KYUNGPOOK Math. J. 59(2019), 617-629
https://doi.org/10.5666/KMJ.2019.59.4.617
pISSN 1225-6951 eISSN 0454-8124

The Monoid of Linear Hypersubstitutions

Thawhat Changphas*, Bundit Pibaljommee and Klaus Denecke
Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
e-mail : thacha@kku.ac.th, banpib@kku.ac.th *and* klausdenecke@hotmail.com

Abstract. A term is called linear if each variable which occurs in the term, occurs only once. A hypersubstitution is said to be linear if it maps any operation symbol to a linear term of the same arity. Linear hypersubstitutions have some importance in Theoretical Computer Science since they preserve recognizability [7]. We show that the collection of all linear hypersubstitutions forms a monoid. Linear hypersubstitutions are used to define linear hyperidentities. The set of all linear term operations of a given algebra forms with respect to certain superposition operations a function algebra. Hypersubstitutions define endomorphisms on this function algebra.

1. Introduction

The concept of a term is one of the fundamental concepts of algebra. To define terms one needs variables and operation symbols. Let $(f_i)_{i\in I}$ be an indexed set of operation symbols. To every operation symbol f_i there belongs an integer n_i as its arity. The type of the formal language of terms is the indexed set $\tau = (n_i)_{i\in I}$ of these arities. Moreover, one needs variables from an alphabet X. Let $X_n := \{x_1, \ldots, x_n\}$ be a finite alphabet and let $X := \{x_1, \ldots, x_n, \ldots\}$ be countably infinite. Then n-ary terms of type τ are defined as follows:
Let $n \geq 1$.

(i) Every variable $x_j \in X_n$ is an n-ary term (of type τ).

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and if f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term (of type τ).

(iii) The set $W_\tau(X_n)$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii).

* Corresponding Author.
Received August 22, 2018; revised December 18, 2018; accepted December 19, 2018.
2010 Mathematics Subject Classification: 08B15, 08B25.
Key words and phrases: linear term, linear hypersubstitution, linear hyperidentity.

The set of all terms of type τ over the countably infinite alphabet X is defined by $W_\tau(X) := \bigcup_{n\geq 1} W_\tau(X_n)$.

On the set $W_\tau(X)$ of all terms of type τ an algebra $\mathcal{F}_\tau(X)$ of type τ can be defined if we consider the operation symbols f_i as n_i-ary operations $\bar{f}_i : W_\tau(X)^{n_i} \to W_\tau(X)$ with $(t_1, \ldots, t_{n_i}) \mapsto \bar{f}_i(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i})$. This is possible by (ii) of the definition of terms. Then the pair $\mathcal{F}(X) := (W_\tau(X); (\bar{f}_i)_{i\in I})$ is an algebra of type τ which is free with respect to the class $Alg(\tau)$ of all algebras of type τ, freely generated by X in the sense that for every algebra $\mathcal{A} \in Alg(\tau)$ any mapping $\varphi : X \to A$ can be extended to a homomorphism $\bar{\varphi} : \mathcal{F}_\tau(X) \to \mathcal{A}$ with $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i\in I})$. Here $f_i^{\mathcal{A}} : A^{n_i} \to A$ are the n_i-ary operations defined on A which correspond to the operation symbols f_i.

A term in which each variable occurs at most once, is said to be linear. For a formal definition of n-ary linear terms we replace (ii) in the definition of terms by a slightly different condition.

Definition 1.1. An *n-ary linear term* of type τ is defined in the following inductive way:

(i) $x_j \in X_n$ is for any $j \in \{1, \ldots, n\}$ an n-ary linear term (of type τ).

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary linear terms and if $var(t_j) \cap var(t_k) = \emptyset$ for all $1 \leq j < k \leq n_i$, then $f_i(t_1 \ldots, t_{n_i})$ is an n-ary linear term.

(iii) The set $W_\tau^{lin}(X_n)$ of all n-ary linear terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii).

The set of all linear terms of type τ over the countably infinite alphabet X is defined by $W_\tau^{lin}(X) := \bigcup_{n\geq 1} W_\tau^{lin}(X_n)$.

The set $W_\tau^{lin}(X)$ is not closed under application of the $\bar{f}_i$'s since $f_i(t_1, \ldots, t_{n_i})$ needs not to be linear if $t_1, \ldots, t_{n_i}$ are linear. But, if we define to every f_i a partial operation $\hat{f}_i$ on $W_\tau^{lin}(X)$ by

$$\hat{f}_i(t_1, \ldots, f_{n_i}) := \begin{cases} f_i(t_1, \ldots, f_{n_i}) & \text{if } var(t_j) \cap var(t_k) = \emptyset \text{ for all} \\ & 1 \leq j < k \leq n_i, \\ \text{not defined,} & \text{otherwise,} \end{cases}$$

then $\mathcal{F}_\tau^{lin}(X) := (W_\tau^{lin}(X); (\hat{f}_i)_{\imath\in I})$ is a partial algebra of type τ. The domain of $\hat{f}_i$ consists of all n_i-tuples $(t_1, \ldots, t_{n_i})$ satisfying the condition $var(t_j) \cap var(t_k) = \emptyset, 1 \leq j < k \leq n_i$.

