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Hybrid Algorithms for Ky Fan Inequalities and Common Fixed Points of Demicontractive Single-valued and Quasi-nonexpansive Multi-valued Mappings

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Abstract. In this paper, we consider a common solution of three problems in real Hilbert spaces: the Ky Fan inequality problem, the variational inequality problem and the fixed point problem for demicontractive single-valued and quasi-nonexpansive multi-valued mappings. To find the solution we present a new iterative algorithm and prove a strong convergence theorem under mild conditions. Moreover, we provide a numerical example to illustrate the convergence behavior of the proposed iterative method.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. Let $f : H \times H \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$ for all $x \in C$. The classical Ky Fan inequality [7] consists of finding a point x^* in C such that

$$f(x^*, y) \geq 0, \ \ \forall y \in C. \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by $Sol(f, C)$. In fact, the Ky Fan inequality can be formulated as an equilibrium problem. If $f(x, y) = \langle Ax, y - x \rangle$, where $A : C \to H$ is a operator, then problem (1.1) become the following variational inequality problem (shortly, $VI(A, C)$): find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \ \ \forall y \in C. \tag{1.2}$$

The equilibrium problem which was considered as the Ky Fan inequality is very general in the sense that it includes, as special cases, the optimization problem,

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the variational inequality problem, the complementarity problem, the saddle point problem, the Nash equilibrium problem in noncooperative games and the Kakutani fixed point problem, etc., see [1, 4, 5, 9, 10, 18] and the references therein. Recently, algorithms for solving the Ky Fan inequality have been studied extensively.

In 2001, Yamada [27] proved that the sequence $\{x_n\}$ generated by the projected gradient algorithm

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda A x_n), \ \ \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

converges to the unique solution x^* of $VI(A,C)$ under the assumption that A is strongly monotone and Lipschitz continuous, the mapping $P_C(I - \lambda A)$ is strictly contractive over C. If A is monotone and Lipschitz, the projected gradient algorithm (1.3) may not be convergent. In order to deal with this situation, Korpelevich [15] introduced an extragradient algorithm:

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \ \ \forall n \in \mathbb{N}. \end{cases} \tag{1.4}$$

He also proved that the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same solution x^* of $VI(A,C)$ under the assumptions that A is L-Lipschitz and monotone, $\lambda \in (0, \frac{1}{L})$.

In 2008, the extragradient algorithm (1.4) has been extended to Ky Fan inequality problem by Muu et al. [17] as follows:

$$\begin{cases} x_1 \in C, \\ y_n = \underset{w \in C}{\operatorname{argmin}} \left[\lambda f(x_n, w) + \dfrac{1}{2}\|w - x_n\|^2 \right], \\ x_{n+1} = \underset{z \in C}{\operatorname{argmin}} \left[\lambda f(y_n, z) + \dfrac{1}{2}\|z - x_n\|^2 \right], \ \ \forall n \in \mathbb{N}. \end{cases} \tag{1.5}$$

Under assumptions that f is pseudomonotone and Lipschitz-type continuous, the authors showed that the sequence $\{x_n\}$ converges to an element of $Sol(f,C)$.

For obtaining a common element of set of solutions of Ky Fan inequality (1.1) and the set of fixed points of a nonexpansive mapping T in a real Hilbert space H, Anh [3] introduced an iterative algorithm by the modified viscosity approximation method. The sequence $\{x_n\}$ is defined by

$$\begin{cases} x_1 \in C, \\ y_n = \underset{w \in C}{\operatorname{argmin}} \left[\lambda_n f(x_n, w) + \dfrac{1}{2}\|w - x_n\|^2 \right], \\ z_n = \underset{z \in C}{\operatorname{argmin}} \left[\lambda_n f(y_n, z) + \dfrac{1}{2}\|z - x_n\|^2 \right], \\ x_{n+1} = \alpha_n h(x_n) + \beta_n x_n + \gamma_n(\mu T x_n + (1-\mu) z_n), \ \ \forall n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where C is a nonempty closed convex subset of H and h is a contractive mapping of C into itself. The author showed that under certain conditions, the sequence $\{x_n\}$ converges strongly to an element of $Sol(f, C) \cap F(T)$.

Later in 2013, Vahidi et al. [24] introduced an iterative algorithm for finding a common element of the sets of fixed points for nonexpansive multi-valued mappings, strict pseudo-contractive single-valued mappings and the set of solutions of Ky Fan inequality for pseudomonotone and Lipschitz-type continuous bifunctions in Hilbert spaces.

In this paper, motivated by the research described above, we propose a new iterative algorithm for finding a common element of the sets of fixed points for demicontractive single-valued mappings, quasi-nonexpansive multi-valued mappings, the set of solutions of Ky Fan inequality for pseudomonotone and Lipschitz-type continuous bifunctions, and the set of solutions of variational inequality for ϕ-inverse strongly monotone mappings in real Hilbert spaces. We obtain strong convergence theorems for the sequence generated by the proposed algorithm in a real Hilbert space. Our results generalize and improve a number of known results including the results of Anh [3] and Vahidi et al. [24].

2. Preliminaries and Useful Lemmas

In this section, we recall some definitions and results for further use. Let C be a nonempty closed convex subset of a real Hilbert space H. We denote the strong convergence and the weak convergence of the sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is also known in [19] that a Hilbert space H satisfies *Opial's condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y\in C} \|x - y\|.$$

Since C is nonempty closed and convex, $P_C x$ exists and is unique. It is also known that P_C has the following characteristic properties, see [11, 23] for more details.

Lemma 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $P_C : H \to C$ be the metric projection. Then*

(i) *for all $x \in C$, $y \in H$,*

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2;$$

(ii) *$P_C x = y$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \ \ \forall z \in C.$$

Lemma 2.2.([23]) *Let C be a nonempty closed convex subset of a Hilbert space H and let A be a mapping of C into H. Let $u \in C$. Then for $\eta > 0$,*

$$u = P_C(I - \eta A)u \Leftrightarrow u \in VI(A, C).$$

Definition 2.3.([13]) A mapping $A : C \to H$ is called *δ-inverse strongly monotone* if there exists a positive real number δ such that

$$\langle x - y, Ax - Ay\rangle \geq \delta\|Ax - Ay\|^2, \ \ \forall x, y \in C.$$

We now give some concepts of the monotonicity of a bifunction.

