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# On A Subclass of Harmonic Multivalent Functions Defined by a Certain Linear Operator

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Abstract. In this paper, we introduce and study a new subclass of p-valent harmonic functions defined by modified operator and obtain the basic properties such as coefficient characterization, distortion properties, extreme points, convolution properties, convex combination and also we apply integral operator for this class.

## 1. Introduction

Harmonic mappings have found several applications in many diverse fields such as operations research, engineering, and other allied branches of applied mathematics. A continous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both $u$ and $v$ are real harmonic in $D$. In any simply connected domain $D \subset \mathbb{C}$, we can write

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ (see [5]). Recently, Jahangiri and Ahuja [9] defined the class $\mathcal{H}_p\,(p \in \mathbb{N} = \{1, 2, 3, ...\})$, consisting of all $p-$valent harmonic functions $f = h + \overline{g}$ that are sense preserving in the open unit disk

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$U = \{z : |z| < 1\}$, and $h$, $g$ are of the form:

$$h(z) = z^p + \sum_{k \geq p+1} a_k z^k, \quad g(z) = \sum_{k \geq p} b_k z^k, \ |b_p| < 1. \tag{1.2}$$

If $g \equiv 0$, the harmonic function $f = h + \overline{g}$ reduces to an analytic function $f = h$. Let $\mathcal{H}_p^-$ denote the subclass of $\mathcal{H}_p$ consisting of functions $f_n = h + \overline{g_n}$ such that $h$ and $g_n$ given by:

$$h(z) = z^p + \sum_{k \geq p+1} a_k z^k, \quad g_n(z) = (-1)^n \sum_{k \geq p} b_k z^k, \ |b_p| < 1. \tag{1.3}$$

The class $\mathcal{H}_1 = \mathcal{H}$ of harmonic univalent functions studied by Jahangiri et al. [10] (see also [6], [12]). For complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$, $j = 1, 2, ..., s$), $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, ...\}$, $\ell, \lambda \geq 0$, the operator $I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) f(z)$ is defined as follows (see El-Ashwah and Aouf [8]):

$$I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) f(z) = I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) h(z) + (-1)^n I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) g(z), \tag{1.4}$$

$$I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) h(z) = z^p + \sum_{k \geq p+1} \left( \frac{p + \lambda (k-p) + \ell}{p + \ell} \right)^n \Gamma_k(\alpha_1) a_k z^k, \tag{1.5}$$

$$I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) g(z) = (-1)^n \sum_{k \geq p} \left( \frac{p + \lambda (k-p) + \ell}{p + \ell} \right)^n \Gamma_k(\alpha_1) b_k z^k, \tag{1.6}$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-p} ... (\alpha_q)_{k-p}}{(\beta_1)_{k-p} ... (\beta_s)_{k-p} (1)_{k-p}}, \tag{1.7}$$

and $(\theta)_\nu$ is the Pochhammer symbol defined, in terms of the Gamma function $\Gamma$, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} = \begin{cases} 1, \text{ if } & (\nu = 0; \theta \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}), \\ \theta(\theta+1)...(\theta + \nu - 1) & (\nu \in \mathbb{N}; \theta \in \mathbb{C}). \end{cases} \tag{1.8}$$

For $1 < \gamma < 2$, and for all $z \in U$, let $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1, \gamma)$ denote the family of harmonic $p-$valent functions $f = h + \overline{g}$ where $h$ and $g$ of the form *(1.2)* such that

$$\Re \left\{ \frac{I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) h(z) + \overline{(-1)^n I_{p,q,s,\lambda}^{n,\ell}(\alpha_1) g(z)}}{z^p} \right\} < \gamma, \tag{1.9}$$

Let $\mathcal{H}_{p,q,s}^-(n, \ell, \lambda, \alpha_1, \gamma)$ be the subclass of $\mathcal{H}_{p,q,s}(n, \ell, \lambda, \alpha_1, \gamma)$ consisting of harmonic functions $f_n = h + \overline{g_n}$ so that $h$ and $g_n$ given by *(1.3)*.