Let $PAlg(\tau)$ be the class of all partial algebras of type τ. If $\mathcal{A}, \mathcal{B} \in PAlg(\tau)$, then a mapping $\bar{h} : A \to B$ is called a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if for all $i \in I$

Here $var(t_j)$ is the set of all variables occurring in t_j.

the following is satisfied:
if $(a_1, \ldots, a_{n_i}) \in dom f_i^{\mathcal{A}}$, then $(\bar{h}(a_1), \ldots, \bar{h}(a_{n_i})) \in dom f_i^{\mathcal{B}}$, and then

$$\bar{h}(f_i^{\mathcal{A}}(a_1, \ldots a_{n_i})) = f_i^{\mathcal{B}}(\bar{h}(a_1), \ldots, \bar{h}(a_{n_i})).$$

(Here $dom f_i^{\mathcal{A}}$ is the domain of the function $f_i^{\mathcal{A}}$). It is easy to see that $\mathcal{F}_\tau^{lin}(X)$ is a free algebra w.r.t. $PAlg(\tau)$, freely generated by X, i.e. that for all $\mathcal{A} \in PAlg(\tau)$ any mapping $h : X \to A$ can be extended to a homomorphism $\bar{h}$.

The set $W_\tau(X)$ of all terms of type τ is closed under composition, i.e. if $t_1, \ldots, t_n$ are m-ary terms of type τ and if $f_i(s_1, \ldots, s_{n_i})$ is an n-ary term of type τ, then the term

$$f_i(s_1(t_1, \ldots, t_n), \ldots, s_{n_i}(t_1, \ldots, t_n))$$

obtained by replacing the variables $x_1, \ldots, x_n$ occurring in $s_1, \ldots s_{n_i}$ by the terms $t_1, \ldots, t_{n_i}$ is a new m-ary term of type τ. This is not true for linear terms.

Pairs of linear terms are said to be linear equations and if a linear equation is satisfied in an algebra or in a variety of algebras we speak of a linear identity. Linear identities are studied in many papers, see e.g. [1, 3, 9]. Let

$$Lin_\tau(X) := \{s \approx t \mid s, t \in W_\tau^{lin}(X)\} = W_\tau^{lin}(X) \times W_\tau^{lin}(X) = (W_\tau^{lin}(X))^2$$

be the set of all linear equations of type τ. It is well-known that the set $(W_\tau(X))^2$ of all equations of type τ forms a fully invariant congruence relation on the term algebra $\mathcal{F}_\tau(X)$.

Theorem 1.2. *$Lin_\tau(X)$ is a congruence on the partial linear term algebra $\mathcal{F}_\tau^{lin}(X) = (W_\tau^{lin}(X); (\hat{f}_i)_{i\in I})$.*

Proof. Clearly, $Lin(\tau)$ is reflexive, symmetric and transitive. Assume that $s_1 \approx t_1, \ldots, s_{n_i} \approx t_{n_i} \in Lin(\tau)$, that $(s_1, \ldots, s_{n_i}) \in dom \hat{f}_i$, i.e. $var(s_j) \cap var(s_k) = \emptyset$, for all $j, k \in \{1, \ldots, n_i\}, j \neq k$ and that $(t_1, \ldots, t_{n_i}) \in dom \hat{f}_i$, i.e. $var(t_j) \cap var(t_k) = \emptyset$, for all $j, k \in \{1, \ldots, n_i\}, j \neq k$. Then $\hat{f}_i(s_1, \ldots, s_{n_i}) = f_i(s_1, \ldots, s_{n_i})$ and $\hat{f}_i(t_1, \ldots, t_{n_i}) = f_i(t_1, \ldots, t_{n_i})$. Then from $f_i(s_1, \ldots, s_{n_i}) \approx f_i(t_1, \ldots, t_{n_i}) \in W_\tau(X)^2$ (since $W_\tau(X)$ is a congruence on $\mathcal{F}_\tau(X)$) there follows $\hat{f}_i(s_1, \ldots, s_{n_i}) \approx \hat{f}_i(t_1, \ldots, t_{n_i}) \in Lin_\tau(X)$. □

Since linearity of the terms on the left-hand side and the right-hand side of a linear identity can get lost under substitution, $Lin(\tau)$ is not fully invariant.

We consider the following example. Let $\tau = (2)$ with the binary operation symbol f. Then $W_{(2)}^{lin}(X_2) = \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}$ and thus
$Lin_{(2)}(X_2) = \{x_1 \approx x_1,\ x_1 \approx x_2,\ x_1 \approx f(x_1, x_2),\ x_1 \approx f(x_2, x_1),\ x_2 \approx x_1,\ x_2 \approx x_2,\ x_2 \approx f(x_1, x_2), x_2 \approx f(x_2, x_1), f(x_1, x_2) \approx x_1,\ f(x_1, x_2) \approx x_2,\ f(x_1, x_2) \approx f(x_1, x_2),\ f(x_1, x_2) \approx f(x_2, x_1),\ f(x_2, x_1) \approx x_1,\ f(x_2, x_1) \approx x_2,\ f(x_2, x_1) \approx f(x_1, x_2),\ f(x_2, x_1) \approx f(x_2, x_1)\}$.
Substituting in $x_1 \approx f(x_1, x_2)$ for x_1 the linear term $f(x_2, x_1)$ we get $f(x_2, x_1) \approx f(f(x_2, x_1), x_2) \notin Lin_{(2)}(X_2)$.

We recall that $s \approx t$ is satisfied as an identity in the algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ of type τ if $s^{\mathcal{A}} = t^{\mathcal{A}}$, i.e. if the induced term operations are equal. In this case we write $\mathcal{A} \models s \approx t$. Let V be a variety of type τ, i.e. a model class of a set Σ of equations:

$$V = Mod\Sigma = \{\mathcal{A} \in Alg(\tau) \mid \forall s \approx t \in \Sigma(\mathcal{A} \models s \approx t)\}.$$

Definition 1.3. A variety V is said to be *linear*, if there is a set $\Sigma \subseteq Lin_\tau(X)$ of linear equations of type τ such that $V = Mod\Sigma$.