Definition 2.4. Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and let $f : H \times H \to \mathbb{R}$ be a bifunction. A bifunction f is said to be:

(i) *strongly monotone* on C if there exists a constant $\alpha > 0$ such that

$$f(x, y) + f(y, x) \leq -\alpha\|x - y\|^2, \ \ \forall x, y \in C;$$

(ii) *monotone* on C if

$$f(x, y) + f(y, x) \leq 0, \ \ \forall x, y \in C;$$

(iii) *pseudomonotone* on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \ \ \forall x, y \in C;$$

(iv) *Lipschitz-type continuous* on C if there exist two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2, \ \ \forall x, y, z \in C.$$

From the definition above we obviously have the following implications: (1) It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), (2) If $f(x, y) = \langle \Phi(x), y - x\rangle$ for a mapping $\Phi : H \to H$. Then the notions of monotonicity of bifunction f collapse to the notions of monotonicity of mapping Φ, respectively. In addition, if mapping Φ is L-Lipschitz on C, *i.e.*, $\|\Phi(x) - \Phi(y)\| \leq L\|x - y\|$ for all $x, y \in C$. Then, f is also Lipschitz-type continuous on C, for example, with constants $L_1 = \frac{L}{2\epsilon}$, $L_2 = \frac{L\epsilon}{2}$, for any $\epsilon > 0$.

Definition 2.5. Let H be a real Hilbert space, and let $f : H \times H \to \mathbb{R}$ be a bifunction. For each $z \in H$, by $\partial f(z, u)$ we denote the *subdifferential* of the function $f(z, \cdot)$ at u, *i.e.*,

$$\partial f(z, u) = \{\xi \in H : f(z, t) - f(z, u) \geq \langle \xi, t - u\rangle, \forall t \in H\}.$$

Definition 2.6. Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The *normal cone* of C at $v \in C$ is defined by

$$N_C(v) = \{z \in H : \langle z, y - v\rangle \leq 0, \forall y \in C\}.$$

Lemma 2.7.([6]) *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $f : H \times H \to \mathbb{R}$ be a bifunction. For each $z \in H$, suppose that $f(z, \cdot)$ is subdifferentiable on C. Then x^* is a solution to the following convex problem:*

$$\min\{f(z, x) : x \in C\}$$

if and only if $0 \in \partial f(z, x^*) + N_C(x^*)$, where $f(z, \cdot)$ denotes the subdifferential of $f(z, \cdot)$ and $N_C(x^*)$ is the normal cone of C at $x^* \in C$.

Lemma 2.8.([2, 17]) *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and let $f : H \times H \to \mathbb{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction. For each $x \in C$, let $f(x, \cdot)$ be convex and subdifferentiable on C. Let $\{x_n\}, \{z_n\}$, and $\{w_n\}$ be the sequences generated by $x_1 \in C$ and by*

$$w_n = \operatorname*{argmin}_{w \in C} \left[\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2\right],$$
$$z_n = \operatorname*{argmin}_{z \in C} \left[\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2\right].$$

Then for each $x^* \in Sol(f, C)$,

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2, \ \forall n \in \mathbb{N}. \tag{2.1}$$

A mapping $h : C \to C$ is a *contraction* if there exists a constant $\eta \in (0, 1)$ such that $\|h(x) - h(y)\| \leq \eta\|x - y\|$ for all $x, y \in C$. Let $T : C \to C$ be a single-valued mapping. An element $x \in C$ is said to be a *fixed point* of T if $x = Tx$. The fixed point set of T is denoted by $F(T) = \{x \in C : x = Tx\}$. A single-valued mapping T is called *strictly pseudononspreading* [20] if there exists $k \in [0, 1)$ such that, for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle,$$

where I denotes the identity mapping. Note that if $k = 0$, a mapping T is called *nonspreading* [14]. As a generalization of the class of strictly pseudononspreading mappings, the class of demicontractive mappings was introduced by Hicks and Kubicek [12] in 1977.

Recall that a single-valued mapping T is said to be *demicontractive* if $F(T) \neq \emptyset$ and there exists $\kappa \in [0, 1)$ such that, for all $x \in C$ and for all $z \in F(T)$,

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \kappa\|x - Tx\|^2.$$

We call κ the *contraction coefficient*. Clearly, strictly pseudononspreading mapping with a nonempty fixed point set is demicontractive.

We now give two examples for the class of demicontractive mappings.

Example 2.9. Let H be the real line and $C = [0, 1]$. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} \dfrac{4}{7}x \sin\left(\dfrac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Obviously, $F(T) = \{0\}$. Also, for all $x \in C$, we have $|Tx - T0|^2 = |Tx|^2 = |\frac{4}{7}x \sin\left(\frac{1}{x}\right)|^2 \leq |\frac{4x}{7}|^2 \leq |x|^2 \leq |x - 0|^2 + k|x - Tx|^2$ for all $k \in [0, 1)$. Therefore, T is demicontractive.

Example 2.10. Let H be the real line and $C = [-1, 1]$. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} \dfrac{9 - x}{10}, & x \in [-1, 0), \\ \dfrac{x + 9}{10}, & x \in [0, 1]. \end{cases}$$

Obviously, $F(T) = \{1\}$ and T is demicontractive.

The following lemma obtained by Suantai and Phuengrattana [22] is useful for our results.

Lemma 2.11. *Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a demicontractive mapping with contraction coefficient κ. Then, the following hold:*

(i) $F(T) = F(P_C(I - \mu(I - T)))$ *for all* $\mu > 0$*;*

(ii) $P_C(I - \mu(I - T))$ *is quasi-nonexpansive, for all* $\mu \in (0, 1 - \kappa]$.

The set C of H is called *proximinal* if for each $x \in H$ there exists $z \in C$ such that

$$\|x - z\| = \inf\{\|x - y\| : y \in C\} = \operatorname{dist}(x, C).$$

It is clear that every nonempty closed convex subset of a real Hilbert space is proximinal. We denote by $CB(C)$ and $KC(C)$ the families of all nonempty closed bounded subsets, and nonempty compact convex subsets of C, respectively. The *Pompeiu-Hausdorff metric* $\mathcal{H}$ on $CB(C)$ is defined by

$$\mathcal{H}(A, B) := \max\left\{\sup_{x \in A} \operatorname{dist}(x, B), \sup_{y \in B} \operatorname{dist}(y, A)\right\}, \ \ \forall A, B \in CB(C).$$

Let $S : C \to CB(C)$ be a multi-valued mapping. An element $x \in C$ is said to be a *fixed point* of S if $x \in Sx$. The fixed point set of S is denoted by $F(S) = \{x \in C : x \in Sx\}$.

