KYUNGPOOK Math. J. 59(2019), 689-702
https://doi.org/10.5666/KMJ.2019.59.4.689
pISSN 1225-6951 eISSN 0454-8124

On the Stability of a Higher Functional Equation in Banach Algebras

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be real (or complex) algebras. We investigate the stability of a sequence $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ of mappings from $\mathcal{A}$ into $\mathcal{B}$ satisfying the higher functional equation:

$$f_n(x+y+zw) = f_n(x) + f_n(y) + \sum_{\substack{i+j=n \\ i\le j}} [f_i(z)f_j(w) + c_{ij} f_i(w) f_j(z)]$$

for each $n = 0, 1, \cdots$ and all $x, y, z, w \in \mathcal{A}$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

1. Introduction and Preliminaries

The study of stability problems originated from a famous talk given by S.M. Ulam [18] in 1940 : *Under what condition does there exists a homomorphism near an approximate homomorphism ?* In the next year 1941, D. H. Hyers [8] was answered affirmatively the question of Ulam for Banach spaces, which states that *if* $\delta > 0$ *and* $f : \mathcal{X} \to \mathcal{Y}$ *is a map with* $\mathcal{X}$ *a normed space,* $\mathcal{Y}$ *a Banach space such that*

$$||f(x+y) - f(x) - f(y)|| \le \delta$$

for all $x, y \in \mathcal{X}$*, then there exists a unique additive mapping* $T : \mathcal{X} \to \mathcal{Y}$ *such that*

$$||f(x) - T(x)|| \le \delta$$

for all $x \in \mathcal{X}$. A generalized version of the theorem of Hyers for approximately additive mappings was first given by T. Aoki [1] in 1950. In 1978, Th.M. Rassias

Received July 25, 2018; accepted May 10, 2019.
2010 Mathematics Subject Classification: 39B52, 46H99, 39B72, 39B82.
Key words and phrases: higher functional equation, stability.

[15] independently introduced the unbounded Cauchy difference and was the first to prove the stability of the linear mapping between Banach spaces: *if there exist a* $\theta \geq 0$ *and* $0 \leq p < 1$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in \mathcal{X}$, *then there exists a unique additive map* $T : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in \mathcal{X}$.

In 1991, Z. Gajda [6] answered the question for the case $p > 1$, which was raised by Rassias. Gajda [6] also gave an example that the Rassias' stability result is not valid for $p = 1$.

In 1992, a generalization of the Rassias theorem was obtained by P. Găvruţă [7]:

Suppose $(\mathcal{G}, +)$ *is an abelian group,* $\mathcal{Y}$ *is a Banach space and the so-called admissible control function* $\varphi : \mathcal{G} \times \mathcal{G} \to [0, \infty)$ *satisfies*

$$\psi(x, y) := \frac{1}{2}\sum_{k=0}^{\infty} \frac{\varphi(2^k x, 2^k y)}{2^k} < \infty,$$

for all $x, y \in \mathcal{G}$. *If* $f : \mathcal{G} \to \mathcal{Y}$ *is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in \mathcal{G}$, *then there exists a unique additive mapping* $T : \mathcal{G} \to \mathcal{Y}$ *such that*

$$\|f(x) - T(x)\| \leq \psi(x, x)$$

for all $x \in \mathcal{G}$.

Throughout this note, let $\mathbb{N}$ be the set of natural numbers and we assume that $\mathcal{A}$ and $\mathcal{B}$ are algebras over the real or complex field $\mathbb{F}(\mathbb{R}$ or $\mathbb{C})$. An additive mapping $h : \mathcal{A} \to \mathcal{B}$ is said to be a *ring homomorphism* if the functional equation $h(xy) = h(x)h(y)$ holds for all $x, y \in \mathcal{A}$. An additive mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a *ring left derivation* (resp. *ring derivation*) if the functional equation $d(xy) = xd(y) + yd(x)$ (resp. $d(xy) = xd(y) + d(x)y$) holds for all $x, y \in \mathcal{A}$. In addition, d is called a *linear left derivation* (resp. *linear derivation*) if the functional equation $d(\lambda x) = \lambda d(x)$ is valid for all $\lambda \in \mathbb{F}$ and for all $x \in \mathcal{A}$.

M. Brešar and J. Vukman [5, Proposition 1.6] showed that every ring left derivation on a semiprime ring is a ring derivation which maps the ring into its center.