Let $Id^{lin}(V)$ be the set of all linear identities in V. Using the fact that IdV is a fully invariant congruence on the term algebra $\mathcal{F}_\tau(X)$, in a similar way as in the proof of Theorem 1.2 we show that $Id^{lin}(V)$ is a congruence on the partial linear term algebra $\mathcal{F}_\tau^{lin}(X)$. This fact gives a derivation concept for linear identities, similar to the derivation concept of identities and allows us to consider and to study a linear equational logic. Not every identity of a linear variety is linear. This means that by using the usual derivation concept of Universal Algebra from linear identities also non-linear identities can be derived.

For example, $M = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1x_2x_3x_4 \approx x_1x_3x_2x_4\}$, the variety of medial semigroups, is linear, $x_1^2x_2x_3x_4 \approx x_1^2x_3x_2x_4$ is an identity in M, but not linear.

For the basic concepts on Universal Algebra see [5] and for partial algebras see [2].

2. Linear Hypersubstitutions and Linear Hyperidentities

A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which maps every n_i-ary operation symbol f_i to an n_i-ary term of type τ is said to be a hypersubstitution (of type τ). The extension $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ is defined inductively by

(i) $\hat{\sigma}[x] := x$ for any variable $x \in X$ and

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ assumed that the results $\hat{\sigma}[t_j], 1 \leq j \leq n_i$, are already defined. Here the right-hand side means the superposition of terms.

Then a product $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ can be defined and the set $Hyp(\tau)$ of all hypersubstitutions of type τ becomes a monoid. Hypersubstitutions can be applied to identities and to algebras of type τ. Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ. Then the identity $s \approx t$ is said to be a hyperidentity satisfied in $\mathcal{A}$, if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are satisfied as identities in $\mathcal{A}$ for all $\sigma \in Hyp(\tau)$. We write $\mathcal{A} \models s \approx t$ if $s \approx t$ is satisfied as an identity in $\mathcal{A}$ and $\mathcal{A} \models_{hyp} s \approx t$ if $s \approx t$ is satisfied as a hyperidentity in $\mathcal{A}$. The algebra $\sigma[\mathcal{A}] := (A; (\sigma(f_i))_{i \in I}^{\mathcal{A}})$ is said to be derived from $\mathcal{A}$ by σ. There holds

$$\mathcal{A} \models \hat{\sigma}[s] \approx \hat{\sigma}[t] \Leftrightarrow \sigma(\mathcal{A}) \models s \approx t.$$

(This equivalence is called the 'conjugate property'). For more information on hyperidentities and hypersubstitutions see e.g. [5, 6, 8].

Now we consider hypersubstitutions mapping the operation symbols to linear terms.

Definition 2.1. A linear hypersubstitution of type τ is a hypersubstitution

$$\sigma : \{f_i \mid i \in I\} \to W_\tau^{lin}(X).$$

Let $Hyp^{lin}(\tau)$ be the set of all linear hypersubstitutions of type τ.

Example 2.2. Let $\tau = (2)$ with the binary operation symbol f. Since $W_{(2)}(X_2) = \{x_1, x_2, f(x_1, x_2), f(x_2, x_1)\}$ is the set of all binary linear terms of type (2), $Hyp^{lin}(2)$ consists of four hypersubstitutions which can be written as $\sigma_{x_1}, \sigma_{x_2}, \sigma_{f(x_1,x_2)}$ and $\sigma_{f(x_2,x_1)}$. If we apply the multiplication $\circ_h$ on this set, we obtain a monoid where the operation $\circ_h$ can be described by the following table:

$\circ_h$	$\sigma_{f(x_1,x_2)}$	$\sigma_{f(x_2,x_1)}$	σ_{x_1}	σ_{x_2}
$\sigma_{f(x_1,x_2)}$	$\sigma_{f(x_1,x_2)}$	$\sigma_{f(x_2,x_1)}$	σ_{x_1}	σ_{x_2}
$\sigma_{f(x_2,x_1)}$	$\sigma_{f(x_2,x_1)}$	$\sigma_{f(x_1,x_2)}$	σ_{x_1}	σ_{x_2}
σ_{x_1}	σ_{x_1}	σ_{x_2}	σ_{x_1}	σ_{x_2}
σ_{x_2}	σ_{x_2}	σ_{x_1}	σ_{x_1}	σ_{x_2}.

Here $\sigma_{f(x_1,x_2)}$ is the identity element of the monoid $(Hyp^{lin}(2); \circ_h, \sigma_{id})$.

In the general case it is not clear whether or not the extension $\hat{\sigma}$ of a linear hypersubstitution maps linear terms to linear terms and then it is also not clear whether or not the linear hypersubstitutions form a monoid. The first fact is well-known (see e. g. [5]).