Definition 2.12. A multi-valued mapping $S : C \to CB(C)$ is said to

(i) be *nonexpansive* if $\mathcal{H}(Sx, Sy) \leq \|x - y\|$ for all $x, y \in C$;

(ii) be *quasi-nonexpansive* if $F(S) \neq \emptyset$ and $\mathcal{H}(Sx, Sz) \leq \|x - z\|$ for all $x \in C$ and $z \in F(S)$;

(iii) satisfy *condition* (E_μ) if there exists $\mu \geq 1$ such that for each $x, y \in C$,

$$\mathrm{dist}(x, Sy) \leq \mu \mathrm{dist}(x, Sx) + \|x - y\|.$$

We say that S satisfies *condition* (E) whenever S satisfies (E_μ) for some $\mu \geq 1$.

From the above definitions, it is clear that:

(i) if S is nonexpansive, then T satisfies the condition (E_1);

(ii) if C is compact, then S is hemicompact.

We now give an example for the class of quasi-nonexpansiveness multi-valued mapping satisfying the condition (E).

Example 2.13. Let $C = [0, \infty)$ and $S : C \to CB(C)$ be defined by

$$Sx = \left[\frac{x}{4}, \frac{x}{2}\right] \text{ for all } x \in C.$$

Then S is quasi-nonexpansive and satisfies condition (E).

Although the condition (E) implies the quasi-nonexpansiveness for single-valued mappings, but it is not true for multi-valued mappings as the following example.

Example 2.14.([25]) Let $C = [0, \infty)$ and $S : C \to CB(C)$ be defined by

$$Sx = [x, 2x] \text{ for all } x \in C.$$

Then S satisfies condition (E) and is not quasi-nonexpansive.

Notice also that the classes of (multi-valued) quasi-nonexpansive mappings and mappings satisfying condition (E) are different (see Examples 2.15).

Example 2.15.([8]) Let $C = [-1, 1]$ and $S : C \to CB(C)$ be defined by

$$Sx = \begin{cases} \left\{\frac{x}{1+|x|}\sin(\frac{1}{x})\right\} & \text{if } x \neq 0; \\ \{0\} & \text{if } x = 0. \end{cases}$$

Then S is quasi-nonexpansive and does not satisfy condition (E).

Lemma 2.16.([16]) *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists* *a nondecreasing sequence* $\{\tau(n)\} \subset \mathbb{N}$ *such that* $\tau(n) \to \infty$*, and the following properties are satisfied by all (sufficiently large) numbers* $n \in \mathbb{N}$*:*

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \quad t_n \leq t_{\tau(n)+1}.$$

In fact,

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

Lemma 2.17.([23]) *In Hilbert space* H*, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \ \forall x, y \in H.$$

Lemma 2.18.([28]) *Let* H *be a Hilbert space. Let* $x_1, x_2, \ldots, x_N \in H$ *and* $\alpha_1, \alpha_2, \ldots, \alpha_N$ *be real numbers in* $[0, 1]$ *such that* $\sum_{i=1}^{N} \alpha_i = 1$*. Then,*

$$\left\|\sum_{i=1}^{N} \alpha_i x_i\right\|^2 = \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \sum_{1 \leq i,j \leq N} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.19.([26]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers, let* $\{b_n\}$ *be a sequence in* $(0, 1)$ *with* $\sum_{n=1}^{\infty} b_n = \infty$*, let* $\{d_n\}$ *be a sequence of nonnegative real numbers with* $\sum_{n=1}^{\infty} d_n < \infty$*, and let* $\{c_n\}$ *be a sequence of real numbers with* $\limsup_{n\to\infty} c_n \leq 0$*. Suppose that the following inequality holds:*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n + d_n, \ \ \forall n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} a_n = 0$*.*

3. Main Results

In this section, we show strong convergence theorems for the sequence generated by the hybrid algorithm (3.1) based on extragradient algorithm which solve the problem of finding of four sets, *i.e.*, $F(T)$, $F(S)$, $Sol(f, C)$, and $VI(B, C)$.

Now, let C be a nonempty, closed and convex subset of a real Hilbert space H and $f : H \times H \to \mathbb{R}$ be a bifunction such that $f(x, x) = 0$, for all $x \in C$. In order to find a point in $F(T) \cap F(S) \cap Sol(f, C) \cap VI(B, C) \neq \emptyset$, we make use of the following blanket assumptions:

Assumptions $\mathcal{A}$

(A1) f is monotone on C;

(A2) F is Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$;

(A3) $f(x, \cdot)$ is convex and subdifferentiable on C, for all $x \in C$;

(A4) f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

We are now in a position to prove our main results.

Theorem 3.1. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let f be a bifunction satisfying assumptions $\mathcal{A}$ on C, $T : C \to C$ be a demicontractive single-valued mapping with contraction coefficient κ, $S : C \to KC(C)$ be a quasi-nonexpansive multi-valued mapping satisfying the condition (E), and $B : C \to H$ be a δ-inverse strongly monotone mapping. Assume that $\mathcal{F} = F(T) \cap F(S) \cap Sol(f, C) \cap VI(B, C) \neq \emptyset$ and $Sp = \{p\}$ for all $p \in \mathcal{F}$. Let $h : C \to C$ be a k-contraction. For $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ be sequences generated by*

$$(3.1)\quad \begin{cases} w_n = \underset{w \in C}{\operatorname{argmin}} \left[\lambda_n f(x_n, w) + \dfrac{1}{2}\|w - x_n\|^2 \right], \\ z_n = \underset{z \in C}{\operatorname{argmin}} \left[\lambda_n f(w_n, z) + \dfrac{1}{2}\|z - x_n\|^2 \right], \\ y_n = \alpha_n z_n + \beta_n u_n + \gamma_n P_C(I - \mu_n(I - T)) z_n + \zeta_n P_C(I - \eta_n B) z_n, \\ x_{n+1} = \sigma_n h(x_n) + (1 - \sigma_n) y_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $u_n \in S z_n$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\zeta_n\}$, $\{\sigma_n\}$, $\{\mu_n\}$, $\{\eta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(C1) $\{\sigma_n\} \subset (0, 1)$, $\lim_{n\to\infty} \sigma_n = 0$, $\sum_{n=1}^{\infty} \sigma_n = \infty$;

(C2) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$;

(C3) $\mu_n \in (0, 1 - \kappa]$ *with* $\lim_{n\to\infty} \mu_n = 0$;

(C4) $\eta_n \in [d, e]$ *for some* $d, e \in (0, 2\delta)$ *and for all* $n \in \mathbb{N}$;

