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Statistical Approximation of Szász Type Operators Based on Charlier Polynomials

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Abstract. In the present note, we study some approximation properties of the Szász type operators based on Charlier polynomials introduced by S. Varma and F. Taşdelen (Math. Comput. Modelling, 56 (5-6) (2012) 108-112). We establish the rates of A-statistical convergence of these operators. Finally, we prove a Voronovskaja type approximation theorem and local approximation theorem via the concept of A-statistical convergence.

1. Introduction

Szász [28] defined the following linear positive operators

$$\mathcal{S}_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \in [0, \infty)$ and $f(x)$ is a continuous function on $[0, \infty)$ whenever the above sum converges uniformly. Many authors have studied approximation properties of these operators and generalized Szász operators involving different type of polynomials. Jakimovski and Leviatan [19] introduced a generalization of Szász operators based on the Appell polynomials and studied the approximation properties of these operators. Varma et al. [29] proposed the generalization of Szász operators based on Brenke type polynomials and gave convergence properties by means of a Korovkin type theorem and an order approximation using classical methods. Aral and Duman [4] studied a statistical asymptotic formula for the Szász-Mirkajan-Kantorovich operators via $A-$stastical convergence. In 2011, Örkcü and Doğru [25] defined a Kantorovich variant of $q-$Szász-Mirakjan operators and obtained weighted statistical convergence theorems for the operators. Özarslan [26] proposed the Mittag-Leffler

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operators and obtained some direct theorems and $A-$statistical approximation theorems for these operators. Altomare et al. [1] considered a new kind of generalization of SMK operators and studied the rate of approximation by interms suitable moduli of continuity. Several researchers have studied various generalizations of these operators and obtained results about their approximation properties; we refer the reader to such papers as [3, 5, 6, 11, 13, 14, 15, 16, 17, 20, 22, 23, 24].

Varma and Taşdelen introduced a relation between orthogonal polynomials and linear positive operators. They considered Szász type operators depending on Charlier polynomials.
These polynomials [18] have generating functions of the form

$$e^t\left(1-\frac{t}{a}\right)^u=\sum_{k=0}^{\infty}C_k^{(a)}(u)\frac{t^k}{k!},\ \ |t|<a, \tag{1.2}$$

where $C_k^{(a)}(u)=\sum_{r=0}^{k}\binom{k}{r}(-u)_r\left(\frac{1}{a}\right)^r$ and $(m)_0=1,(m)_j=m(m+1)\cdots(m+j-1)$ for $j\geq 1$.

For $C_\gamma[0,\infty):=\{f\in C[0,\infty):|f(t)|\leq Me^{\gamma t}$ for some $\gamma>0,M>0$ and $t\in[0,\infty)\}$, Varma and Taşdelen [30] studied the following Szász type operators based on Charlier polynomials

$$\mathcal{L}_n(f;x,a)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}f\left(\frac{k}{n}\right), \tag{1.3}$$

where $a>1$ and $x\in[0,\infty)$. If $a\to\infty$ and $x-\frac{1}{n}$ instead of x, these operators reduce to the operators (1.1). They studied uniform convergence of these operators by using Korovkin's theorem on compact subsets of $[0,\infty)$ and the order of approximation by using the modulus of continuity. Very recently, Kajla and Agrawal [21] discussed some approximation properties of these operators e.g. Lipschitz type space, weighted approximation theorems and rate of approximation of functions having derivatives of bounded variation.

The aim of the present article is to study $A-$statistical convergence to prove a Korovkin type convergence theorem. We also obtain a Voronovskaja type approximation theorem and local approximation theorem via $A-$statistical convergence.