Lemma 2.3. *For any hypersubstitution σ and any term t we have*

$$var(t) \supseteq var(\hat{\sigma}[t]).$$

Proof. We give a proof by induction on the complexity of t. If $t = x_i \in X$ is a variable, then

$$var(t) = var(x_i) = \{x_i\} = var(\hat{\sigma}[x_i]).$$

Assume that $t = f_i(t_1, \ldots, t_{n_i})$ and that $var(t_j) \supseteq var(\hat{\sigma}[t_j]), j = 1, \ldots, n_i$. Then

$$\begin{aligned} var(t) &= \bigcup_{j=1}^{n_i} var(t_j) && \supseteq \bigcup_{j=1}^{n_i} var(\hat{\sigma}[t_j]) \\ &\supseteq var(\sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])) && = var(\hat{\sigma}[t]) \end{aligned}$$

by properties of the superposition of terms. □

Then we obtain:

Lemma 2.4. *For any linear term $t = f_i(t_1, \ldots, t_{n_i})$ and any linear hypersubstitution σ we get*

$$var(\hat{\sigma}[t_j]) \cap var(\hat{\sigma}[t_k]) = \emptyset$$

for all $1 \le j < k \le n_i$.

Proof. The previous lemma gives $var(t_l) \supseteq var(\hat{\sigma}[t_l])$ for any $1 \le l \le n_i$ and thus

$$\emptyset = var(t_j) \cap var(t_k) \supseteq var(\hat{\sigma}[t_j]) \cap var(\hat{\sigma}[t_k]) = \emptyset. \quad \square$$

Lemma 2.5. *The extension of a linear hypersubstitution maps linear terms to linear terms.*

Proof. Let $t \in W_\tau^{lin}(X)$ and let $\sigma \in Hyp^{lin}(\tau)$. If $t = x_i$, then $\hat{\sigma}[x_i] = x_i$ is linear. Otherwise, i.e. if $t = f_i(t_1, \ldots, t_{n_i})$, by the previous lemma we have $var(\hat{\sigma}[t_j]) \cap var(\hat{\sigma}[t_k]) = \emptyset$ for all $1 \le j < k \le n_i$. Inductively we may assume that $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]$ are linear. Altogether, $\hat{\sigma}[t] = \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ is linear. $\square$

One more consequence is

Theorem 2.6. $(Hyp^{lin}(\tau); \circ_h, \sigma_{id})$ *is a submonoid of* $(Hyp(\tau); \circ_h, \sigma_{id})$.

Proof. If $\sigma_1, \sigma_2 \in Hyp^{lin}(\tau)$, we have to show that $\sigma_1 \circ_h \sigma_2 \in Hyp^{lin}(\tau)$. Indeed, $(\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)]$. Since $\sigma_2(f_i)$ is linear and since σ_1 is linear, by the previous lemma $\hat{\sigma}_1[\sigma_2(f_i)]$ is linear. The identity hypersubstitution σ_{id} is linear since $\sigma_{id}(f_i) = f_i(x_1, \ldots x_{n_i})$ is linear. $\square$

Now we will give two more examples of linear hypersubstitions.

Example 2.7. Let $\tau = (4, 2)$ be the type with a quaternary operation symbol g and a binary operation symbol f. Let σ be the hypersubstitution which maps g to $f(f(x_1, x_3), f(x_2, x_4))$ and f to $f(x_1, x_2)$. Then $\sigma \in Hyp^{lin}(4, 2)$.

Example 2.8. Let $\tau = (4, 2, 1)$ be the type with a quaternary operation symbol g, a binary operation symbol f and a unary operation symbol h and let σ be the hypersubstitution which maps g to $f(f(x_1, x_3), h(x_4))$, which maps f to $f(x_1, x_2)$ and h to $h(x_1)$. Obviously, $\sigma \in Hyp^{lin}(4, 2, 1)$.

The second example shows that there are linear hypersubstitutions which map n_i-ary operation symbols to terms, which do not use all variables from $\{x_1, \ldots, x_{n_i}\}$.

As for an arbitrary mapping also for the extension of a hypersubstitution one may consider the kernel of this mapping (see e.g. [8]). In particular, by Lemma 2.4, this can be done for a linear hypersubstitution σ since its extension $\hat{\sigma}$ maps $W_\tau^{lin}(X)$ to $W_\tau^{lin}(X)$.

Definition 2.9. Let $\sigma \in Hyp^{lin}(\tau)$. Then for $s, t \in W_\tau^{lin}(X)$

$$Ker\sigma := \{(s, t) \mid \hat{\sigma}[s] = \hat{\sigma}[t]\}$$

is said to be the *kernel* of the linear hypersubstitution σ.

For $\sigma \in Hyp(\tau)$ the kernel $Ker\sigma$ is a fully invariant congruence on the absolutely free algebra $\mathcal{F}_\tau(X) = (W_\tau(X); (\bar{f}_i)_{i \in I})$. For linear hypersubstitutions we get

Proposition 2.10. *Let* $\sigma \in Hyp^{lin}(\tau)$. *Then* $Ker\sigma$ *is a congruence on the partial algebra* $\mathcal{F}_\tau^{lin}(X) = (W_\tau^{lin}(X); (\hat{f}_i)_{i \in I})$.