We note that by the special choices of $\alpha_i$ $(i = 1, 2, ..., q)$ and $\beta_j$ $(j = 1, 2, ..., s), n, \ell$ and $\lambda$ our class $\mathcal{H}^{-}_{p,q,s}(n, \ell, \lambda, \alpha_1, \gamma)$ gives rise the following new subclasses of the class $\mathcal{H}_p$ :

(i) For $p = 1,\ q = s + 1, \alpha_i = 1(i = 1, ..., s + 1), \beta_j\ = 1\ (j = 1, 2, ..., s)$, we get $\mathcal{H}^{-}_{1,s+1,s}(n, \ell, \lambda, \alpha_1, \gamma) = \mathcal{H}^{-}_{1}(n, \ell, \lambda, \gamma)$

$$= \left\{ f \in \mathcal{H} :\ \Re\left\{ \frac{I^n(\lambda, \ell)h(z) + \overline{(-1)^n I^n(\lambda, \ell)g_n(z)}}{z} \right\} < \gamma \right.$$

$$\left. , 1 < \gamma < 2, \ell, \lambda \geq 0, n \in \mathbb{N}_0, z \in U \right\}$$

where $I^n(\lambda, \ell)$ is the modified Cata's operator (see [14]).

(ii) For $p = 1,\ \lambda = 1, \ell = 0,\ q = s + 1, \alpha_i = 1(i = 1, ..., s + 1), \beta_j\ = 1\ (j = 1, 2, ..., s)$, we get $\mathcal{H}^{-}_{1,s+1,s}(n, 0, 1, \alpha_1, \gamma) = \mathcal{H}^{-}_{1}(n, \gamma)$

$$= \left\{ f \in \mathcal{H} :\ \Re\left\{ \frac{D^n h(z) + \overline{(-1)^n D^n g_n(z)}}{z} \right\} < \gamma,\ 1 < \gamma < 2, n \in \mathbb{N}_0, z \in U \right\},$$

where $D^n$ is the modified Salagean operator (see [11]), the differential opertor $D^n$ was introduced by Salagean (see [15]);

(iii) For $p = 1,\ \lambda = 1, \ell = 1,\ q = s + 1, \alpha_i = 1(i = 1, ..., s + 1), \beta_j\ = 1\ (j = 1, 2, ..., s)$, we get $\mathcal{H}^{-}_{1,s+1,s}(n, 1, 1, \alpha_1, \gamma) = \mathcal{H}^{-}_{1}(n, \gamma)$

$$= \left\{ f \in \mathcal{H} :\ \Re\left\{ \frac{I^n h(z) + \overline{(-1)^n I^n g_n(z)}}{z} \right\} < \gamma,\ 1 < \gamma < 2, \right.$$

$$\left. n \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}, z \in U \right\},$$

where $I^n$ is the modified Uralegaddi-Somanatha operator (see [16]), defined as follows:

$$I^n f(z) = I^n h(z) + \overline{(-1)^n I^n g_n(z).}$$

(iv) For $p = 1,\ \lambda = 1,\ q = s + 1, \alpha_i = 1(i = 1, ..., s + 1), \beta_j\ = 1\ (j = 1, 2, ..., s)$, we get $\mathcal{H}^{-}_{1,s+1,s}(n, \ell, 1, \alpha_1, \gamma) = \mathcal{H}^{-}_{1}(n, \ell, \gamma)$

$$= \left\{ f \in \mathcal{H} :\ \Re\left\{ \frac{I^n_\ell h(z) + \overline{(-1)^n I^n_\ell g_n(z)}}{z} \right\} < \gamma,\ 1 < \gamma < 2, n \in \mathbb{R}, \ell > -1, z \in U \right\},$$

where $I^n_\ell$ is the modified Cho-Kim operator [3] (also see [4]), defined as follows:

$$I^n_\ell f(z) = I^n_\ell h(z) + \overline{(-1)^n I^n_\ell g_n(z).}$$

(v) For $p = 1,\ \ell = 0,\ q = s + 1, \alpha_i = 1(i = 1, ..., s + 1), \beta_j\ = 1\ (j = 1, 2, ..., s)$, we get $\mathcal{H}^{-}_{1,s+1,s}(n, 0, \lambda, \alpha_1, \gamma) = \mathcal{H}^{-}_{1}(n, \lambda, \gamma)$

