+\frac{1}{\sqrt{\mu_*}}+\frac{1}{\sqrt{2\lambda}}\right)\bar{C}_*\sqrt{2T}\exp(T\tilde{D}_T(M))\, |\varepsilon|^{N+1},$$

or

$$\left\| u_\varepsilon - \sum_{r=0}^{N} u_r \varepsilon^r \right\|_{W_1(T)} \le C_T \left|\varepsilon\right|^{N+1}. \tag{4.38}$$

Thus, we have the following theorem 4.4.

Theorem 4.4. *Let* (H_1), (H_2), (H_7) *and* (H_8) *hold. Then there exist constants* $M > 0$ *and* $T > 0$ *such that, for every* ε, *with* $|\varepsilon| < 1$, *Prob.* (P_ε) *has a unique weak solution* $u_\varepsilon \in W_1(M, T)$ *satisfying an asymptotic estimation up to order* $N + 1$ *as in* (4.38), *where the functions* u_r, $r = 0, 1, ..., N$ *are weak solutions of Prob.* (P_0), $(\tilde{P}_r)$, $r = 0, 1, ..., N$, *respectively.*

Remark 4.2. Typical examples about asymptotic expansion of solutions in a small parameter can be found in many papers, such as [14, 15, 16]. In the case of many small parameters, there is only partial results, for example, we refer to [17, 24, 25] for the asymptotic expansion of solutions in two or three small parameters.

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