Definition 1.1. A sequence $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ of additive mappings from $\mathcal{A}$ into $\mathcal{B}$ is called a *higher ring left derivation* (resp. *higher ring derivation*) from $\mathcal{A}$ into $\mathcal{B}$ if the functional equation

(1.1)

$$h_n(xy) = \sum_{\substack{i+j=n \\ i \leq j}} [h_i(x)h_j(y) + c_{ij}h_i(y)h_j(x)] \left(\text{resp. } h_n(xy) = \sum_{i+j=n} h_i(x)h_j(y)\right)$$

holds for each $n = 0, 1, \cdots$ and for all $x, y \in \mathcal{A}$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Let $\mathcal{A} = \mathcal{B}$. The sequence $H = \{h_1, h_2, \cdots, h_n, \cdots\}$ of additive mappings on $\mathcal{A}$ satisfying the relation (1.1), particulary, is called a *strong higher ring left derivation* (resp. *strong higher ring derivation*) on $\mathcal{A}$ if h_0 acts as an identity mapping on $\mathcal{A}$ in (1.1). If each h_n in H satisfies the functional equation $h_n(\lambda x) = \lambda h_n(x)$ for all $\lambda \in \mathbb{F}$ and all $x \in \mathcal{A}$, then we say that H is a *higher linear left derivation* (resp. *higher linear derivation*).

Remark 1.2. K.-H. Park [14] proved that every strong higher ring left derivation $H = \{h_1, h_2, \cdots, h_n, \cdots\}$ on a semiprime ring is a strong higher ring derivation such that each h_n in H maps the ring into its center. In Definition 1.1, the higher ring left derivation H from $\mathcal{A}$ into $\mathcal{B}$ (resp. the strong higher ring left derivation on $\mathcal{A}$) is a ring homomorphism if $n = 0$ (resp. a ring left derivation if $n = 1$).

Consider a sequence $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that the functional equation

$$f_n(x + y + zw) = f_n(x) + f_n(y) + \sum_{\substack{i+j=n \\ i \leq j}} [f_i(z)f_j(w) + c_{ij}f_i(w)f_j(z)] \tag{1.2}$$

holds for each $n = 0, 1, \cdots$ and all $x, y, z, w \in \mathcal{A}$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

For convenience' sake, we will say that the relation (1.2) is a *higher functional equation*. In particular, if $f_n(0) = 0$ for each $n = 0, 1, \cdots$, then we see that F is a higher ring left derivation from $\mathcal{A}$ into $\mathcal{B}$.

Remark 1.3. Let $F = \{f_1, f_2, \cdots, f_n, \cdots\}$ be a sequence of mappings on $\mathcal{A}$ and f_0 an identity mapping on $\mathcal{A}$. Then F satisfies the functional equation (1.2) if and only if F is a strong higher ring left derivation on $\mathcal{A}$. In fact, F is a strong higher ring left derivation on $\mathcal{A}$ since it follows from induction that for each f_n in F, we get $f_n(0) = 0$.

Example 1.4. Given any ring left derivation d on an algebra $\mathcal{A}$ with unit and an invertible element $c \in \mathcal{A}$, let $\delta : \mathcal{A} \to \mathcal{A}$ be the mapping defined by $\delta(x) = cd(x)$ for all $x \in \mathcal{A}$. Then $\delta(xy) = \vartheta(x)\delta(y) + \vartheta(y)\delta(x)$ for all $x, y \in \mathcal{A}$, where the relation $\vartheta(x) = cxc^{-1}, x \in \mathcal{A}$, defines an inner automorphism of $\mathcal{A}$. Let $f_0 = \vartheta$, $f_n = \delta$ if $n = m$ for some $m \in \mathbb{N}$, and $f_n = 0$ if $n \geq 1$ and $n \neq m$. Then we see that the sequence $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ of mappings on $\mathcal{A}$ satisfies the equation (1.2).

In 1949, D.G. Bourgin [4] proved the following stability result, which is sometimes called the superstability of ring homomorphisms: *suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *are*

Banach algebras with unit. If $f : \mathcal{A} \to \mathcal{B}$ is a surjective mapping such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon,$$
$$\|f(xy) - f(x)f(y)\| \leq \delta$$

for some $\varepsilon > 0$, $\delta > 0$ and all $x, y \in \mathcal{A}$, then f is a ring homomorphism.

R. Badora [2] gave a generalization of the Bourgin's result. Badora [3] also obtained the following results for the stability in the sense of Hyers and Ulam and for the superstability of ring derivations: *let $\mathcal{A}_1$ be a subalgebra of a Banach algebra $\mathcal{A}$. Assume that $f : \mathcal{A}_1 \to \mathcal{A}$ is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon,$$
$$\|f(xy) - xf(y) - f(x)y\| \leq \delta$$

for some $\varepsilon \geq 0$, $\delta \geq 0$ and all $x, y \in \mathcal{A}_1$. Then there exists a unique ring derivation $d : \mathcal{A}_1 \to \mathcal{A}$ such that

$$\|f(x) - d(x)\| \leq \varepsilon$$

for all $x \in \mathcal{A}_1$. Moreover,

$$x\{f(y) - d(y)\} = 0$$

for all $x, y \in \mathcal{A}_1$. In addition, if $\mathcal{A}_1$ and $\mathcal{A}$ have the unit element, then f is a ring derivation.

On the other hand, T. Miura *et al.* [13] proved the stability in the sense of Hyers, Ulam and Rassias and the superstability of ring derivations on Banach algebras.

In [11] and [12], we dealt with the stability of higher ring derivations and higher ring left derivations, respectively.