Lemma 1.1.([21]) *For the operators $\mathcal{L}_n(t^s;x,a)$, $s=3,4$, we have*

(i) $\mathcal{L}_n(t^3;x,a)=x^3+\frac{x^2}{n}\left(6+\frac{3}{a-1}\right)+\frac{2x}{n^2}\left(\frac{1}{(a-1)^2}+\frac{3}{a-1}+5\right)+\frac{5}{n^3};$

(ii) $$\mathcal{L}_n(t^4;x,a)=x^4+\frac{x^3}{n}\left(10+\frac{6}{a-1}\right)+\frac{x^2}{n^2}\left(31+\frac{30}{a-1}+\frac{11}{(a-1)^2}\right)+\frac{x}{n^3}\left(67+\frac{31}{a-1}+\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right)+\frac{15}{n^4}.$$

Lemma 1.2.([21]) *For the operators $\mathcal{L}_n(f;x,a)$, we have*

(i) $\mathcal{L}_n((t-x);x,a) = \frac{1}{n}$;

(ii) $\mathcal{L}_n((t-x)^2;x,a) = \frac{ax}{n(a-1)} + \frac{2}{n^2}$;

(iii) $\mathcal{L}_n((t-x)^4;x,a) = \frac{x}{n^3}\left(17 + \frac{49}{(a-1)} - \frac{20}{(a-1)^2} + \frac{6}{(a-1)^3}\right) + \frac{x^2}{n^2}\left(19 - \frac{46}{(a-1)} + \frac{3}{(a-1)^2}\right) + \frac{15}{n^4}.$

In what follows, let $\tilde{C_B}[0,\infty)$ be the space of all real valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm $||f||_{\tilde{C_B}[0,\infty)} = \sup_{x\in[0,\infty)} |f(x)|$.
Further, let us define the following Peetre's K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{||f-g|| + \delta||g''||\},\ \delta > 0,$$

where $W^2 = \{g \in \tilde{C_B}[0,\infty) : g', g'' \in \tilde{C_B}[0,\infty)\}$. By [7, p.177, Theorem 2.4], there exists an absolute constant $M > 0$ such that

$$K_2(f,\delta) \le M\left\{\omega_2(f,\sqrt{\delta}) + \min(1,\delta) \parallel f \parallel_{\tilde{C_B}[0,\infty)}\right\}, \tag{1.4}$$

where the second order modulus of smoothness is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<|h|\le\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

We define the usual modulus of continuity of $f \in \tilde{C_B}[0,\infty)$ as

$$\omega(f,\delta) = \sup_{0<|h|\le\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

Let $B_\varphi[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f\varphi(x)$, where M_f is a positive constant depending only on f and $\varphi(x) = 1 + x^2$ is a weight function. Let $C_\varphi[0,\infty)$ be the space of all continuous functions in $B_\varphi[0,\infty)$ with the norm $\|f\|_\varphi := \sup_{x\in[0,\infty)} \frac{|f(x)|}{\varphi(x)}$ and $C^*_\varphi[0,\infty) := \left\{f \in C_\varphi[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{\varphi(x)} \text{ is finite}\right\}$.

2. Main Results

2.1. A-statistical Convergence

First, we give some basic definitions and notations on the concept of A-statistical convergence. Let $A = (a_{nk}), (n, k \in \mathbb{N})$, be a non-negative infinite summability matrix. For a given sequence $x := (x_k)$, the A-transform of x denoted by Ax : $((Ax)_n)$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k,$$

provided the series converges for each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_n x_n = L$. The sequence $x = (x_n)$ is said to be a A-statistically convergent to L i.e. $st_A - \lim_n x_n = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geq \epsilon} a_{nk} = 0$.

Replacing A by C_1, the Cesáro matrix of order one, the A-statistical convergence reduces to the statistical convergence. Similarly, if we take $A = I$, the identity matrix, then A-statistical convergence coincides with the ordinary convergence. Many researchers have studied the statistical convergence of different types of operators (cf. [2, 8, 9, 10, 12, 25, 27]). In the following result we prove a weighted Korovkin theorem via A-statistical convergence.

Theorem 2.1. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and $x \in [0, \infty)$. Then, for all $f \in C^*_\varphi[0, \infty)$ we have*

$$st_A - \lim_n \|\mathcal{L}_n(f, ., a) - f\|_{\varphi_\alpha} = 0,$$

where $\varphi_\alpha(x) = 1 + x^{2+\alpha}$, $\alpha > 0$.

Proof. From [10, p. 191, Th. 3], it is sufficient to show that $st_A - \lim_n \|\mathcal{L}_n(e_j, ., a) - e_j\|_\varphi = 0$, where $e_j(x) = x^j$, $j = 0, 1, 2$.