$$= \left\{ f \in \mathcal{H} :\ \Re\left\{ \frac{D^n_\lambda h(z) + \overline{(-1)^n D^n_\lambda g_n(z)}}{z} \right\} < \gamma,\ 1 < \gamma < 2, \lambda \geq 0, n \in \mathbb{N}_0, z \in U \right\},$$

where $D_\lambda^n$ is the modified Al-Oboudi operator (see [1]), defined as follows:

$$D_\lambda^n f(z) = D_\lambda^n h(z) + (-1)^n D_\lambda^n g_n(z).$$

## 2. Coefficient Characterization

Unless otherwise mentioned, we assume throughout this article that $1 < \gamma \leq 2, \ell > -p,\ p \in \mathbb{N},\ \lambda \geq 0,\ n \in \mathbb{N}_0$ and $\Gamma_k(\alpha_1)$ is given by (1.7). In our first theorem, we introduce a sufficient condition for the coefficient bounds of harmonic functions in $\mathcal{H}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$.

**Theorem 2.1.** *Let $f = h + \overline{g}$ where $h$ and $g$ are of the form (1.2). Then $f \in \mathcal{H}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ if*

$$\sum_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)a_k|$$

$$+\sum_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)b_k| \leq \gamma - 1 \tag{2.1}$$

*where $a_p = 1$.*

*Proof.* Using the fact that $\Re\{w(z)\} < \gamma$ iff $|w(z)-1| < |w(z)-(2\gamma-1)|$, it suffices to show that

$$\left|\frac{\frac{I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)h(z)+\overline{(-1)^n I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)g(z)}}{z^p}-1}{\frac{I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)h(z)+\overline{(-1)^n I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)g(z)}}{z^p}-(2\gamma-1)}\right| < 1.$$

We have

$$\left|\frac{\frac{I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)h(z)+\overline{(-1)^n I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)g(z)}}{z^p}-1}{\frac{I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)h(z)+\overline{(-1)^n I_{p,q,s,\lambda}^{n,\ell}(\alpha_1)g(z)}}{z^p}-(2\gamma-1)}\right|$$

$$=\left|\frac{\begin{array}{c}\sum\limits_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n\Gamma_k(\alpha_1)a_k z^{k-p}\\ +(-1)^n\sum\limits_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n\Gamma_k(\alpha_1)\overline{b_k z^{k-p}}\end{array}}{\begin{array}{c}-2(\gamma-1)+\sum\limits_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n\Gamma_k(\alpha_1)a_k z^{k-p}\\ +(-1)^n\sum\limits_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n\Gamma_k(\alpha_1)\overline{b_k z^{k-p}}\end{array}}\right|$$

$$\leq \frac{\left[\sum_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)a_k|\left|z^{k-p}\right| + \sum_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)b_k|\left|z^{k-p}\right|\right]}{\left[2(\gamma-1)-\sum_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)a_k|\left|z^{k-p}\right| - \sum_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)b_k|\left|z^{k-p}\right|\right]}$$

$$< \frac{\left[\sum_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)a_k| + \sum_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)b_k|\right]}{\left[2(\gamma-1)-\sum_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)a_k| - \sum_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)b_k|\right]} \leq 1,$$

which is bounded above by 1 by using (2.1). This completes the proof of Theorem 2.1. □

**Theorem 2.2.** *Let $f_n = h + \overline{g_n}$ where $h$ and $g_n$ are of the form (1.3). Then $f_n \in \mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ if and only if*

$$\sum_{k\geq p+1}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)a_k|$$

$$+\sum_{k\geq p}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n|\Gamma_k(\alpha_1)b_k| \leq \gamma-1 \tag{2.2}$$