Here it is natural to ask that there exists an approximate sequence of mappings which is not an exactly sequence of mappings satisfying the functional equation (1.2). We observe the following example.

Example 1.5. Let A be a compact Hausdorff space and let $C(A)$ be the commutative Banach algebra of complex-valued continuous functions on A under pointwise operations and the supremum norm $\|\cdot\|_\infty$. Assume that $\vartheta : C(A) \to C(A)$ is a nonzero algebra homomorphism. We define $g : C(A) \to C(A)$ by

$$g(z)(a) = \begin{cases} \vartheta(z)(a)\log|\vartheta(z)(a)| & \text{if } \vartheta(z)(a) \neq 0, \\ 0 & \text{if } \vartheta(z)(a) = 0 \end{cases}$$

for all $z \in C(A)$ and $a \in A$. It is easy to see that $g(zw) = \vartheta(z)g(w) + g(z)\vartheta(w)$ for all $z, w \in C(A)$. Let $f_0 = \vartheta$, $f_n = g$ if $n = m$ for some $m \in \mathbb{N}$, and $f_n = 0$ if $n \geq 1$ and $n \neq m$. Then we see that the sequence $F = \{f_0, f_1, \cdots, f_n \cdots\}$ satisfies the relation

$$f_n(zw) = \sum_{\substack{i+j=n \\ i \leq j}} [f_i(z)f_j(w) + c_{ij}f_i(w)f_j(z)]$$

for each $n = 0, 1, \cdots$, and all $z, w \in C(A)$, where

$$c_{ij} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

We claim that

$$\|f_n(x+y+zw) - f_n(x) - f_n(y) - f_n(zw)\|_\infty \leq 2(\|x\|_\infty + \|y\|_\infty + \|zw\|_\infty)$$

for all $x, y, z, w \in C(A)$.

Observe that for all $x, y, z, w \in \mathbb{F} \setminus \{0\}$ with $x + y + zw \neq 0$,

$$\big|(x+y+zw)\log|x+y+zw| - x\log|x| - y\log|y| - zw\log|zw|\big| \leq 2(|x|+|y|+|zw|)$$

Indeed, fix $x, y, z, w \in \mathbb{F} \setminus \{0\}$, $x + y + zw \neq 0$ arbitrarily. Since $\log(1 + x) \leq x$ for all $x \geq 0$, we get

$$\begin{aligned}
&\big|(x+y+zw)\log|x+y+zw| - x\log|x| - y\log|y| - zw\log|zw|\big| \\
&\leq |x|\left|\log\frac{|x+y+zw|}{|x|}\right| + |y|\left|\log\frac{|x+y+zw|}{|y|}\right| + |zw|\left|\log\frac{|x+y+zw|}{|zw|}\right| \\
&\leq |x|\log\left(1+\frac{|y+zw|}{|x|}\right) + |y|\log\left(1+\frac{|x+zw|}{|y|}\right) + |zw|\log\left(1+\frac{|x+y|}{|zw|}\right) \\
&\leq |x|\frac{|y+zw|}{|x|} + |y|\frac{|x+zw|}{|y|} + |zw|\frac{|x+y|}{|zw|} \\
&= |y+zw| + |x+zw| + |x+y| \\
&\leq 2(|x|+|y|+|zw|).
\end{aligned}$$

This yields, for each $n = 0, 1, \cdots$,

$$\|f_n(x+y+zw) - f_n(x) - f_n(y) - f_n(zw)\|_\infty \leq 2(\|x\|_\infty + \|y\|_\infty + \|zw\|_\infty)$$

for all $x, y, z, w \in C(A)$.

Now, it follows that

$$\begin{aligned}
&\Big\|f_n(x+y+zw) - f_n(x) - f_n(y) - \sum_{\substack{i+j=n \\ i\leq j}} [f_i(z)f_j(w) + c_{ij}f_i(w)f_j(z)]\Big\|_\infty \\
&= \|f_n(x+y+zw) - f_n(x) - f_n(y) - f_n(zw)\|_\infty \\
&\leq 2(\|x\|_\infty + \|y\|_\infty + \|zw\|_\infty)
\end{aligned}$$

for each $n = 0, 1, \cdots$ and all $x, y, z, w \in C(A)$. Thus we may regard F as an approximate sequence of mappings on $C(A)$ with respect to the equation (1.2).

2. Main results

Our objective is to investigate the stability of the higher functional equation (1.2) in the sense of the generalized version of Hyers-Ulam-Rassias due to [7]. Furthermore, we will show the superstability of the equation (1.2).