In view of [30, Lemma 1], it follows that

$$st_A - \lim_n \|\mathcal{L}_n(e_0, ., a) - e_0\|_\varphi = 0. \tag{2.1}$$

Again, using [30, Lemma 1], we find that

$$\|\mathcal{L}_n(e_1, ., a) - e_1\|_\varphi = \sup_{x \geq 0} \frac{\left| x + \frac{1}{n} - x \right|}{(1 + x^2)} \leq \frac{1}{n}.$$

For $\epsilon > 0$, let us define the following sets:

$$\mathcal{F} : \;\; = \;\; \left\{ n \in \mathbb{N} : \|\mathcal{L}_n(e_1; ., a) - e_1\|_\varphi \geq \epsilon \right\}$$

and

$$\mathcal{F}_1 : \;=\; \left\{n \in \mathbb{N} : \frac{1}{n} \geq \epsilon\right\}.$$

Then, we obtain $\mathcal{F} \subseteq \mathcal{F}_1$ which implies that $\sum_{k\in\mathcal{F}} a_{nk} \leq \sum_{k\in\mathcal{F}_1} a_{nk}$ and hence

$$st_A - \lim_n \|\mathcal{L}_n(e_1; ., a) - e_1\|_\varphi = 0. \tag{2.2}$$

Next, we can write

$$\begin{aligned} \| \mathcal{L}_n(e_2; ., a) - e_2 \|_\varphi &= \sup_{x\geq 0} \frac{1}{1+x^2}\left| x^2 + \frac{x}{n}\left(3 + \frac{1}{a-1}\right) + \frac{2}{n^2} - x^2 \right| \\ &\leq \frac{1}{n}\frac{(3a-2)}{(a-1)} + \frac{2}{n^2}. \end{aligned}$$

Now, we define the following sets:

$$\begin{aligned} \mathcal{G} : &= \left\{n \in \mathbb{N} : \|\mathcal{L}_n(e_2; ., a) - e_2\|_\varphi \geq \epsilon\right\}, \\ \mathcal{G}_1 : &= \left\{n \in \mathbb{N} : \frac{1}{n}\frac{(3a-2)}{(a-1)} \geq \frac{\epsilon}{2}\right\} \\ &\text{and} \\ \mathcal{G}_2 : &= \left\{n \in \mathbb{N} : \frac{2}{n^2} \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

Then, we get $\mathcal{G} \subseteq \mathcal{G}_1 \cup \mathcal{G}_2$, which implies that

$$\sum_{k\in\mathcal{G}} a_{nk} \leq \sum_{k\in\mathcal{G}_1} a_{nk} + \sum_{k\in\mathcal{G}_2} a_{nk}$$

and hence

$$st_A - \lim_n \|\mathcal{L}_n(e_2; ., a) - e_2\|_\varphi = 0. \tag{2.3}$$

This completes the proof of the theorem. □

Similarly, from Lemma 1.2, we have

$$st_A - \lim_n \|\mathcal{L}_n((e_1 - xe_0)^j; x, a)\|_\varphi = 0, \quad j = 0, 1, 2, 3, 4. \tag{2.4}$$

Next, we prove a Voronovskaja type theorem for the operators $\mathcal{L}_n$.

Theorem 2.2. *Let $A = (a_{nk})$ be a nonnegative regular summability matrix. Then, for every $f \in C^*_\varphi[0,\infty)$ such that $f', f'' \in C^*_\varphi[0,\infty)$, we have*

$$st_A - \lim_{n\to\infty} n\left(\mathcal{L}_n(f; x, a) - f(x)\right) = f'(x) + \frac{1}{2}\frac{ax}{(a-1)} f''(x),$$

uniformly with respect to $x \in [0, E], (E > 0)$.