Proof. $Ker\sigma$ is clearly an equivalence relation on $W_\tau^{lin}(X)$. For $i \in I$ let $(s_1, t_1) \in Ker\sigma, \ldots, (s_{n_i}, t_{n_i}) \in Ker\sigma$ and thus

$$\hat{\sigma}[s_1] = \hat{\sigma}[t_1], \ldots, \hat{\sigma}[s_{n_i}] = \hat{\sigma}[t_{n_i}]. \qquad (*)$$

Assume that $\hat{f}_i(s_1, \ldots, s_{n_i})$ and $\hat{f}_i(t_1, \ldots, t_{n_i})$ exist. Then

$$\hat{f}_i(s_1, \ldots, s_{n_i}) = f_i(s_1, \ldots, s_{n_i}) \text{ and } \hat{f}_i(t_1, \ldots, t_{n_i}) = f_i(t_1, \ldots, t_{n_i}) \qquad (**)$$

and by Lemma 2.3, $var(\hat{\sigma}[s_j]) \cap var(\hat{\sigma}[s_k]) = \emptyset$ and $var(\hat{\sigma}[t_j]) \cap var(\hat{\sigma}[t_k]) = \emptyset$ for all $1 \leq j < k \leq n_i$. Applying the linear term $\sigma(f_i)$ on $(*)$ we obtain

$$\sigma(f_i)(\hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_{n_i}]) = \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]).$$

Both sides of this equation are linear terms and by the definition of the extension and we obtain

$$\hat{\sigma}[f_i(s_1, \ldots, s_{n_i})] = \hat{\sigma}[f_i(t_1, \ldots, t_{n_i})].$$

By $(**)$ we have

$$\hat{\sigma}[\hat{f}_i(s_1, \ldots, s_{n_i})] = \hat{\sigma}[\hat{f}_i(t_1, \ldots, t_{n_i})]$$

and then

$$(\hat{f}_i(s_1, \ldots, s_{n_i}), \hat{f}_i(t_1, \ldots, t_{n_i})) \in Ker\sigma.$$ □

We remark that Theorem 2.6 has important consequences. Let $\mathcal{M}$ be an arbitrary submonoid of $\mathcal{H}yp(\tau)$ with the universe M. A variety V of algebras of type τ is said to be M-solid if for every $s \approx t \in IdV$ and every $\sigma \in M, \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$. The collection of all M-solid varieties of type τ forms a complete sublattice of the lattice of all varieties of type τ. Since the set of all linear identities of a variety $V$is not closed under substitution of terms, i.e. is not an equational theory, we cannot apply the general theory of M-hyperidentities and M-solid varieties. Therefore, not all of our next definitions follow the general theory.

Definition 2.11. Let $\mathcal{A}$ be an algebra and let K be a class of algebras, both of type τ. A linear identity $s \approx t$ is said to be a *linear hyperidentity* in $\mathcal{A}$ (respectively, in K) if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{A}$ (respectively, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdK$) for every $\sigma \in Hyp^{lin}(\tau)$. In this case we write $\mathcal{A} \models_{lin} s \approx t$ in the first, and $K \models_{lin} s \approx t$ in the second case.

We define an operator χ_{lin}^E by

$$\chi_{lin}^E[s \approx t] = \{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid \sigma \in M\}.$$

This extends, additively, to sets of identities, so that for any set Σ of linear identities we set

$$\chi_{lin}^E[\Sigma] = \bigcup \{\chi_{lin}^E[s \approx t] \mid s \approx t \in \Sigma \text{ and } s \approx t \in (W_\tau^{lin}(X))^2\}.$$

Using derived algebras we define now an operator χ^A_{lin} on the set $Alg(\tau)$, first on individual algebras and then on classes K of algebras, by

$$\chi^A_{lin}[\mathcal{A}] = \{\sigma(\mathcal{A}) \mid \sigma \in Hyp^{lin}(\tau)\} \text{ and } \chi^A_{lin}[K] = \bigcup\{\chi^A_{lin}[\mathcal{A}] \mid \mathcal{A} \in \mathcal{K}\}.$$

The identity hypersubstitution is linear. Using this, one shows that both operators are extensive, i.e. $\Sigma \subseteq \chi^E_{lin}[\Sigma]$ and $K \subseteq \chi^A_{lin}[K]$. Monotonicity of both operators is a consequence of their definition and idempotency follows from the fact that $Hyp^{lin}(\tau)$ forms a monoid. By definition, both operators are additive. From the conjugate property there follows:

$$\chi^A_{lin}[\mathcal{A}] \models s \approx t \Leftrightarrow \mathcal{A} \models \chi^E_{lin}[s \approx t].$$

Altogether, as for arbitrary monoids of hypersubstitutions, also for linear hypersubstitutions, we have:

Proposition 2.12. *Let τ be a fixed type. The two operators χ^E_{lin} and χ^A_{lin} are additive closure operators and are conjugate with respect to the relation $\models \subseteq Alg(\tau) \times (W^{lin}_\tau(X))^2$ of satisfaction.*

The sets of all fixed points $\{K \mid \chi^A_{lin}[K] = K, K \subseteq Alg(\tau)\}$ and $\{\Sigma \mid \chi^K_{lin}[\Sigma], \Sigma \subseteq W^{lin}_\tau(X))^2\}$ form complete sublattices of the power set lattices $\mathcal{P}(Alg(\tau))$ and $\mathcal{P}((W^{lin}_\tau(X))^2)$, respectively.

The relation $\models$ of satisfaction of an equation as linear identity of an algebra $\mathcal{A}$ defines the Galois connections (Id, Mod) and (Mod, Id).

The relation of linear hypersatisfaction induces a new Galois connection $(H_{lin}Id, H_{lin}Mod)$, defined on classes K and sets Σ of linear equations as follows:

$$H_{lin}IdK = \{s \approx t \in (W^{lin}_\tau(X))^2 \mid s \approx t \text{ is a linear hyperidentity in } \mathcal{A} \text{ for all } \mathcal{A} \text{ in } K\},$$

$$H_{lin}Mod\Sigma = \{\mathcal{A} \in Alg(\tau) \mid \text{all identities in } \Sigma \text{ are linear hyperidentities of } \mathcal{A}\}.$$

Sets of equations of the form $H_{lin}IdK$ are called linear hyperequational theories and classes of algebras of the same type having the form $H_{lin}Mod\Sigma$ are called linear hyperequational classes. As a property of a Galois connection, the combinations $H_{lin}IdH_{lin}Mod$ and $H_{lin}ModH_{lin}Id$ are closure operators and their fixed points form two complete sublattices of the power set lattices $\mathcal{P}(W_\tau(X))^2$ and $\mathcal{P}(Alg(\tau))$. Now we may apply the general theory of conjugate pairs of additive closure operators [11], (see [8]).