Theorem 2.1. *Let $\mathcal{A}$ be an algebra and $\mathcal{B}$ a Banach algebra. For each $n = 0, 1, 2, \cdots$, let $\varphi_n : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ be a function such that*

$$\psi_n(x, y, z, w) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\varphi_n(2^k x, 2^k y, 2^k z, w)}{2^k} < \infty \tag{2.1}$$

for all $x, y, z, w \in \mathcal{A}$. Suppose that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that each $n = 0, 1, \cdots$,

$$\left\| f_n(x+y+zw) - f_n(x) - f_n(y) - \sum_{\substack{i+j=n \\ i \leq j}} [f_i(z) f_j(w) + c_{ij} f_i(w) f_j(z)] \right\| \leq \varphi_n(x, y, z, w) \tag{2.2}$$

holds for all $x, y, z, w \in \mathcal{A}$. Then there exists a unique higher ring left derivation $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n = 0, 1, \cdots$,

$$\|f_n(x) - h_n(x)\| \leq \psi_n(x, x, 0, 0) + c_n \tag{2.3}$$

holds for all $x \in \mathcal{A}$, where

$$c_n = \left\| \sum_{\substack{i+j=n \\ i \leq j}} [f_i(0) f_j(0) + c_{ij} f_i(0) f_j(0)] \right\|$$

for each $n = 0, 1, \cdots$.

Moreover,

$$\sum_{\substack{i+j=n \\ i \leq j}} h_i(x)[f_j(y) - h_j(y)] + \sum_{\substack{i+j=n \\ i \leq j}} c_{ij} [f_i(y) - h_i(y)] h_j(x) = 0 \tag{2.4}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$.

Proof. Putting $z = w = 0$ in (2.2), we have

$$\left\| f_n(x+y) - f_n(x) - f_n(y) - \sum_{\substack{i+j=n \\ i \leq j}} [f_i(0) f_j(0) + c_{ij} f_i(0) f_j(0)] \right\| \leq \varphi_n(x, y, 0, 0)$$

which yields

$$\begin{aligned}
&\|f_n(x+y) - f_n(x) - f_n(y)\| \\
&\leq \left\| f_n(x+y) - f_n(x) - f_n(y) - \sum_{\substack{i+j=n \\ i\leq j}} [f_i(0)f_j(0) + c_{ij}f_i(0)f_j(0)] \right\| \\
&\quad + \left\| \sum_{\substack{i+j=n \\ i\leq j}} [f_i(0)f_j(0) + c_{ij}f_i(0)f_j(0)] \right\| \\
&\leq \varphi_n(x,y,0,0) + c_n,
\end{aligned}$$

i.e.,

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \leq \varphi_n(x,y,0,0) + c_n \tag{2.5}$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$.

Using Hyers' direct method on inequality (2.5), it follows from induction on l that

$$\left\| \frac{1}{2^l} f_n(2^l x) - f_n(x) \right\| \leq \frac{1}{2} \sum_{k=0}^{l-1} \frac{\varphi_n(2^k x, 2^k x, 0, 0) + c_n}{2^k} \tag{2.6}$$

for each $n = 0, 1, \cdots$ and all $x \in \mathcal{A}$ and that

$$\left\| \frac{1}{2^l} f_n(2^l x) - \frac{1}{2^m} f_n(2^m x) \right\| \leq \frac{1}{2} \sum_{k=m}^{l-1} \frac{\varphi_n(2^k x, 2^k x, 0, 0) + c_n}{2^k}$$

for each $l > m$ and all $x \in \mathcal{A}$. Hence the convergence of (2.1) tells us that the sequence $\{\frac{1}{2^l} f_n(2^l x)\}$ is Cauchy for each $n = 0, 1, \cdots$ and all $x \in \mathcal{A}$. Let

$$h_n(x) = \lim_{l\to\infty} \frac{1}{2^l} f_n(2^l x) \tag{2.7}$$

for each $n = 0, 1, \cdots$ and all $x \in \mathcal{A}$. Taking $l \to \infty$ in (2.6), we obtain (2.3). In view of the same process as Hyers' method [8], we see that each mapping h_n, $n = 0, 1, \cdots$, is additive and unique.

Next, we need to show that the sequence $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ satisfies the identity

$$h_n(xy) = \sum_{\substack{i+j=n \\ i\leq j}} [h_i(x)h_j(y) + c_{ij}h_i(y)h_j(x)]$$

for each $n = 0, 1, \cdots$ and all $z, w \in \mathcal{A}$. Setting $x = y = 0$ in (2.2), we get

$$\left\| f_n(zw) - 2f_n(0) - \sum_{\substack{i+j=n \\ i\leq j}} [f_i(z)f_j(w) + c_{ij}f_i(w)f_j(z)] \right\| \leq \varphi_n(0,0,z,w)$$

which implies that

$$\left\| f_n(zw) - \sum_{\substack{i+j=n \\ i\le j}} [f_i(z)f_j(w) + c_{ij} f_i(w)f_j(z)] \right\| \le \varphi_n(0,0,z,w) + 2\|f_n(0)\| \tag{2.8}$$

for each $n = 0, 1, \cdots$ and all $z, w \in \mathcal{A}$. Let a function $\Delta_n : \mathcal{A} \times \mathcal{A} \to \mathcal{B}$ be defined by