*where $a_p = 1$.*

*Proof.* Since $\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma) \subseteq \mathcal{H}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$, we only need to prove the "only if" part of this theorem. For functions $f_n(z)$ of the form (1.3), the condition

$$\Re\left\{\frac{I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)h(z)+\overline{(-1)^n I^{n,\ell}_{p,q,s,\lambda}(\alpha_1)g_n(z)}}{z^p}\right\} < \gamma$$

is equivalent to

$$\Re\left\{1+\sum_{k\geq p+1}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n\Gamma_k(\alpha_1)a_k z^{k-p}\right.$$

$$+(-1)^{2n}\sum_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^{n}\Gamma_k(\alpha_1)\overline{b_k z^{k-p}}\Bigg\}$$

$$\leq 1+\sum_{k\geq p+1}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^{n}|\Gamma_k(\alpha_1)a_k|\left|z^{k-p}\right|$$

$$+\sum_{k\geq p}\left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^{n}|\Gamma_k(\alpha_1)b_k|\,\overline{|z^{k-p}|}<\gamma.$$

Letting $z\longrightarrow 1^{-}$, we obtain the inequality (2.1), and so the proof of Theorem 2.2 is completed. □

**Remark 2.1.**

(i) If $p=1, q=s+1, \alpha_i=1 (i=1,...,s+1)$ and $\beta_j=1$ $(j=1,2,...,s)$ in Theorem 2.2, then we get the result obtained by Mostafa et al. [14, Theorem 2].

(ii) If $\lambda=1, \ell=0, p=1, q=s+1, \alpha_i=1 (i=1,...,s+1), \beta_j=1$ $(j=1,2,...,s)$ and $n=1$, in Theorem 2.2, then we get the result obtained by Dixit and Porwal [7, Theorem 2.1].

## 3. Extreme Points and Distortion Theorem

In the following theorem we give the extreme points of the closed convex hulls of the class $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ denoted by clco$\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$.

**Theorem 3.1.** *Let $f_n=h+\overline{g_n}$ where $h$ and $g_n$ are of the form (1.3). Then $f_n\in clco\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ if and only if*

$$f_n(z)=\sum_{k\geq p}\mu_k h_k(z)+\eta_k g_{k_n}(z), \tag{3.1}$$

*where*

$$h_p(z)=z^p, \tag{3.2}$$

$$h_k(z)=z^p+\frac{(\gamma-1)}{|\Gamma_k(\alpha_1)|}\left[\frac{(p+\ell)}{p+\lambda(k-p)+\ell}\right]^{n}z^k \quad (k\geq p+1, n\in\mathbb{N}_0) \tag{3.3}$$

*and*

$$g_{k_n}(z):=z^p+(-1)^n\frac{(\gamma-1)}{|\Gamma_k(\alpha_1)|}\left[\frac{(p+\ell)}{p+\lambda(k-p)+\ell}\right]^{n}\overline{z}^k \quad (k\geq p, n\in\mathbb{N}_0) \tag{3.4}$$

$$\mu_k,\eta_k\geq 0,\quad \mu_p=1-\sum_{k\geq p+1}\mu_k-\sum_{k\geq p}\eta_k.$$

*In particular, the extreme points of the class* $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ *are* $\{h_k(z)\}$ *and* $\{g_{k_n}(z)\}$.

*Proof.* Suppose that

$$f_n(z) = \sum_{k\geq p}\left(\mu_k h_k(z) + \eta_k g_{k_n}(z)\right)$$

$$= z^p + \sum_{k\geq p+1}\frac{(\gamma-1)}{|\Gamma_k(\alpha_1)|}\left[\frac{(p+\ell)}{p+\lambda(k-p)+\ell}\right]^n \mu_k z^k$$

$$+(-1)^n\sum_{k\geq p}\frac{(\gamma-1)}{|\Gamma_k(\alpha_1)|}\left[\frac{(p+\ell)}{p+\lambda(k-p)+\ell}\right]^n \eta_k \overline{z}^k.$$

Then

$$\sum_{k\geq p+1}\frac{[p+\lambda(k-p)+\ell]^n|\Gamma_k(\alpha_1)|}{(\gamma-1)[p+\ell]^n}\left(\frac{(\gamma-1)[p+\ell]^n}{[p+\lambda(k-p)+\ell]^n|\Gamma_k(\alpha_1)|}\mu_k\right)$$

$$+\sum_{k\geq p}\frac{[p+\lambda(k-p)+\ell]^n|\Gamma_k(\alpha_1)|}{(\gamma-1)[p+\ell]^n}\left(\frac{(\gamma-1)[p+\ell]^n}{[p+\lambda(k-p)+\ell]^n|\Gamma_k(\alpha_1)|}\eta_k\right)$$

$$= \sum_{k\geq p+1}\mu_k + \sum_{k\geq p}\eta_k = 1-\mu_p \leq 1$$

and so $f_n \in \mathrm{clco}\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$.