(C5) $0 < a \le \alpha_n, \beta_n, \gamma_n, \zeta_n \le b < 1$ *and* $\alpha_n + \beta_n + \gamma_n + \zeta_n = 1$ *for all* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\langle q - h(q), x - q \rangle \ge 0, \ \ \forall x \in \mathcal{F}.$$

Proof. Let $Q = P_{\mathcal{F}}$ and it easy to see that Qh is contraction. By the Banach contraction principle, there exists $q \in \mathcal{F}$ such that $q = (Qh)(q)$. Applying Lemma 2.8, we have

$$(3.2)\quad \|z_n - q\|^2 \le \|x_n - q\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2.$$

This implies that

$$(3.3)\quad \|z_n - q\| \le \|x_n - q\|.$$

Since S is quasi-nonexpansive and $Sq = \{q\}$, by (3.3), we have

$$\|u_n - q\| = \operatorname{dist}(u_n, Sq) \leq \mathcal{H}(Sz_n, Sq) \leq \|z_n - q\| \leq \|x_n - q\|. \tag{3.4}$$

By Lemma 2.11(ii), $P_C(I-\mu_n(I-T))$ is quasi-nonexpansive for all $n \in \mathbb{N}$. It implies by $P_C(I - \mu_n(I - T))q = q$ and (3.3) that

$$\|P_C(I - \mu_n(I - T))z_n - q\| \leq \|z_n - q\| \leq \|x_n - q\|. \tag{3.5}$$

Let $x, y \in C$. Since B is δ-inverse strongly monotone, by condition $(C4)$, we have

$$\begin{aligned}
\|P_C(I - \eta_n B)x - P_C(I - \eta_n B)y\|^2 &\leq \|(I - \eta_n B)x - (I - \eta_n B)y\|^2 \\
&= \|x - y\|^2 - 2\eta_n\langle x - y, Bx - By\rangle \\
&\quad + \eta_n^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - 2\eta_n\delta\|Bx - By\|^2 \\
&\quad + \eta_n^2\|Bx - By\|^2 \\
&= \|x - y\|^2 - \eta_n(2\delta - \eta_n)\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - d(2\delta - e)\|Bx - By\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

This shows that $P_C(I-\eta_n B)$ is nonexpansive for all $n \in \mathbb{N}$. Thus, by $P_C(I-\eta_n B)q = q$ and (3.3), we have

$$\|P_C(I - \eta_n B)z_n - q\| \leq \|z_n - q\| \leq \|x_n - q\|. \tag{3.6}$$

From (3.3)-(3.6), we get that

$$\begin{aligned}
\|y_n - q\| &= \|\alpha_n z_n + \beta_n u_n + \gamma_n P_C(I - \mu_n(I - T))z_n + \zeta_n P_C(I - \eta_n B)z_n - q\| \\
&\leq \alpha_n\|z_n - q\| + \beta_n\|u_n - q\| + \gamma_n\|P_C(I - \mu_n(I - T))z_n - q\| \\
&\quad + \zeta_n\|P_C(I - \eta_n B)z_n - q\| \\
&\leq (\alpha_n + \beta_n + \gamma_n + \zeta_n)\|x_n - q\| \\
&= \|x_n - q\|. \qquad (3.7)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\sigma_n h(x_n) + (1 - \sigma_n)y_n - q\| \\
&\leq \sigma_n\|h(x_n) - q\| + (1 - \sigma_n)\|y_n - q\| \\
&\leq \sigma_n\big(\|h(x_n) - h(q)\| + \|h(q) - q\|\big) + (1 - \sigma_n)\|x_n - q\| \\
&\leq \sigma_n\big(\|h(x_n) - h(q)\| + \|h(q) - q\|\big) + (1 - \sigma_n)\|x_n - q\| \\
&\leq \sigma_n k\|x_n - q\| + \sigma_n\|h(q) - q\| + (1 - \sigma_n)\|x_n - q\| \\
&= (1 - \sigma_n(1 - k))\|x_n - q\| + \sigma_n\|h(q) - q\| \\
&\leq \max\left\{\|x_n - q\|, \frac{\|h(q) - q\|}{1 - k}\right\}.
\end{aligned}$$

By induction, we get

$$\|x_n - q\| \le \max\left\{\|x_1 - q\|, \frac{\|h(q) - q\|}{1-k}\right\}, \ \ \forall n \in \mathbb{N}.$$

This implies that $\{x_n\}$ is bounded, and we also obtain that $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{h(x_n)\}$ are bounded.

By Lemma 2.18, (3.1), (3.2), and (3.3), we obtain that

$$\begin{aligned}
\|y_n - q\|^2 &\le \alpha_n\|z_n - q\|^2 + \beta_n\|u_n - q\|^2 + \gamma_n\|P_C(I - \mu_n(I-T))z_n - q\|^2 \\
&\quad + \zeta_n\|P_C(I - \eta_n B)z_n - q\|^2 - \alpha_n\beta_n\|z_n - u_n\|^2 \\
&\quad - \alpha_n\gamma_n\|z_n - P_C(I - \mu_n(I-T))z_n\|^2 \\
&\quad - \alpha_n\zeta_n\|z_n - P_C(I - \eta_n B)z_n\| - \beta_n\gamma_n\|u_n - P_C(I - \mu_n(I-T))z_n\|^2 \\
&\quad - \beta_n\zeta_n\|u_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - \gamma_n\zeta_n\|P_C(I - \mu_n(I-T))z_n - P_C(I - \eta_n B)z_n\|^2 \\
&\le \alpha_n\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\|z_n - q\|^2 \\
&\quad + \zeta_n\|z_n - q\|^2 - \alpha_n\beta_n\|z_n - u_n\|^2 \\
&\quad - \alpha_n\gamma_n\|z_n - P_C(I - \mu_n(I-T))z_n\|^2 - \alpha_n\zeta_n\|z_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - \beta_n\gamma_n\|u_n - P_C(I - \mu_n(I-T))z_n\|^2 - \beta_n\zeta_n\|u_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - \gamma_n\zeta_n\|P_C(I - \mu_n(I-T))z_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - \alpha_n(1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - \alpha_n(1 - 2\lambda_n c_2)\|w_n - z_n\|^2 \\
&\le \|x_n - q\|^2 - \alpha_n\beta_n\|z_n - u_n\|^2 - \alpha_n\gamma_n\|z_n - P_C(I - \mu_n(I-T))z_n\|^2 \\
&\quad - \alpha_n\zeta_n\|z_n - P_C(I - \eta_n B)z_n\|^2 - \beta_n\gamma_n\|u_n - P_C(I - \mu_n(I-T))z_n\|^2 \\
&\quad - \beta_n\zeta_n\|u_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - \gamma_n\zeta_n\|P_C(I - \mu_n(I-T))z_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - \alpha_n(1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - \alpha_n(1 - 2\lambda_n c_2)\|w_n - z_n\|^2. \qquad (3.8)
\end{aligned}$$