$$\Delta_n(z,w) = f_n(zw) - \sum_{\substack{i+j=n \\ i\le j}} [f_i(z)f_j(w) + c_{ij} f_i(w)f_j(z)] \tag{2.9}$$

for each $n = 0, 1, \cdots$ and all $z, w \in \mathcal{A}$. Using (2.1) and (2.8), we have

$$\lim_{l\to\infty} \frac{1}{2^l} \Delta_n(2^l z, w) = 0 \tag{2.10}$$

for each $n = 0, 1, \cdots$ and all $z, w \in \mathcal{A}$. Now, from (2.7), (2.9) and (2.10), we deduce that

$$\begin{aligned}
h_n(zw) &= \lim_{l\to\infty} \frac{1}{2^l} f_n(2^l(zw)) = \lim_{l\to\infty} \frac{1}{2^l} f_n((2^l z)w)) \\
&= \lim_{l\to\infty} \frac{1}{2^l} \left\{ \sum_{\substack{i+j=n \\ i\le j}} [f_i(2^l z)f_j(w) + c_{ij} f_i(w)f_j(2^l z)] + \Delta_n(2^l z, w) \right\} \\
&= \lim_{l\to\infty} \sum_{\substack{i+j=n \\ i\le j}} \frac{1}{2^l} [f_i(2^l z)f_j(w) + c_{ij} f_i(w)f_j(2^l z)] + \lim_{l\to\infty} \frac{1}{2^l} \Delta_n(2^l z, w) \\
&= \sum_{\substack{i+j=n \\ i\le j}} \left\{ \lim_{l\to\infty} \frac{1}{2^l} f_i(2^l z) f_j(y) + c_{ij} \lim_{l\to\infty} \frac{1}{2^l} f_i(w) f_j(2^l z) \right\} \\
&= \sum_{\substack{i+j=n \\ i\le j}} [h_i(z)f_j(w) + c_{ij} f_i(w)h_j(z)]
\end{aligned}$$

That is, we obtain that

$$h_n(zw) = \sum_{\substack{i+j=n \\ i\le j}} [h_i(z)f_j(w) + c_{ij} f_i(w)h_j(z)] \tag{2.11}$$

for each $n = 0, 1, \cdots$ and all $z, w \in \mathcal{A}$. Let $l \in \mathbb{N}$ be fixed. Applying the relation (2.11) and the additivity of each h_n, $n = 0, 1, \cdots$, we get

$$\begin{aligned}
&\sum_{\substack{i+j=n \\ i\le j}} [h_i(z)f_j(2^l w) + c_{ij} f_i(2^l w)h_j(z)] = h_n(z(2^l w)) = h_n((2^l z)w) \\
&= \sum_{\substack{i+j=n \\ i\le j}} [h_i(2^l z)f_j(w) + c_{ij} f_i(w)h_j(2^l z)] = 2^l \sum_{\substack{i+j=n \\ i\le j}} [h_i(z)f_j(y) + c_{ij} f_i(w)h_j(z)]
\end{aligned}$$

Hence we get
(2.12)
$$\sum_{\substack{i+j=n\\ i\le j}}[h_i(z)f_j(w)+c_{ij}f_i(w)h_j(z)]=\sum_{\substack{i+j=n\\ i\le j}}\left[h_i(z)\frac{1}{2^l}f_j(2^lw)+c_{ij}\frac{1}{2^l}f_i(2^lw)h_j(z)\right]$$

for each $n=0,1,\cdots$ and all $z,w\in\mathcal{A}$. Taking $l\to\infty$ in (2.12), we see that

$$\sum_{\substack{i+j=n\\ i\le j}}[h_i(z)f_j(w)+c_{ij}f_i(w)h_j(z)]=\sum_{\substack{i+j=n\\ i\le j}}[h_i(z)h_j(w)+c_{ij}h_i(w)h_j(z)] \tag{2.13}$$

for each $n=0,1,\cdots$ and all $z,w\in\mathcal{A}$ which means (2.4). Combining (2.11) with (2.13), it follows that $H=\{h_0,h_1,\cdots,h_n,\cdots\}$ satisfies the relation

$$h_n(xy)=\sum_{\substack{i+j=n\\ i\le j}}[h_i(z)h_j(w)+c_{ij}h_i(w)h_j(z)].$$

for each $n=0,1,\cdots$ and all $z,w\in\mathcal{A}$, i.e., H is a higher ring left derivation from $\mathcal{A}$ into $\mathcal{B}$. This completes the proof of the theorem. □

Remark 2.2. Let $\mathcal{A}$ be an algebra and $\varphi_n:\mathcal{A}\times\mathcal{A}\times\mathcal{A}\times\mathcal{A}\to[0,\infty)$ a function such that

$$\psi_n(x,y,z,w)=\frac{1}{2}\sum_{k=0}^{\infty}2^k\varphi_n(2^{-k}x,2^{-k}y,2^{-k}z,w)<\infty$$

for all $x,y,z,w\in\mathcal{A}$. If we replace (2.10) in Theorem 2.1 by

$$\lim_{l\to\infty}2^l\Delta_n(2^{-l}z,w)=0,$$

then (2.4) in Theorem 2.1 does not generally hold in case of $\psi_n(x,y,z,w)<\infty$. For, we see that

$$\lim_{l\to\infty}2^l\Delta_n(2^{-l}z,w)=0$$

is not valid since $\lim_{n\to\infty}2^{l+1}\|f_n(0)\|\neq 0$.