Conversely, if $f_n \in \mathrm{clco}\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$. Set

$$\mu_k = \frac{|\Gamma_k(\alpha_1)|[p+\lambda(k-p)+\ell]^n}{(\gamma-1)[p+\ell]^n}a_k, \quad (k\geq p+1),$$

and

$$\eta_k = \frac{|\Gamma_k(\alpha_1)|[p+\lambda(k-p)+\ell]^n}{(\gamma-1)[p+\ell]^n}b_k, \quad (k\geq p).$$

Then note that by Theorem 2.2, $0\leq\mu_k\leq 1$, $(k\geq p+1)$, and $0\leq\eta_k\leq 1$, $(k\geq p)$. Let $\mu_p = 1-\sum\limits_{k\geq p+1}\mu_k - \sum\limits_{k\geq p}\eta_k$ and $\mu_p\geq 0$.

The required representation is obtained as

$$f_n(z) = z^p + \sum_{k\geq p+1}a_k z^k + (-1)^n\sum_{k\geq p}b_k\overline{z}^k$$

$$= z^p + \sum_{k\geq p+1}\frac{(\gamma-1)[p+\ell]^n}{[p+\lambda(k-p)+\ell]^n|\Gamma_k(\alpha_1)|}\mu_k z^k$$

$$+(-1)^n\sum_{k\geq p}\frac{(\gamma-1)[p+\ell]^n}{[p+\lambda(k-p)+\ell]^n|\Gamma_k(\alpha_1)|}\eta_k\overline{z}^k$$

$$= z^p + \sum_{k\geq p+1} \left(h_k(z) - z^p\right)\mu_k + \sum_{k\geq p} \left(g_k(z) - z^p\right)\eta_k$$

$$= \left(1 - \sum_{k\geq p+1} \mu_k - \sum_{k\geq p} \eta_k\right) z^p + \sum_{k\geq p+1} h_k(z)\mu_k + \sum_{k\geq p} (g_k(z)\,\eta_k$$

$$= \sum_{k\geq p} \left(\mu_k h_k(z) + \eta_k g_k(z)\right).$$

This completes the proof of Theorem 3.1. □

The following theorem gives the distortion bounds for functions in the class $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ which yields a covering result for this class.

**Theorem 3.2.** *Let $f_n \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ with $|b_p| < \gamma - 1$. Then for $|z| = r < 1$, we have*

$$(1 - |b_p|)\, r^p - \frac{[p+\ell]^n}{|\Gamma_{p+1}(\alpha_1)|\,[p+\lambda+\ell]^n}\left\{\gamma - 1 - |b_p|\right\} r^{p+1} \leq |f_n(z)|$$

$$\leq (1 + |b_p|)\, r^p + \frac{[p+\ell]^n}{|\Gamma_{p+1}(\alpha_1)|\,[p+\lambda+\ell]^n}\left\{\gamma - 1 - |b_p|\right\} r^{p+1} \tag{3.5}$$