Consequently, utilizing (3.8), we conclude that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \sigma_n\|h(x_n) - q\|^2 + (1-\sigma_n)\|y_n - q\|^2 \\
&\le \sigma_n\|h(x_n) - q\|^2 + (1-\sigma_n)\|x_n - q\|^2 - (1-\sigma_n)\alpha_n\beta_n\|z_n - u_n\|^2 \\
&\quad - (1-\sigma_n)\alpha_n\gamma_n\|z_n - P_C(I - \mu_n(I-T))z_n\|^2 \\
&\quad - (1-\sigma_n)\alpha_n\zeta_n\|z_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - (1-\sigma_n)\beta_n\gamma_n\|u_n - P_C(I - \mu_n(I-T))z_n\|^2 \\
&\quad - (1-\sigma_n)\beta_n\zeta_n\|u_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - (1-\sigma_n)\gamma_n\zeta_n\|P_C(I - \mu_n(I-T))z_n - P_C(I - \eta_n B)z_n\|^2 \\
&\quad - (1-\sigma_n)\alpha_n(1 - 2\lambda_n c_1)\|x_n - w_n\|^2 \\
&\quad - (1-\sigma_n)\alpha_n(1 - 2\lambda_n c_2)\|w_n - z_n\|^2. \qquad (3.9)
\end{aligned}$$

Therefore, we have

$$(1-\sigma_n)\alpha_n\beta_n\|z_n-u_n\|^2 \le \|x_n-q\|^2-\|x_{n+1}-q\|^2+\sigma_n\|h(x_n)-q\|^2. \tag{3.10}$$

In order to prove that $x_n \to q$ as $n\to\infty$, we have consider the following two cases.

Case 1. Suppose that there exists n_0 such that $\{\|x_n-q\|\}$ is nonincreasing, for all $n\ge n_0$. Boundedness of $\{\|x_n-q\|\}$ implies that $\{\|x_n-q\|\}$ is convergent. Since $\{h(x_n)\}$ is bounded and $\sigma_n\to 0$ as $n\to\infty$, from (3.10) and condition $(C5)$, we obtain that

$$\lim_{n\to\infty}\|z_n-u_n\|=0. \tag{3.11}$$

By (3.9), we have

$$\begin{aligned}(1-\sigma_n)\alpha_n\gamma_n\|z_n-P_C(I-\mu_n(I-T))z_n\|^2 \le \|x_n-q\|^2-\|x_{n+1}-q\|^2\\ +\sigma_n\|h(x_n)-q\|^2.\end{aligned}$$

This implies by conditions $(C1)$ and $(C5)$ that

$$\lim_{n\to\infty}\|z_n-P_C(I-\mu_n(I-T))z_n\|=0. \tag{3.12}$$

By similar argument we can obtain that

$$\lim_{n\to\infty}\|z_n-P_C(I-\eta_nB)z_n\|=0, \tag{3.13}$$

$$\lim_{n\to\infty}\|x_n-w_n\|=0,\ \ \lim_{n\to\infty}\|w_n-z_n\|=0. \tag{3.14}$$

Also, by (3.14), we have

$$\|x_n-z_n\|\le\|x_n-w_n\|+\|w_n-z_n\|\to 0,\ \text{ as } n\to\infty. \tag{3.15}$$

Next, we will show that

$$\limsup_{n\to\infty}\langle h(q)-q,x_n-q\rangle\le 0$$

where $q=Qh(q)$. To show this inequality, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle h(q)-q,x_n-q\rangle=\lim_{i\to\infty}\langle h(q)-q,x_{n_i}-q\rangle.$$

Without loss of generality, we may assume that $x_{n_i}\rightharpoonup x^*$ as $i\to\infty$ where $x^*\in C$. Since $\|x_{n_i}-z_{n_i}\|\to 0$ as $i\to\infty$, we have $z_{n_i}\rightharpoonup x^*$. We will show that $x^*\in\mathcal{F}$. Assume $x^*\notin F(T)$. From Lemma 2.11(i), we have that $x^*\in F(P_C(I-\mu_{n_i}(I-T)))$ for all $i\in\mathbb{N}$. That is $x^*\ne P_C(I-\mu_{n_i}(I-T))x^*$. By Opial's property, condition

$(C3)$, and (3.12), we have

$$\begin{aligned}
\liminf_{i\to\infty}\|z_{n_i}-x^*\| &< \liminf_{i\to\infty}\|z_{n_i}-P_C(I-\mu_{n_i}(I-T))x^*\|\\
&= \liminf_{i\to\infty}(\|z_{n_i}-P_C(I-\mu_{n_i}(I-T))z_{n_i}\|\\
&\quad+\|P_C(I-\mu_{n_i}(I-T))z_{n_i}-P_C(I-\mu_{n_i}(I-T))x^*\|)\\
&\leq \liminf_{i\to\infty}(\|z_{n_i}-P_C(I-\mu_{n_i}(I-T))z_{n_i}\|\\
&\quad+\|z_{n_i}-x^*\|+\mu_{n_i}\|(I-T)z_{n_i}-(I-T)x^*\|)\\
&\leq \liminf_{i\to\infty}\|z_{n_i}-x^*\|.
\end{aligned}$$

This is a contradiction. Then $x^*\in F(T)$.