Theorem 2.13. *For any variety V of type τ, the following conditions are equivalent:*

(i) $V = H_{lin}ModH_{lin}IdV$.

(ii) $\chi^A_{lin}[V] = V$.

(iii) $IdV = H_{lin}IdV$.

(iv) $\chi^E_{lin}[IdV] = IdV$.

And dually, for any equational theory Σ *of type* τ*, the following conditions are equivalent:*

(i') $\Sigma = H_{lin}IdH_{lin}Mod\Sigma$.

(ii') $\chi^E_{lin}[\Sigma] = \Sigma$.

(iii') $Mod\Sigma = H_{lin}Mod\Sigma$.

(iv') $\chi^A_{lin}[Mod\Sigma] = Mod\Sigma$.

From the general theory of conjugate pairs of additive closure operators for $K \subseteq Alg(\tau)$ and $\Sigma \subseteq (W^{lin}_\tau(X))^2$ one obtains also the following conditions:

(i) $\chi^A_{lin}[K] \subseteq ModIdK \Leftrightarrow ModIdK = H_{lin}ModH_{lin}IdK \Leftrightarrow \chi^A_{lin}[ModIdK] = ModIdK$.

(ii) $\chi^E_{lin}[\Sigma] \subseteq IdMod\Sigma \Leftrightarrow IdMod\Sigma = H_{lin}IdH_{lin}Mod\Sigma \Leftrightarrow \chi^E_{lin}[IdMod\Sigma] = IdMod\Sigma$.

The second proposition can be used if a variety $V = Mod\Sigma$ is defined by a linear equational basis Σ. If we want to check, whether IdV is a fixed point under the operator χ^E_{lin}, it is enough to apply all linear hypersubstitutions to the equational basis Σ: $\chi^E_{lin}[IdV] = IdV \Leftrightarrow \chi^E_{lin}[\Sigma] \subseteq IdV$ if $V = Mod\Sigma$.

For any subset of equations and for any monoid $\mathcal{M}$ of hypersubstitutions we defined a variety V to be M-solid, if V is a fixed point under the corresponding operator χ^A_M. Because of property (ii) we may define

Definition 2.14. A variety V of type τ is said to be *linear-solid* if it is linear and if the defining linear identities are linear hyperidentities.

Not every identity in a linear-solid variety V must be a linear hyperidentity. We consider the following example:

Example 2.15. Let $\tau = (2)$ and let $M = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3, x_1x_2x_3x_4 \approx x_1x_3x_2x_4\}$ be the variety of medial semigroups. If we apply the four linear hypersubstitutions $\sigma_{id}, \sigma_{x_1}, \sigma_{x_2}, \sigma_{f(x_2,x_1)}$ to each of the both defining identities of variety M, we get identities satisfied in M. Therefore, M is linear-solid, but the equation $x_1^2x_2x_3x_4 \approx x_1^2x_3x_2x_4$ is an identity in M, but not linear, therefore it cannot be a linear hyperidentity since $\mathcal{H}yp(\tau)$ as a monoid contains the identity hypersubstitution and thus each linear hyperidentity is a linear identity.

3. The Monoid of Linear Hypersubstitutions of Type $\tau_n^{|I|}$

In this section we consider monoids of linear hypersubstitutions when f_i is n-ary, $n \geq 2$, for every $i \in I$, i.e. if the type contains $|I|$ operation symbols of the same arity n, $n \geq 2$. Such types will be denoted by $\tau_n^{|I|}, n \geq 2$. We recall that projection hypersubstitutions map each operation symbol to a variable.

To determine all linear hypersubstitutions of this type, we describe the form of n-ary terms. We denote the number of occurences of operation symbols in a term t by $op(t)$, i.e. $op(t)$ is defined inductively by

(i) $op(t) = 0$ if $t = x_i \in X_n$ is a variable and

(ii) $op(f_i(t_1, \ldots, t_{n_i})) = \sum\limits_{j=1}^{n_i} op(t_j) + 1$ if $t = f_i(t_1, \ldots, t_{n_i})$ is a composed n-ary term.

Proposition 3.1. *Let t be an n-ary linear term of type $\tau_n^{|I|}, n \geq 2$. Then $op(t) \leq 1$.*

Proof. If $t = x_i \in X_n$, then $op(t) = 0$. Let $t = f_i(t_1, \ldots, t_n)$ be an n-ary linear term. If there were a number $k, 1 \leq k \leq n_i$ with $op(t_k) = 1$, then $|var(t_k)| \geq 2$ since the arity of all operation symbols in t_k is greater than 1 and t_k is linear. Assume that $x_m \neq x_l \in var(t_k)$. These variables cannot occur in another subterm of t. But in t there occur $n - 1$ pairwise different variables which are different from x_m and x_l. This contradiction shows that $op(t_k) = 0$ for all $1 \leq k \leq n_i$ and thus $op(t) = 1$. □

This observation allows us to describe the form of all n-ary linear terms and we obtain

$$W^{lin}_{\tau_n^{|I|}}(X_n) = \{f_i(x_{s(1)}, \ldots, x_{s(n)}) \mid i \in I \text{ and } s \text{ is a permutation on} \{1, \ldots, n\}\}.$$

Then it is also clear that there are precisely $|I|n! + n$- many n-ary linear terms.