*Proof.* Let $f_n(z) \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$. Taking the absolute value of $f_n(z)$ we have

$$|f_n(z)| \leq (1 + |b_p|)\, r^p + \sum_{k\geq p+1} \left[|a_k| + |b_k|\right] r^k \leq (1 + |b_p|)\, r^p + r^{p+1} \sum_{k\geq p+1} \left[|a_k| + |b_k|\right]$$

$$= (1 + |b_p|)\, r^p + \frac{(\gamma-1)\,[p+\ell]^n}{[p+\lambda+\ell]^n\,|\Gamma_{p+1}(\alpha_1)|} r^{p+1} \sum_{k\geq p+1} \frac{[p+\lambda+\ell]^n\,|\Gamma_{p+1}(\alpha_1)|}{(\gamma-1)\,[p+\ell]^n} \left[|a_k| + |b_k|\right]$$

$$\leq (1 + |b_p|)\, r^p + \frac{(\gamma-1)\,[p+\ell]^n}{[p+\lambda+\ell]^n\,|\Gamma_{p+1}(\alpha_1)|} r^{p+1} \left\{ \sum_{k\geq p+1} \frac{[p+\lambda(k-p)+\ell]^n}{(\gamma-1)\,[p+\ell]^n} |\Gamma_k(\alpha_1) a_k| \right.$$

$$\left. + \sum_{k\geq p+1} \frac{[p+\lambda(k-p)+\ell]^n}{(\gamma-1)\,[p+\ell]^n} |\Gamma_k(\alpha_1) b_k| \right\}$$

$$\leq (1 + |b_p|)\, r^p + \frac{(\gamma-1)\,[p+\ell]^n}{[p+\lambda+\ell]^n\,|\Gamma_{p+1}(\alpha_1)|} \left\{1 - \frac{|b_p|}{(\gamma-1)}\right\} r^{p+1}$$

$$= (1 + |b_p|)\, r^p + \frac{[p+\ell]^n}{[p+\lambda+\ell]^n\,|\Gamma_{p+1}(\alpha_1)|} \left\{\gamma - 1 - |b_p|\right\} r^{p+1}.$$

Similarly we can prove

$$|f(z)| \geq (1-|b_p|)\, r^p - \frac{[p+\ell]^n}{|\Gamma_{p+1}(\alpha_1)|\,[p+\lambda+\ell]^n}\,\{\gamma-1-|b_p|\}\, r^{p+1}. \qquad \square$$

**Remark 2.2.** The bounds given in Theorem 3.2 for functions $f_n = h + \overline{g_n}$, where $h$ and $g_n$ are given by (1.3), also hold for functions of the form $f = h + g$, where $h$ and $g$ are given by (1.2) if the coefficient condition (2.1) is satisfied. The upper bound given for $f(z) \in \mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ is sharp and the equality occurs for the functions

$$f(z) = z^p + |b_p|\,\overline{z}^p + \frac{[p+\ell]^n}{|\Gamma_{p+1}(\alpha_1)|\,[p+\lambda+\ell]^n}\,\{\gamma-1-|b_p|\}\,\overline{z}^{p+1} \tag{3.6}$$

and

$$f(z) = z^p - |b_p|\,\overline{z}^p - \frac{[p+\ell]^n}{|\Gamma_{p+1}(\alpha_1)|\,[p+\lambda+\ell]^n}\,\{\gamma-1-|b_p|\}\, z^{p+1}, \tag{3.7}$$

showing that the bounds given in Theorem 3.2 are sharp.

## 4. Closure Property of the Class $\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$

In the next two theorems, we prove that the class $\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ is invariant under convolution and convex combinations of its members. The convolution of two harmonic functions,

$$f_n(z) = z^p + \sum_{k\geq p+1} a_k z^k + (-1)^n \sum_{k\geq p} b_k \overline{z}^k\ , \tag{4.1}$$

and

$$F_n(z) = z^p + \sum_{k\geq p+1} A_k z^k + (-1)^n \sum_{k\geq p} B_k \overline{z}^k\ , \tag{4.2}$$

is defined as

$$(f_n * F_n)(z) = (F_n * f_n)(z) = z^p + \sum_{k\geq p+1} a_k A_k z^k + (-1)^n \sum_{k\geq p} b_k B_k \overline{z}^k. \tag{4.3}$$

Using this definition, the next theorem shows that the class $\mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ is closed under convolution.