Since Sx^* is compact and convex, for all $i\in\mathbb{N}$, we can choose $q_{n_i}\in Sx^*$ such that $\|z_{n_i}-q_{n_i}\|=\text{dist}(z_{n_i},Sx^*)$ and the sequence $\{q_{n_i}\}$ has a convergent subsequence $\{q_{n_k}\}$ with $\lim_{k\to\infty}q_{n_k}=q\in Sx^*$. By condition (E), there exists $\mu\geq 1$ such that

$$\text{dist}(z_{n_i},Sx^*)\leq \mu\text{dist}(z_{n_i},Sz_{n_i})+\|z_{n_i}-x^*\|.$$

Suppose that $q\neq x^*$. Since $z_{n_i}\rightharpoonup x^*$, it follows by the Opial's condition and (3.11) that

$$\begin{aligned}
\limsup_{k\to\infty}\|z_{n_k}-x^*\| &< \limsup_{k\to\infty}\|z_{n_k}-q\|\\
&\leq \limsup_{k\to\infty}(\|z_{n_k}-q_{n_k}\|+\|q_{n_k}-q\|)\\
&= \limsup_{k\to\infty}(\text{dist}(z_{n_k},Sx^*)+\|q_{n_k}-q\|)\\
&\leq \limsup_{k\to\infty}(\mu\text{dist}(z_{n_k},Sz_{n_k})+\|z_{n_k}-x^*\|+\|q_{n_k}-q\|)\\
&\leq \limsup_{k\to\infty}(\mu\|z_{n_k}-u_{n_k}\|+\|z_{n_k}-x^*\|+\|q_{n_k}-q\|)\\
&= \limsup_{k\to\infty}\|z_{n_k}-x^*\|.
\end{aligned}$$

This is a contradiction. Then $x^*\in F(S)$.

Assume $x^*\notin VI(B,C)$. From Lemma 2.2, we have that $x^*\notin F(P_C(I-\eta_nB))$ for all $n\in\mathbb{N}$. That is $x^*\neq P_C(I-\eta_nB)x^*$. Now, since $z_{n_i}\rightharpoonup x^*$, it follows by (3.13) and Opial's property that

$$\begin{aligned}
\liminf_{i\to\infty}\|z_{n_i}-x^*\| &< \liminf_{i\to\infty}\|z_{n_i}-P_C(I-\eta_{n_i}B)x^*\|\\
&= \liminf_{i\to\infty}(\|z_{n_i}-P_C(I-\eta_{n_i}B)z_{n_i}\|\\
&\quad+\|P_C(I-\eta_{n_i}B)z_{n_i}-P_C(I-\eta_{n_i}B)x^*\|)\\
&\leq \liminf_{i\to\infty}(\|z_{n_i}-P_C(I-\eta_{n_i}B)z_{n_i}\|+\|z_{n_i}-x^*\|)\\
&= \liminf_{i\to\infty}\|z_{n_i}-x^*\|.
\end{aligned}$$

This is a contradiction. Then $x^* \in VI(B, C)$.

It follows from Lemma 2.7 and $f(x, \cdot)$ is convex on C for each $x \in C$, we see that

$$w_n = \operatorname*{argmin}_{y \in C} \left[\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \right]$$

if and only if

$$0 \in \partial \left(\lambda_n f(x_n, w_n) + \frac{1}{2}\|w_n - x_n\|^2 \right) + N_C(w_n),$$

where $N_C(w_n)$ is the normal cone of C at $w_n \in C$. Then there exists $v \in \partial f(x_n, w_n)$ and $u_n \in N_C(w_n)$ such that

$$0 = \lambda_n v + w_n - x_n + u_n.$$

Using successively the definition of the normal cone to C at w_n and the subdifferential of the convex function $f(x_n, \cdot)$ at w_n, we can write the following two inequalities

$$\langle w_n - x_n, y - w_n \rangle \geq \lambda_n \langle v, w_n - y \rangle, \;\; y \in C,$$

and

$$f(x_n, y) - f(x_n, w_n) \geq \langle v, y - w_n \rangle, \;\; y \in C$$

Thus, we have

$$\lambda_n (f(x_n, y) - f(x_n, w_n)) \geq \langle w_n - x_n, w_n - y \rangle, \;\; y \in C.$$

Hence

$$f(x_{n_i}, y) - f(x_{n_i}, w_{n_i}) \geq \frac{1}{\lambda_{n_i}} \langle w_{n_i} - x_{n_i}, w_{n_i} - y \rangle, \;\; y \in C. \tag{3.16}$$

Since $\lim_{i \to \infty} \|x_{n_i} - w_{n_i}\| = 0$, we have $w_{n_i} \rightharpoonup x^*$. Passing to the limit in the inequality (3.16) as $i \to \infty$ and using the hypothesis $(A4)$ and $(C2)$, we obtain $f(x^*, y) \geq 0$ for all $y \in C$. This implies that $x^* \in Sol(f, C)$ and hence $x^* \in \mathcal{F}$. Since $q = (Qh)(q)$ and $x^* \in \mathcal{F}$, it follows that

$$\limsup_{n \to \infty} \langle h(q) - q, x_n - q \rangle = \lim_{i \to \infty} \langle h(q) - q, x_{n_i} - q \rangle = \langle h(q) - q, x^* - q \rangle \leq 0.$$

By using Lemma 2.17 and (3.7), we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\sigma_n h(x_n) + (1-\sigma_n)y_n - q\|^2 \\
&\le \|(1-\sigma_n)(y_n-q)\|^2 + 2\sigma_n\langle h(x_n)-q, x_{n+1}-q\rangle \\
&\le (1-\sigma_n)^2\|y_n-q\|^2 + 2\sigma_n\langle h(x_n)-h(q), x_{n+1}-q\rangle \\
&\quad + 2\sigma_n\langle h(q)-q, x_{n+1}-q\rangle \\
&\le (1-\sigma_n)^2\|x_n-q\|^2 + 2\sigma_n k\|x_n-q\|\|x_{n+1}-q\| \\
&\quad + 2\sigma_n\langle h(q)-q, x_{n+1}-q\rangle \\
&\le (1-\sigma_n)^2\|x_n-q\|^2 + \sigma_n k(\|x_n-q\|^2 + \|x_{n+1}-q\|^2) \\
&\quad + 2\sigma_n\langle h(q)-q, x_{n+1}-q\rangle \\
&= ((1-\sigma_n)^2 + \sigma_n k)\|x_n-q\|^2 + \sigma_n k\|x_{n+1}-q\|^2 \\
&\quad + 2\sigma_n\langle h(q)-q, x_{n+1}-q\rangle \\
&= (1-\sigma_n k - 2\sigma_n(1-k) + \sigma_n^2)\|x_n-q\|^2 + \sigma_n k\|x_{n+1}-q\|^2 \\
&\quad + 2\sigma_n\langle h(q)-q, x_{n+1}-q\rangle.
\end{aligned}$$

This implies that

$$\|x_{n+1}-q\|^2 \le \left(1 - \frac{2(1-k)\sigma_n - \sigma_n^2}{1-\sigma_n k}\right)\|x_n-q\|^2 + \frac{2\sigma_n}{1-\sigma_n k}\langle h(q)-q, x_{n+1}-q\rangle.$$

Putting $b_n = \frac{2(1-k)\sigma_n - \sigma_n^2}{1-\sigma_n k}$ and $c_n = \frac{2}{2(1-k)-\sigma_n}\langle h(q)-q, x_{n+1}-q\rangle$, we have $\sum_{n=1}^{\infty} b_n = \infty$ and $\limsup_{n\to\infty} c_n \le 0$. Hence, by Lemma 2.19, we conclude the the sequence $\{x_n\}$ converge strongly to q.