Corollary 2.3. *Let $\mathcal{A}$ be a normed algebra and $\mathcal{B}$ a Banach algebra. Let $\theta_n\in(0,\infty)$ for each $n=0,1,\cdots$ and p,q real numbers such that $p\neq 1$. Suppose that $F=\{f_0,f_1,\cdots,f_n,\cdots\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n=0,1,\cdots$,*

$$\left\|f_n(x+y+zw)-f_n(x)-f_n(y)-\sum_{\substack{i+j=n\\ i\le j}}[f_i(z)f_j(w)+c_{ij}f_i(w)f_j(z)]\right\|$$
$$\le\theta_n(\|x\|^p+\|y\|^p+\|z\|^p\,\|w\|^q)$$

holds for all $x, y, z, w \in \mathcal{A}$. Then there exists a unique higher ring left derivation $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n = 0, 1, \cdots$,

$$\|f_n(x) - h_n(x)\| \le \begin{cases} \frac{2\theta_n}{2-2^p}\|x\|^p + c_n & \text{if } p < 1, \\ \frac{2^p\theta_n}{2^p-2}\|x\|^p + c_n & \text{if } p > 1 \end{cases}$$

holds all $x \in \mathcal{A}$, where

$$c_n = \left\| \sum_{\substack{i+j=n \\ i \le j}} [f_i(0)f_j(0) + c_{ij} f_i(0) f_j(0)] \right\|$$

for each $n = 0, 1, \cdots$.

Proof. Let $\varphi_n(x, y, z, w) = \theta_n(\|x\|^p + \|y\|^p + \|z\|^p \, \|w\|^q)$ for each $n = 0, 1, \cdots$ and all $x, y, z, w \in \mathcal{A}$. Suppose that $p < 1$. Since we have

$$\begin{aligned} \psi_n(x, x, 0, 0) &= \frac{1}{2}\sum_{k=0}^{\infty} \frac{\varphi_n(2^k x, 2^k x, 0, 0)}{2^k} = \frac{\theta_n}{2}\sum_{k=0}^{\infty} \frac{\|2^k x\|^p + \|2^k x\|^p}{2^k} \\ &= \theta_n \|x\|^p \sum_{k=0}^{\infty} 2^{(p-1)k} = \theta_n \|x\|^p \frac{1}{1 - 2^{p-1}} = \frac{2\theta_n}{2 - 2^p}\|x\|^p, \end{aligned}$$

it follows from (2.3) in Theorem 2.1 that

$$\|f_n(x) - h_n\| \le \psi_n(x, x, 0, 0) + c_n = \frac{2\theta_n}{2 - 2^p}\|x\|^p + c_n.$$

for each $n = 0, 1, \cdots$ and all $x \in \mathcal{A}$.

Assume that $p > 1$. Since we have

$$\begin{aligned} \psi_n(x, x, 0, 0) &= \frac{1}{2}\sum_{k=0}^{\infty} 2^k \varphi_n(2^{-k}x, 2^{-k}x, 0, 0) = \frac{\theta_n}{2}\sum_{k=0}^{\infty} 2^k(\|2^{-k}x\|^p + \|2^{-k}x\|^p) \\ &= \theta_n \|x\|^p \sum_{k=0}^{\infty} 2^{(1-p)k} = \theta_n \|x\|^p \frac{1}{1 - 2^{1-p}} = \frac{2^p\theta_n}{2^p - 2}\|x\|^p, \end{aligned}$$

it follows from (2.3) in Theorem 2.1 that

$$\|f_n(x) - h_n\| \le \psi_n(x, x, 0, 0) + c_n = \frac{2^p\theta_n}{2^p - 2}\|x\|^p + c_n.$$

for each $n = 0, 1, \cdots$ and all $x \in \mathcal{A}$. □

By setting $\varphi_n(x, y, z, w) = \varepsilon_n$ for each $n = 0, 1, \cdots$ and all $x, y, z, w \in \mathcal{A}$, Theorem 2.1 also gives us the following corollary.