**Theorem 4.1.** *For $1 < \beta \leq \gamma \leq 2$, Let $f_n \in \mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ and $F_n \in \mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\beta)$. Then*

$$f_n * F_n \in \mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma) \subseteq \mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\beta).$$

*Proof.* Let the functions $f_n(z)$ defined by (4.1) be in $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ and the functions $F_n(z)$ defined by (4.2) be in $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\beta)$. Then the convolution $f_n * F_n$ is given by (4.3). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n(z) \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\beta)$, we note that $|A_k| < 1$ and $|B_k| < 1$. Now for the convolution function $f_n * F_n$, we obtain

$$\begin{aligned}
&\sum_{k\geq p+1} \frac{1}{\beta-1}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)a_k|\,|A_k| \\
&\quad+\sum_{k\geq p}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)b_k|\,|B_k| \\
&\leq \sum_{k\geq p+1} \frac{1}{\beta-1}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)a_k| \\
&\quad+\sum_{k\geq p}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)b_k| \\
&\leq \sum_{k\geq p+1} \frac{1}{\gamma-1}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)a_k| \\
&\quad+\sum_{k\geq p}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)b_k| \\
&\leq 1,
\end{aligned}$$

since $1 < \beta \leq \gamma \leq 2$, and $f_n \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$. □

Now, we show that the class $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ is closed under convex combination of its members.

**Theorem 4.2.** *The family* $\mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$ *is closed under convex combination.*

*Proof.* For i = 1, 2, 3, ..., suppose $f_{n_i} \in \mathcal{H}^{-}_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$, where $f_{n_i}$ is given by

$$f_{n_i}(z) = z^p + \sum_{k\geq p+1} a_{k_i} z^k + (-1)^n \sum_{k\geq p} b_{k_i} z^{-k}.$$

Then by (2.2), we have

$$\begin{aligned}
&\sum_{k\geq p+1} \frac{1}{(\gamma-1)}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)a_{k_i}| \\
&+\sum_{k\geq p} \frac{1}{(\gamma-1)}\left(\frac{p+\lambda\,(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)b_{k_i}| \leq 1.
\end{aligned} \tag{4.4}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i \leq 1$, the convex combination of $f_{n_i}$ may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z^p + \sum_{k\geq p+1} \left( \sum_{i=1}^{\infty} t_i a_{k_i} \right) z^k + (-1)^n \sum_{k\geq p} \left( \sum_{i=1}^{\infty} t_i b_{k_i} \right) z^{-k}. \tag{4.5}$$

Using the inequality (4.4), we have

$$\begin{aligned} &\sum_{k\geq p+1} \frac{1}{(\gamma-1)} \left( \frac{p+\lambda(k-p)+\ell}{p+\ell} \right)^n |\Gamma_k(\alpha_1)| \left( \sum_{i=1}^{\infty} t_i |a_{k_i}| \right) \\ &+ \sum_{k\geq p} \frac{1}{(\gamma-1)} \left( \frac{p+\lambda(k-p)+\ell}{p+\ell} \right)^n |\Gamma_k(\alpha_1)| \left( \sum_{i=1}^{\infty} t_i |b_{k_i}| \right) \\ &= \sum_{i=1}^{\infty} t_i \left( \sum_{k\geq p+1} \frac{1}{(\gamma-1)} \left( \frac{p+\lambda(k-p)+\ell}{p+\ell} \right)^n |\Gamma_k(\alpha_1)| \, |a_{k_i}| \right. \\ &\left. + \sum_{k\geq p} \frac{1}{(\gamma-1)} \left( \frac{p+\lambda(k-p)+\ell}{p+\ell} \right)^n |\Gamma_k(\alpha_1)| \, |b_{k_i}| \right) \\ &\leq \sum_{i=1}^{\infty} t_i = 1, \end{aligned}$$

which is the required coefficient condition. □

Finally, we examine the closure property of the class $\mathcal{H}^{-}_{p,q,s}(n, \ell, \lambda, \alpha_1, \gamma)$ under the generalized Bernardi-Libera-Livingston integral operator (see [2, 13]), $I_c(f)$ which is defined by,

$$I_c(f) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt, \ c > -p. \tag{4.6}$$