Case 2. Assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\|x_{n_i} - q\| < \|x_{n_{i+1}} - q\|,$$

for all $i \in \mathbb{N}$. In this case from Lemma 2.16, there exists a nondecreasing sequence $\{\tau(n)\}$ of $\mathbb{N}$ for all $n \ge n_0$, for some n_0 large enough, such that $\tau(n) \to \infty$ as $n \to \infty$, and the following inequalities hold for all $n \ge n_0$,

$$\|x_{\tau(n)} - q\| < \|x_{\tau(n)+1} - q\|, \quad \|x_n - q\| < \|x_{\tau(n)+1} - q\|.$$

From (3.10), we have $\lim_{n\to\infty}\|z_{\tau(n)} - u_{\tau(n)}\| = 0$, and similarly we obtain

$$\lim_{n\to\infty}\|z_{\tau(n)} - P_C(I - \mu_{\tau(n)}(I-T))z_{\tau(n)}\| = 0,$$
$$\lim_{n\to\infty}\|z_{\tau(n)} - P_C(I - \eta_{\tau(n)}B)z_{\tau(n)}\| = 0,$$
$$\lim_{n\to\infty}\|x_{\tau(n)} - w_{\tau(n)}\| = 0, \quad \lim_{n\to\infty}\|w_{\tau(n)} - z_{\tau(n)}\| = 0.$$

Following an argument similar to that in *Case 1*, we have

$$\lim_{n\to\infty}\|x_{\tau(n)} - q\| = 0, \quad \lim_{n\to\infty}\|x_{\tau(n)+1} - q\| = 0.$$

Thus, by Lemma 2.16, we have

$$0 \le \|x_n - q\| \le \max\left\{\|x_{\tau(n)} - q\|, \|x_n - q\|\right\} \le \|x_{\tau(n)+1} - q\|.$$

Therefore, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$. □

Recall that a multi-valued mapping $S : C \subseteq H \to CB(C)$ is said to satisfy *Condition (A)* if $\|x - p\| = \operatorname{dist}(x, Sp)$ for all $x \in H$ and $p \in F(S)$; see [21]. We see that S satisfies Condition (A) if and only if $Sp = \{p\}$ for all $p \in F(S)$. Then the following result can be obtained from Theorem 3.1 immediately.

Theorem 3.2. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let f be a bifunction satisfying assumptions $\mathcal{A}$ on C, $T : C \to C$ be a demicontractive single-valued mapping with contraction coefficient κ, $S : C \to KC(C)$ be a quasi-nonexpansive multi-valued mapping satisfying the condition (E), and $B : C \to H$ be a δ-inverse strongly monotone mapping. Assume that $\mathcal{F} = F(T) \cap F(S) \cap Sol(f, C) \cap VI(B, C) \neq \emptyset$ and S satisfies Condition (A). Let $h : C \to C$ be a k-contraction. For $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ be generated by* (3.1)*, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\zeta_n\}$, $\{\sigma_n\}$, $\{\mu_n\}$, $\{\eta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:*

(C1) $\{\sigma_n\} \subset (0,1)$, $\lim_{n\to\infty} \sigma_n = 0$, $\sum_{n=1}^{\infty} \sigma_n = \infty$;

(C2) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$;

(C3) $\mu_n \in (0, 1-\kappa]$ *with* $\lim_{n\to\infty} \mu_n = 0$;

(C4) $\eta_n \in [d, e]$ *for some* $d, e \in (0, 2\delta)$ *and for all* $n \in \mathbb{N}$;

(C5) $0 < a \le \alpha_n, \beta_n, \gamma_n, \zeta_n \le b < 1$ *and* $\alpha_n + \beta_n + \gamma_n + \zeta_n = 1$ *for all* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\langle q - h(q), x - q\rangle \ge 0, \ \ \forall x \in \mathcal{F}.$$

Remark 3.3.

(1) Theorems 3.1 and 3.2 extends based on the work of Anh [3] and Vahidi et al. [24], that is, we present a hybrid algorithm for finding a common element of the sets of fixed points for demicontractive single-valued mappings, quasi-nonexpansive multi-valued mappings, the set of solutions of an equilibrium problem for a pseudomonotone, Lipschitz-type continuous bifunctions and variational inequality for ϕ-inverse strongly monotone mappings in real Hilbert spaces.

(2) It is know that the class of demicontractive single-valued mappings contains the classes of nonexpansive single-valued mappings, nonspreading single-valued mappings, quasi-nonexpansive single-valued mappings, and strictly pseudononspreading single-valued mappings. Thus, Theorems 3.1 and 3.2 can be applied to these classes of mappings.

4. Application to Variational Inequalities

In this section, we discuss about an application of Theorem 3.1 to finding a common element of the set of fixed points for demicontractive single-valued mappings and quasi-nonexpansive multi-valued mappings and the set of solutions of variational inequalities for ϕ-inverse strongly monotone and monotone Lipschitz-type continuous mappings.

We consider the particular Ky Fan inequality, corresponding to the bifunction f, defined by $f(x,y) = \langle Ax, y-x\rangle$ for all $x, y \in C$ with $A : C \to H$. Then, the solution w_n in algorithm (3.1) can be expressed as

$$\begin{aligned}
w_n &= \operatorname*{argmin}_{w\in C}\left[\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2\right]\\
&= \operatorname*{argmin}_{w\in C}\left[\lambda_n \langle Ax_n, w - x_n\rangle + \frac{1}{2}\|w - x_n\|^2\right]\\
&= \operatorname*{argmin}_{w\in C}\left[\frac{1}{2}\|w - (x_n - \lambda_n Ax_n)\|^2 - \frac{\lambda_n^2}{2}\|Ax_n\|^2\right]\\
&= \operatorname*{argmin}_{w\in C}\left[\frac{1}{2}\|w - (x_n - \lambda_n Ax_n)\|^2\right]\\
&= P_C(x_n - \lambda_n Ax_n).
\end{aligned}$$