Corollary 2.4. *Let $\mathcal{A}$ be an algebra and $\mathcal{B}$ a Banach algebra. Suppose that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n = 0, 1, \cdots$, there exists $\varepsilon_n > 0$ such that*

$$\left\| f_n(x+y+zw) - f_n(x) - f_n(y) - \sum_{\substack{i+j=n \\ i \le j}} [f_i(z)f_j(w) + c_{ij} f_i(w) f_j(z)] \right\| \le \varepsilon_n \tag{2.14}$$

holds for all $x, y, z, w \in \mathcal{A}$. Then there exists a unique higher ring left derivation $H = \{h_0, h_1, \cdots, h_n, \cdots\}$ from $\mathcal{A}$ into $\mathcal{B}$ such that for each $n = 0, 1, \cdots$ and all $x \in \mathcal{A}$,

$$\|f_n(x) - h_n(x)\| \le \varepsilon_n + c_n,$$

where

$$c_n = \left\| \sum_{\substack{i+j=n \\ i \le j}} [f_i(0)f_j(0) + c_{ij} f_i(0) f_j(0)] \right\|$$

for each $n = 0, 1, \cdots$.

As a consequence of Corollary 2.4, we get the following Bourgin-type superstability [4] of the higher functional equation (1.2).

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with unit. Suppose that $F = \{f_0, f_1, \cdots, f_n, \cdots\}$ is a sequence of mappings from $\mathcal{A}$ into $\mathcal{B}$ satisfying the inequality* (2.14)*, where f_0 is onto. Then F is a higher ring left derivation from $\mathcal{A}$ into $\mathcal{B}$.*

Proof. As in (2.5) and (2.8), the relation (2.14) yields that

$$\|f_n(x+y) - f_n(x) - f_n(y)\| \le \varepsilon_n + c_n$$

for each $n = 0, 1, \cdots$ and all $x, y \in \mathcal{A}$, where

$$c_n = \left\| \sum_{\substack{i+j=n \\ i \le j}} [f_i(0)f_j(0) + c_{ij} f_i(0) f_j(0)] \right\|$$

for each $n = 0, 1, \cdots$, and that

$$\left\| f_n(zw) - \sum_{\substack{i+j=n \\ i \le j}} [f_i(z)f_j(w) + c_{ij} f_i(w) f_j(z)] \right\| \le \varepsilon_n + 2\|f_n(0)\|$$

for each $n = 0, 1, \cdots$ and all $z, w \in \mathcal{A}$. By induction, we lead the conclusion. From the Bourgin's theorem [4], we see that f_0 is a ring homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ and so (2.7) gives that $h_0(z) = \lim_{l \to \infty} \frac{1}{2^l} f_0(2^l z) = f_0(z)$ for all $z \in \mathcal{A}$, i.e., $f_0 = h_0$. If $n = 1$, then it follows from the relation (2.4) that $f_1(z) = h_1(z)$ holds for all $z \in \mathcal{A}$ since h_0 is onto. Let us assume that $f_m(z) = h_m(z)$ is valid for all $z \in \mathcal{A}$

and all $m < n$. Then (2.4) implies that $h_0(z)\{f_n(w) - h_n(w)\} = 0$ for all $z, w \in \mathcal{A}$. Since h_0 is onto, we have $f_n(w) = h_n(w)$ for all $w \in \mathcal{A}$. Hence we conclude that $f_n(z) = h_n(z)$ holds for all $n = 0, 1, \cdots$ and all $z \in \mathcal{A}$. Now, Corollary 2.4 tells us that F is a higher ring left derivation from $\mathcal{A}$ into $\mathcal{B}$. The proof of the theorem is complete. $\square$

We continue the next result.

Theorem 2.6. *Let $\mathcal{A}$ be a semiprime Banach algebra and f_0 an identity mapping on $\mathcal{A}$. Suppose that $F = \{f_1, f_2, \cdots, f_n, \cdots\}$ is a sequence of mappings on $\mathcal{A}$ satisfying the inequality* (2.14). *Then F is a strong higher ring left derivation on $\mathcal{A}$. Furthermore, F is a strong higher ring derivation on $\mathcal{A}$ such that each f_n in F maps $\mathcal{A}$ into its center.*

Proof. For all $z \in \mathcal{A}$, we have, by (2.7),

$$h_0(z) = \lim_{l \to \infty} \frac{1}{2^l} f_0(2^l z) = z$$

and so h_0 is an identity mapping on $\mathcal{A}$.

From the similar way as in the proof of Theorem 2.5 using the induction and the relation (2.4), we get

$$z\{f_n(w) - h_n(w)\} = 0$$

for all $n \in \mathbb{N}$ and all $z, w \in \mathcal{A}$. This implies that

$$\{f_n(w) - h_n(w)\}z\{f_n(w) - h_n(w)\} = 0$$

for all $n \in \mathbb{N}$ and all $z, w \in \mathcal{A}$. Since $\mathcal{A}$ is semiprime, it follows that $f_n(w) = h_n(w)$ for all $n \in \mathbb{N}$ and all $w \in \mathcal{A}$. Therefore, F is a strong higher ring left derivation on $\mathcal{A}$. It follows from [14] that F is a strong higher ring derivation on $\mathcal{A}$ such that each f_n in F maps $\mathcal{A}$ into its center. This completes the proof. $\square$

The Singer-Wermer theorem [16], which is a fundamental result in a Banach algebra theory, states that every continuous linear derivation (or linear left derivation) on a commutative Banach algebra maps into the Jacobson radical. They also conjectured that the assumption of continuity is unnecessary. M.P. Thomas [17] proved the conjecture. According to the Thomas' result, it is easy to see that every linear derivation (or linear left derivation) on a commutative semisimple Banach algebra is identically zero which is the result of B.E. Johnson [9].