**Theorem 4.3.** *Let* $f_n(z) \in \mathcal{H}^{-}_{p,q,s}(n, \ell, \lambda, \alpha_1, \gamma)$. *Then*

$$I_c(f_n(z)) \in \mathcal{H}^{-}_{p,q,s}(n, \ell, \lambda, \alpha_1, \gamma).$$

*Proof.* From the representation of $I_c(f_n(z))$, it follows that

$$I_c(f_n(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} \left[ t^p + \sum_{k\geq p+1} a_k t^k + \overline{(-1)^n \sum_{k\geq p} b_k t^k} \right] dt$$

$$= z^p + \sum_{k\geq p+1} \Phi_k z^k + (-1)^n \sum_{k\geq p} \Psi_k \overline{z^k},$$

where

$$\Phi_k = \left(\frac{c+p}{c+k}\right) a_k \qquad \text{and} \qquad \Psi_k = \left(\frac{c+p}{c+k}\right) b_k \ .$$

Therefore, we have

$$\sum_{k\geq p+1} \left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)|\,|\Phi_k| + \sum_{k\geq p} \left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)|\,|\Psi_k|$$

$$= \sum_{k\geq p+1} \left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)| \left(\frac{c+p}{c+k}\right) |a_k|$$

$$+ \sum_{k\geq p} \left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)| \left(\frac{c+p}{c+k}\right) |b_k|$$

$$\leq \sum_{k\geq p+1} \left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)|\,|a_k| + \sum_{k\geq p} \left(\frac{p+\lambda(k-p)+\ell}{p+\ell}\right)^n |\Gamma_k(\alpha_1)|\,|b_k|$$

$$\leq (\gamma-1) \ \text{by (2.2).}$$

Hence by Theorem 2.2, $I_c(f_n(z)) \in \mathcal{H}^-_{p,q,s}(n,\ell,\lambda,\alpha_1,\gamma)$. □

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# References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci., **(25–28)**(2004), 1429–1436.

[2] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135**(1969), 429–446.

[3] N. E. Cho and T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., **40(3)**(2003), 399–410.

[4] N. E. Cho and H. M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, **37(1–2)**(2003), 39–49.

[5] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., **9(3)**(1984), 3–25.

[6] H. E. Darwish, A. Y. Lashin and S. M. Soileh, *Subclass of harmonic starlike functions associated with Salagean derivative*, Matematiche, **69(2)**(2014), 147–158.

[7] K. K. Dixit and S. Porwal, *A subclass of harmonic univalent functions with positive coefficients*, Tamkang J. Math., **41(3)**(2010), 261–269.

[8] R. M. El-Ashwah and M. K. Aouf, *Differential subordination and superordination for certain subclasses of p-valent functions*, Math. Comput. Modelling, **51(5–6)**(2010), 349–360.

[9] J. M. Jahangiri and O. P. Ahuja, *Multivalent harmonic starlike functions*, Ann. Univ. Marie Curie-Sklodowska Sect. A. **55**(2001), 1–13.

[10] J. M. Jahangiri, Y. C. Kim and H. M. Srivastava, *Construction of a certain class of harmonic close-to-convex functions associated with the Alexander integral transform*, Integral Transforms Spec. Funct., **14**(2003), 237–242.

[11] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijaya, *Salagean type harmonic univalent functions*, Southwest J. Pure Appl. Math., **(2)**(2002), 77–82.

[12] A. Y. Lashin, *On certain subclass of harmonic starlike functions*, Abstr. Appl. Anal., (2014), Art. ID 467929, 7 pp,

[13] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**(1965), 755–758.

[14] A. O. Mostafa, M. K. Aouf, A. Shamandy and E. A. Adwan, *Subclass of harmonic univalent functions defined by modified Cata's operator*, Acta Univ. Apulensis Math. Inform., **39**(2014), 249–261.

[15] G. S. Salagean, *Subclasses of univalent functions*, Complex Analysis—fifth Romanian-Finnish seminar, 362-372, Lecture Notes in Math. **1013**, Springer, Berlin, 1983.

[16] A. Uralegaddi and C. Somanatha, *Certain classes of univalent functions*, Current Topics in Analytic Function Theory(edited by H. M. Srivastava and S. Owa), 371-374, World Scientific Publishing Co., Singaporr, New Jersey, London, Hong Kong,1992.