Also, the solution z_n can be expressed as

$$\begin{aligned}
z_n &= \operatorname*{argmin}_{z\in C}\left[\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2\right]\\
&= \operatorname*{argmin}_{z\in C}\left[\lambda_n \langle Aw_n, z - w_n\rangle + \frac{1}{2}\|z - x_n\|^2\right]\\
&= \operatorname*{argmin}_{z\in C}\left[\frac{1}{2}\|z - (x_n - \lambda_n Aw_n)\|^2 - \frac{\lambda_n^2}{2}\|Aw_n\|^2 - \lambda_n\langle Aw_n, w_n - x_n\rangle\right]\\
&= \operatorname*{argmin}_{z\in C}\left[\frac{1}{2}\|z - (x_n - \lambda_n Aw_n)\|^2\right]\\
&= P_C(x_n - \lambda_n Aw_n).
\end{aligned}$$

Let A be L-Lipschitz-type continuous on C, that is $\|Ax - Ay\| \le L\|x - y\|$ for all $x, y \in C$. Then, for $x, y, z \in C$, we have

$$\begin{aligned}
f(x,y) + f(y,z) - f(x,z) &= -\langle Ay - Ax, y - z\rangle\\
&\ge -\|Ax - Ay\|\|y - z\|\\
&\ge -L\|x - y\|\|y - z\|\\
&\ge -\frac{L}{2}\|x - y\|^2 - \frac{L}{2}\|y - z\|^2.
\end{aligned}$$

Therefore, f is Lipschitz-type continuous on C with $c_1 = c_2 = \frac{L}{2}$.

Now, using Theorem 3.1, we obtain the following strong convergence theorem for finding a common element of the set of common fixed points of a quasi-nonexpansive multi-valued mapping and a demicontractive single-valued mapping and the solution set of two variational inequalities.

Theorem 4.1. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $A : C \to H$ be a monotone and L-Lipschitz-type continuous function, $T : C \to C$ be a demicontractive single-valued mapping with contraction coefficient κ, $S : C \to KC(C)$ be a quasi-nonexpansive multi-valued mapping satisfying the condition (E), and $B : C \to H$ be a δ-inverse strongly monotone mapping. Assume that $\mathcal{F} = F(T) \cap F(S) \cap VI(A, C) \cap VI(B, C) \neq \emptyset$ and $Sp = \{p\}$ for all $p \in \mathcal{F}$. Let $h : C \to C$ be a k-contraction. For $x_1 \in C$, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ be sequences generated by*

$$\begin{cases} w_n = P_C(x_n - \lambda_n A x_n), \\ z_n = P_C(x_n - \lambda_n A w_n), \\ y_n = \alpha_n z_n + \beta_n u_n + \gamma_n P_C(I - \mu_n(I - T))z_n + \zeta_n P_C(I - \eta_n B)z_n, \\ x_{n+1} = \sigma_n h(x_n) + (1 - \sigma_n)y_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $u_n \in S z_n$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\zeta_n\}$, $\{\sigma_n\}$, $\{\mu_n\}$, $\{\eta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(C1) $\{\sigma_n\} \subset (0, 1)$, $\lim_{n\to\infty} \sigma_n = 0$, $\sum_{n=1}^{\infty} \sigma_n = \infty$*;*

(C2) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$*;*

(C3) $\mu_n \in (0, 1 - \kappa]$ *with* $\lim_{n\to\infty} \mu_n = 0$*;*

(C4) $\eta_n \in [d, e]$ *for some* $d, e \in (0, 2\delta)$ *and for all* $n \in \mathbb{N}$*;*

(C5) $0 < a \leq \alpha_n, \beta_n, \gamma_n, \zeta_n \leq b < 1$ *and* $\alpha_n + \beta_n + \gamma_n + \zeta_n = 1$ *for all* $n \in \mathbb{N}$.

Then the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\langle q - h(q), x - q\rangle \geq 0, \ \ \forall x \in \mathcal{F}.$$

5. Numerical Example

In this section, we give an example which shows numerical experiment for supporting our main results.

Example 5.1. Let H be a real line with the Euclidean norm and $C = [0, 10]$. For all $x \in C$, we define mappings T, S, B, h on C as follows:

$$Tx = \begin{cases} \frac{4}{7}x\sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}, \ \ Sx = \left[\frac{x}{4}, \frac{x}{2}\right], \ \ Bx = \frac{x}{15}, \ \ hx = \frac{x}{2}.$$

For each $x, y \in C$, define the bifunction f by $f(x, y) = \langle Ax, y - x\rangle$, where $Ax = \frac{x}{5}$. Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ be generated by (3.1), where $u_n = \frac{x_n}{4}$, $\alpha_n = \frac{2n}{5n+1}$,

$\beta_n = \frac{n}{10n+3}$, $\gamma_n = \frac{3n}{50n+1}$, $\zeta_n = 1 - \frac{2n}{5n+1} - \frac{n}{10n+3} - \frac{3n}{50n+1}$, $\sigma_n = \frac{1}{n+2}$, $\mu_n = \frac{1}{n+3}$, $\eta_n = 4$, and $\lambda_n = 2$. It can be observed that all the assumptions of Theorem 3.1 are satisfied and $F(T) \cap F(S) \cap Sol(f, C) \cap VI(B, C) = \{0\}$. By using SciLab, we compute the iterates of (3.1) for the initial point $x_1 = 9$. The numerical experiment's results of our iteration for approximating the point 0 are given in Table 1.

Table 1: Numerical results of Example 5.1 for the algorithm (3.1)

n	x_n	w_n	z_n	y_n	$\lvert x_n - x_{n-1}\rvert$
1	9.0000000	5.4000000	6.8400000	5.4263101	-
2	5.1175401	3.0705240	3.8893305	3.1146764	3.8825e+00
3	2.9756998	1.7854199	2.2615319	1.8204432	2.1418e+00
4	1.7539246	1.0523547	1.3329827	1.0770736	1.2218e+00
5	1.0437217	0.6262330	0.7932285	0.6427364	7.1020e-01
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	0.0005792	0.0003475	0.0004402	0.0003575	3.6920e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
31	0.0000027	0.0000016	0.0000020	0.0000017	1.6776e-06
32	0.0000016	0.0000010	0.0000012	0.0000010	1.0298e-06
33	0.0000010	0.0000006	0.0000008	0.0000006	6.3427e-07
34	0.0000006	0.0000004	0.0000005	0.0000004	3.8953e-07
35	0.0000004	0.0000002	0.0000003	0.0000002	2.3928e-07

Remark 5.2. Table 1 shows that the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ converge to a unique point 0, where $\{0\} = F(T) \cap F(S) \cap Sol(f, C) \cap VI(B, C)$.

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