Proof. Let $f, f', f'' \in C_{\varphi}^{*}[0, \infty)$. For each $x \geq 0$, define a function

$$\Theta(t,x)=\begin{cases} \dfrac{f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(x)}{(t-x)^2} & if\ t \neq x \\ 0, & if\ t = x. \end{cases}$$

Then

$$\Theta(x, x) = 0 \ \text{ and } \ \Theta(\cdot, x) \in C_{\varphi}^{*}[0, \infty).$$

Thus, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + (t-x)^2\Theta(t, x).$$

Operating by $\mathcal{L}_n$ on the above equality, we obtain

$$\begin{aligned} n\left(\mathcal{L}_n(f; x, a) - f(x)\right) &= f'(x) n\mathcal{L}_n((t-x); x, a) \\ &+ \frac{1}{2} f''(x) n\mathcal{L}_n((t-x)^2; x, a) \\ &+ n\mathcal{L}_n((t-x)^2\Theta(t, x); x, a). \end{aligned}$$

In view of Lemma 1.2, we get

$$st_A - \lim_{n\to\infty} n\mathcal{L}_n((t-x); x, a) = 1, \tag{2.5}$$

$$st_A - \lim_{n\to\infty} n\mathcal{L}_n((t-x)^2; x, a) = \frac{ax}{(a-1)}, \tag{2.6}$$

and

$$st_A - \lim_{n\to\infty} n^2\mathcal{L}_n((t-x)^4; x, a) = x^2\left(19 - \frac{46}{(a-1)} + \frac{3}{(a-1)^2}\right), \tag{2.7}$$

uniformly with respect to $x \in [0, E]$.

Applying Cauchy-Schwarz inequality, we have

$$n\mathcal{L}_n((t-x)^2\Theta(t, x); x, a) \leq \sqrt{n^2\mathcal{L}_n((t-x)^4; x, a)}\sqrt{\mathcal{L}_n(\Theta^2(t, x); x, a)}.$$

Let $\eta(t, x) = \Theta^2(t, x)$, we observe that $\eta(x, x) = 0$ and $\eta(\cdot, x) \in C_{\varphi}^{*}[0, \infty)$. It follows from the proof of Theorem 2.1 that

$$st_A - \lim_{n\to\infty} \mathcal{L}_n(\Theta^2(t, x); x, a) = st_A - \lim_{n\to\infty} \mathcal{L}_n(\eta(t, x); x, a) = \eta(x, x) = 0,$$

uniformly with respect to $x \in [0, E]$. Now, using (2.7), we obtain

$$st_A - \lim_{n\to\infty} n\mathcal{L}_n((t-x)^2\Theta(t,x); x, a) = 0. \tag{2.8}$$

Combining (2.5), (2.6) and (2.8), we get the desired result. □

Now, we obtain the rate of A-statistical convergence for the operators $\mathcal{L}_n$ with the help of Peetre's K-functional.

Theorem 2.3. *Let $f \in W^2$. Then, we have*

$$st_A - \lim_{n} \|\mathcal{L}_n(f; x, a) - f(x)\|_{\tilde{C_B}[0,\infty)} = 0. \tag{2.9}$$

Proof. By our hypothesis, from Taylor's expansion we find that

$$\mathcal{L}_n(f; x, a) - f(x) = f'(x)\mathcal{L}_n((e_1 - x); x, a) + \frac{1}{2} f''(\chi)\mathcal{L}_n((e_1 - x)^2; x, a);$$

where χ lies between t and x.
Thus, we get

$$\begin{aligned} \|\mathcal{L}_n(f; x, a) - f(x)\|_{\tilde{C_B}[0,\infty)} &\le \|f'\|_{\tilde{C_B}[0,\infty)} \|\mathcal{L}_n((e_1 - x); x, a)\|_{C[0,b]} \\ &\quad + \|f''\|_{\tilde{C_B}[0,\infty)} \|\mathcal{L}_n((e_1 - x)^2; x, a)\|_{C[0,b]} \end{aligned} \tag{2.10}$$

Using (2.4) for $\epsilon > 0$, we have

$$\lim_n \sum_{k \in \mathbb{N} : \|f'\|_{\tilde{C_B}[0,\infty)} \|\mathcal{L}_k((e_1 - x); x, a)\|_{C[0,b]} \ge \frac{\epsilon}{2}} a_{nk} = 0,$$

$$\lim_n \sum_{k \in \mathbb{N} : \|f''\|_{\tilde{C_B}[0,\infty)} \|\mathcal{L}_k((e_1 - x)^2; x, a)\|_{C[0,b]} \ge \frac{\epsilon}{2}} a_{nk} = 0.$$