Since every linear hypersubstitution maps the operation symbol f_i to an n-ary term we have a full description of $Hyp^{lin}(\tau_n^{|I|})$. Any linear hypersubstitution maps the operation symbol f_i to a variable $x_j \in X_n$ or to a term of the form $f_j(x_{s(1)}, \ldots, x_{s(n)})$ where s is a permutation on $\{1, \ldots, s_n\}$.

For the product of two linear hypersubstitutions σ_1 and σ_2 there are precisely the following three possibilities:

1. $\sigma_2(f_i) = x_k \in X_n$: Then
$$\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1[\sigma_2(f_i)] = \hat{\sigma}_1[x_k] = x_k = \sigma_{x_k}(f_i).$$

2. $\sigma_2(f_i) = f_j(x_{s(1)}, \ldots, x_{s(n)})$ and $\sigma_1(f_j) = x_k$. Then $(\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)] = \hat{\sigma}_1[f_j(x_{s(1)}, \ldots, x_{s(n)})] = \sigma_1(f_j)(x_{s(1)}, \ldots, x_{s(n)})$ $= x_{s(k)} = \sigma_{x_{s(k)}}(f_i)$.

3. $\sigma_2(f_i) = f_j(x_{s(1)}, \ldots, x_{s(n)})$ and $\sigma_1(f_j) = f_k(x_{s'(1)}, \ldots, x_{s'(n)})$. Then $(\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)] = \hat{\sigma}_1[f_j(x_{s(1)}, \ldots, x_{s(n)})] = \sigma_1(f_j)(x_{s(1)}, \ldots, x_{s(n)}) =$ $f_k(x_{s'(1)}, \ldots, x_{s'(n)})(x_{s(1)}, \ldots, x_{s(n)}) = f_k(x_{(s \circ s')(1)}, \ldots, x_{(s \circ s')(n)})$.

4. Interpretation of Linear Hypersubstitutions on Single Algebras

Term operations induced by linear terms are defined in the usual way. Let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ and let $W_\tau^{lin}(X)$ be the set of all linear terms of type τ. Then the set $(W_\tau^{lin}(X))^{\mathcal{A}}$ of all linear term operations of $\mathcal{A}$ is defined as follows:

(i) $x_i^{\mathcal{A}} := e_i^{n,\mathcal{A}}$,

(ii) if $f_i(t_1, \ldots, t_{n_i}) \in W_\tau^{lin}(X)$ and assumed that $t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}}$ are defined, then

$$(f_i(t_1, \ldots, t_{n_i}))^{\mathcal{A}} = f_i^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}}).$$

Here the right-hand side is the superposition of $f_i^{\mathcal{A}}$ and $t_1^{\mathcal{A}}, \ldots, t_{n_i}^{\mathcal{A}}$.

Let $\mathcal{A}$ be an algebra of type τ and let $W_\tau(X)$ be the set of all terms of type τ. Then the set $(W_\tau(X))^{\mathcal{A}}$ of all term operations induced by terms of type τ is closed under some superposition operations. These operations can be defined on the set

Let $O(A) := \bigcup_{n \geq 1} O^n(A)$ be the set of all operations on A. Here $O^n(A)$ is the set of all n-ary operations defined on A. Then $(W_\tau^{lin}(X))^{\mathcal{A}} \subset O(A)$. The set $O(A)$ is closed under some superposition operations which were introduced by A. I. Mal'cev (see [10]). Here we will use Mal'cev's original notation in spite of the fact that the letter τ was already used for the type of a language or an algebra.

Let $f \in O^n(A)$ and $g \in O^m(A)$. Then

$$\begin{array}{rcl}
(f * g)(x_1, \ldots, x_{m+n-1}) & := & f(g(x_1, \ldots, x_{lin}), x_{m+1}, \ldots, x_{m+n-1}), \\
(\tau f)(x_1, \ldots, x_n) & := & f(x_2, x_1, x_3, \ldots, x_n), \\
(\zeta f)(x_1, \ldots, x_n) & := & f(x_2, x_3, \ldots, x_n, x_1), \\
(\Delta f)(x_1, \ldots, x_{n-1}) & := & f(x_1, x_1, \ldots, x_{n-1}), \\
(\nabla f)(x_1, x_2, \ldots, x_{n+1}) & := & f(x_2, \ldots, x_{n+1}), \\
 & & \text{if } f \in O^n(A) \text{with } n > 1 \text{ and} \\
(\tau f)(x_1) = (\zeta f)(x_1) & = & (\Delta f)(x_1) = (\nabla f)(x_1) = f(x_1)
\end{array}$$

if f is a unary function.

The algebra $((W_\tau(X))^{\mathcal{A}}; *, \zeta, \tau, \Delta, \nabla, e_1^{2,A})$ is said to be the clone of term operations of the algebra $\mathcal{A}$.

Now we ask for the algebraic structure of $((W_\tau^{lin}(X))^{\mathcal{A}}$. The answer was given by Couceiro and Lehtonen in [4].