The following is similar to B.E. Johnson' result [9] in the sense of Hyers-Ulam [8].

Theorem 2.7. *Let $\mathcal{A}$ be a semisimple Banach algebra and f_0 an identity mapping on $\mathcal{A}$. Suppose that $F = \{f_1, f_2 \cdots, f_n \cdots\}$ is a sequence of mappings on $\mathcal{A}$ satisfying the following:*

For each $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ such that

(2.15)
$$\left\| f_n(\alpha x + \beta y + zw) - \alpha f_n(x) - \beta f_n(y) - \sum_{\substack{i+j=n \\ i \le j}} [f_i(z) f_j(w) + c_{ij} f_i(w) f_j(z)] \right\| \le \varepsilon_n$$

holds for all $x, y, z, w \in \mathcal{A}$ *and all* $\alpha, \beta \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$. *Then we have* $F = \{0\}_{n\in\mathbb{N}}$ *on* $\mathcal{A}$.

Proof. Put $\alpha = \beta = 1 \in \mathbb{U}$ in (2.15). Then it follows from Theorem 2.6 that F is a strong higher ring left derivation on $\mathcal{A}$. So, each f_n, $n \in \mathbb{N}$, is additive on $\mathcal{A}$. Setting $y = x$, $z = w = 0$ in (2.15) and then following the same process as in (2.5), we obtain that

$$\|f_n((\alpha+\beta)x) - (\alpha+\beta)f_n(x)\| \le \varepsilon_n + c_n$$

for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$, where

$$c_n = \left\| \sum_{\substack{i+j=n \\ i\le j}} [f_i(0)f_j(0) + c_{ij}f_i(0)f_j(0)] \right\|$$

for each $n \in \mathbb{N}$. Thus we see that

$$\frac{1}{2^l}\|f_n(2^l(\alpha+\beta)x) - (\alpha+\beta)f_n(2^l x)\| \to 0$$

as $l \to \infty$ which implies that for each $n \in \mathbb{N}$,

$$f_n((\alpha+\beta)x) = \lim_{l\to\infty} \frac{1}{2^l} f_n(2^l(\alpha+\beta)x) = (\alpha+\beta)\lim_{l\to\infty}\frac{1}{2^l}f_n(2^l x) = (\alpha+\beta)f_n(x)$$

for all $x \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$.

Clearly, $f_n(0x) = 0 = 0f_n(x)$ for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$. Now, let $\lambda \in \mathbb{C}$ ($\lambda \ne 0$), and let $N \in \mathbb{N}$ greater than $|\lambda|$. By appying a geometric argument, we see that there exist $\lambda_1, \lambda_2 \in \mathbb{U}$ such that $2\frac{\lambda}{N} = \lambda_1 + \lambda_2$. By the additivity of each f_n, $n \in \mathbb{N}$, we get $f_n(\frac{1}{2}x) = \frac{1}{2}f_n(x)$ for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$.

Therefore, we have

$$\begin{aligned} f_n(\lambda x) &= f_n\left(\frac{N}{2}\cdot 2\cdot\frac{\lambda}{N}x\right) = Nf_n\left(\frac{1}{2}\cdot 2\cdot\frac{\lambda}{N}x\right) = \frac{N}{2}f_n((\lambda_1+\lambda_2)x) \\ &= \frac{N}{2}(\lambda_1+\lambda_2)f_n(x) = \frac{N}{2}\cdot 2\cdot\frac{\lambda}{N}f_n(x) = \lambda h_f(x) \end{aligned}$$

for each $n \in \mathbb{N}$ and all $x \in \mathcal{A}$, so that f_n is $\mathbb{C}$-linear for each $n \in \mathbb{N}$. This means that that F is a strong higher *linear* left derivation on $\mathcal{A}$.

Since f_1 is a linear left derivation on $\mathcal{A}$, we have $f_1 = 0$ on $\mathcal{A}$ by [10, Corollary 3.7]. Assume that $n \ge 2$ and $f_m = 0$ for all $m < n$. Since we have

$$f_n(xy) = xf_n(y) + yf_n(x) + \sum_{\substack{i+j=n \\ i\le j,\, i\ne 0,n}} [f_i(x)f_j(y) + c_{ij}f_i(y)f_j(x)],$$

it follows from the hypothesis that

$$f_n(xy) = xf_n(y) + yf_n(x)$$

for all $x, y \in \mathcal{A}$. This implies that f_n is a linear left derivation on $\mathcal{A}$. Therefore, [10, Corollary 3.7] again gives $f_n = 0$ on $\mathcal{A}$. By the induction, we have $F = \{0\}_{n\in\mathbb{N}}$ on $\mathcal{A}$ which completes the proof. □

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