From (2.10), we may write

$$\begin{aligned} &\sum_{k \in \mathbb{N} : \|\mathcal{L}_k(f; x, a) - f(x)\|_{\tilde{C_B}[0,\infty)} \ge \epsilon} a_{nk} \\ &\le \sum_{k \in \mathbb{N} : \|f'\|_{\tilde{C_B}[0,\infty)} \|\mathcal{L}_k((e_1 - x); x, a)\|_{C[0,b]} \ge \frac{\epsilon}{2}} a_{nk} \\ &+ \sum_{k \in \mathbb{N} : \|f''\|_{\tilde{C_B}[0,\infty)} \|\mathcal{L}_k((e_1 - x)^2; x, a)\|_{C[0,b]} \ge \frac{\epsilon}{2}} a_{nk}. \end{aligned}$$

Hence taking limit as $n \to \infty$, we get the desired result. □

Theorem 2.4. *Let $f \in \tilde{C}_B[0,\infty)$, we have*

$$\|\mathcal{L}_n(f;x,a) - f(x)\|_{\tilde{C}_B[0,\infty)} \leq M\omega_2(f, \sqrt{\delta_n^a}),$$

where $\delta_n^a = \|\mathcal{L}_n((e_1-x);x,a)\|_{\tilde{C}_B[0,\infty)} + \|\mathcal{L}_n((e_1-x)^2;x,a)\|_{\tilde{C}_B[0,\infty)}$.

Proof. Let $g \in W^2$, by (2.10), we have

$$\begin{aligned}\|\mathcal{L}_n(g;x,a) - g(x)\|_{\tilde{C}_B[0,\infty)} \leq\ & \|\mathcal{L}_n((e_1-x);x,a)\|_{\tilde{C}_B[0,\infty)}\|g'\|_{\tilde{C}_B[0,\infty)} \\ & + \frac{1}{2}\|\mathcal{L}_n((e_1-x)^2;x,a)\|_{\tilde{C}_B[0,\infty)}\|g''\|_{\tilde{C}_B[0,\infty)} \\ \leq\ & \delta_n^a\|g\|_{W^2}.\end{aligned}$$

For every $f \in \tilde{C}_B[0,\infty)$ and $g \in W^2$, we get

$$\begin{aligned}\|\mathcal{L}_n(f;x,a) - f(x)\|_{\tilde{C}_B[0,\infty)} \leq\ & \|\mathcal{L}_n(f;x,a) - \mathcal{L}_n(g;x,a)\|_{\tilde{C}_B[0,\infty)} \\ & + \|\mathcal{L}_n(g;x,a) - g(x)\|_{\tilde{C}_B[0,\infty)} + \|g-f\|_{\tilde{C}_B[0,\infty)} \\ \leq\ & 2\|g-f\|_{\tilde{C}_B[0,\infty)} + \|\mathcal{L}_n(g;x,a) - g(x)\|_{\tilde{C}_B[0,\infty)} \\ \leq\ & 2\|g-f\|_{\tilde{C}_B[0,\infty)} + \delta_n^a\|g\|_{W^2}.\end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2$, we obtain

$$\|\mathcal{L}_n(f;x,a) - f(x)\|_{\tilde{C}_B[0,\infty)} \leq 2K_2(f, \delta_n^a).$$

Using (1.4), we have

$$\|\mathcal{L}_n(f;x,a) - f(x)\|_{\tilde{C}_B[0,\infty)} \leq M\left\{\omega_2(f, \sqrt{\delta_n^a}) + \min(1, \delta_n^a)||f||_{\tilde{C}_B[0,\infty)}\right\}$$

From (2.4), we get $st_A - \lim_n \delta_n^a = 0$, hence $st_A - \omega_2(f, \sqrt{\delta_n^a}) = 0$. Therefore we get the rate of A-statistical convergence of the sequence of the operators $\mathcal{L}_n(f;x,a)$ defined by (1.3) to $f(x)$ in the space $\tilde{C}_B[0,\infty)$. If we take $A = I$, we obtain the ordinary rate of convergence of these operators. □

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