Theorem 4.1.([4]) *Let $\mathcal{A}$ be an algebra of type τ and let $(W_\tau^{lin}(X))^{\mathcal{A}}$ be the set of all linear term operations induced on $\mathcal{A}$, i.e. term operations induced by linear terms on $\mathcal{A}$. Then $(W_\tau^{lin}(X))^{\mathcal{A}}$ is closed under the operations $*, \zeta, \tau$ and ∇ and contains all projections.*

Couceiro and Lehtonen proved in [4] also that the subalgebra $((W_\tau^{lin}(X))^{\mathcal{A}};\zeta,\tau,\nabla,*)$ of $((W_\tau(X))^{\mathcal{A}};\zeta,\tau,\nabla,*)$ is generated by the set $\{f_i^{\mathcal{A}} \mid i\in I\}\cup J_A$, where J_A is the set of all projections.

Theorem 4.2. *Let $\mathcal{A}$ be any algebra of type τ. If $\sigma\in Hyp^{lin}(\tau)$ satisfies $\hat{\sigma}[Id\mathcal{A}]\subseteq Id\mathcal{A}$. Then*

$$\Phi:(W_\tau^{lin}(X))^{\mathcal{A}}\to(W_\tau^{lin}(X))^{\mathcal{A}} \text{ defined by } t^{\mathcal{A}}\to(\hat{\sigma}[t])^{\mathcal{A}}$$

is an endomorphism of $((W_\tau^{lin}(X))^{\mathcal{A}};*,\zeta,\tau,\nabla)$.

Proof. 1. Φ is well-defined: assume that $t_1^{\mathcal{A}}=t_2^{\mathcal{A}}, t_1,t_2\in(W_\tau^{lin}(X))^{\mathcal{A}}$ There follows $t_1\approx t_2\in Id\mathcal{A}\cap(W_\tau^{lin}(X))^2$ and then

$$\hat{\sigma}[t_1]\approx\hat{\sigma}[t_2]\in\hat{\sigma}[Id\mathcal{A}\cap(W_\tau^{lin}(X))^2]=\hat{\sigma}[Id^{lin}\mathcal{A}].$$

This means $(\hat{\sigma}[t_1])^{\mathcal{A}}=(\hat{\sigma}[t_2])^{\mathcal{A}}$.
2. Now we show the compatibility of Φ with the operations ζ and τ. Let $t^{\mathcal{A}}\in(W_\tau^{lin}(X))^{\mathcal{A}}$, then $\zeta(t^{\mathcal{A}})=(t(x_2,x_3,\ldots,x_n,x_1))^{\mathcal{A}}$ and

$$\Phi(\zeta(t^{\mathcal{A}}))=(\hat{\sigma}[t(x_2,x_3,\ldots,x_n,x_1)])^{\mathcal{A}}=(\hat{\sigma}[t](x_2,x_3,\ldots,x_n,x_1))^{\mathcal{A}}.$$

On the other hand we have

$$\Phi(t^{\mathcal{A}})=(\hat{\sigma}[t])^{\mathcal{A}} \text{ and } \zeta(\Phi(t^{\mathcal{A}}))=(\hat{\sigma}[t](x_2,x_3,\ldots,x_n,x_1))^{\mathcal{A}}.$$

This shows $\Phi(\zeta(t^{\mathcal{A}}))=\zeta(\Phi(t^{\mathcal{A}}))$. In the same way we obtain $\Phi(\tau(t^{\mathcal{A}}))=\tau(\Phi(t^{\mathcal{A}}))$.
3. $\nabla(t^{\mathcal{A}})=(t(x_2,x_3,\ldots,x_{n+1}))^{\mathcal{A}}$ and

$$\begin{aligned}\Phi(\nabla(t^{\mathcal{A}}))&=(\hat{\sigma}[t(x_2,x_3,\ldots,x_{n+1})])^{\mathcal{A}}\\&=(\hat{\sigma}[t](x_2,x_3,\ldots,x_{n+1}))^{\mathcal{A}}\\&=\nabla(\Phi(t^{\mathcal{A}})).\end{aligned}$$

4. For the binary operation $*$ we get

$$\begin{aligned}\Phi(t^{\mathcal{A}}*s^{\mathcal{A}})&=\Phi(t(s,x_{m+1},\ldots,x_{m+n-1})^{\mathcal{A}}))\\&=(\hat{\sigma}[t(s,x_{m+1},\ldots,x_{m+n-1})])^{\mathcal{A}}))\\&=(\hat{\sigma}[t](\hat{\sigma}[s],x_{m+1},\ldots,x_{m+n-1}))^{\mathcal{A}}\\&=(\hat{\sigma}[t])^{\mathcal{A}}*(\hat{\sigma}[s])^{\mathcal{A}}\\&=\Phi(t^{\mathcal{A}})*\Phi(s^{\mathcal{A}}).\end{aligned}$$ □

Because of $e_2^{2,\mathcal{A}}=\tau(e_1^{2,\mathcal{A}})$ and
$(\nabla f)(x_1,\ldots,x_{n+1})$
$=\quad(f*e_2^{2,\mathcal{A}})(x_1,\ldots,x_{n+t})$
$=\quad f(e_2^{2,\mathcal{A}}(x_1,x_2),x_3,\ldots,x_{n+1})$ $(\nabla f)=f*e_2^{2,\mathcal{A}}=f*\tau(e_1^{2,\mathcal{A}})$, the algebra
$=\quad f(x_2,x_3,\ldots,x_{n+1})$, i.e.
$((W_\tau^{lin})^{\mathcal{A}};*,\zeta,\tau,\nabla)$ is rationally equivalent to $((W_\tau^{lin})^{\mathcal{A}};*,\zeta,\tau,e_2^{2,\mathcal{A}})$ and the following theorem holds also for this algebra.

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