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The Zero-divisor Graph of the Ring of Integers Modulo n

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Abstract. Let $\mathbb{Z}_n$ be the ring of integers modulo n and $\Gamma(\mathbb{Z}_n)$ the zero-divisor graph of $\mathbb{Z}_n$. In this paper, we study some properties of $\Gamma(\mathbb{Z}_n)$. More precisely, we completely characterize the diameter and the girth of $\Gamma(\mathbb{Z}_n)$. We also calculate the chromatic number of $\Gamma(\mathbb{Z}_n)$.

1. Introduction

1.1. Preliminaries

In this subsection, we review some concepts from basic graph theory. Let G be a (undirected) graph. Recall that G is *connected* if there is a path between any two distinct vertices of G. The graph G is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices is denoted by K_n. The graph G is a *complete bipartite graph* if G can be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton set, then we call

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G a *star*. We denote the complete bipartite graph by $K_{m,n}$, where m and n are the cardinal numbers of A and B, respectively. For vertices a and b in G, $d(a,b)$ denotes the length of the shortest path from a to b. If there is no such path, then $d(a,b)$ is defined to be ∞; and $d(a,a)$ is defined to be zero. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the supremum of $\{d(a,b) \mid a \text{ and } b \text{ are vertices of } G\}$. The *girth* of G, denoted by $\mathrm{g}(G)$, is defined as the length of the shortest cycle in G. If G contains no cycles, then $\mathrm{g}(G)$ is defined to be ∞. A subgraph H of G is an *induced subgraph* of G if two vertices of H are adjacent in H if and only if they are adjacent in G. The *chromatic number* of G is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, and is denoted by $\chi(G)$. A *clique* C in G is a subset of the vertex set of G such that the induced subgraph of G by C is a complete graph. The *clique number* of G, denoted by $cl(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$. If $K_n \subseteq G$ for all integers $n \geq 1$, then $cl(G)$ is defined to be ∞. A *maximal clique* in G is a clique that cannot be extended by including one more adjacent vertex. It is easy to see that $\chi(G) \geq cl(G)$.

1.2. The Zero-divisor Graph of a Commutative Ring

Let R be a commutative ring with identity and $\mathrm{Z}(R)$ the set of nonzero zero-divisors of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the simple graph with vertex set $\mathrm{Z}(R)$, and for distinct $a, b \in \mathrm{Z}(R)$, a and b are adjacent if and only if $ab = 0$. Clearly, $\Gamma(R)$ is the null graph if and only if R is an integral domain.

In [4], Beck first introduced the concept of the zero-divisor graphs of commutative rings and in [3], Anderson and Nazeer continued the study. In their papers, all elements of R are vertices of the graph and they were mainly interested in colorings. In [2], Anderson and Livingston gave the present definition of $\Gamma(R)$ in order to emphasize the study of the interplay between graph-theoretic properties of $\Gamma(R)$ and ring-theoretic properties of R. It was shown that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$ [2, Theorem 2.3]; and $\mathrm{g}(\Gamma(R)) \leq 4$ [5, (1.4)].

For more on the zero-divisor graph of a commutative ring, the readers can refer to a survey article [1].

Let $\mathbb{Z}_n$ be the ring of integers modulo n. The purpose of this paper is to study some properties of the zero-divisor graph of $\mathbb{Z}_n$. If n is a prime number, then $\mathbb{Z}_n$ has no zero-divisors; so $\Gamma(\mathbb{Z}_n)$ is the null graph. Hence in this paper, we only consider the case that n is a composite. In Section 2, we completely characterize the diameter and the girth of $\Gamma(\mathbb{Z}_n)$. In Section 3, we calculate the chromatic number of $\Gamma(\mathbb{Z}_n)$. Note that all figures are drawn via website http://graphonline.ru/en/.

2. The Diameter and the Girth of $\Gamma(\mathbb{Z}_n)$

Our first result in this section is the complete characterization of the diameter of $\Gamma(\mathbb{Z}_n)$.

Theorem 2.1. *The following statements hold.*

(1) $\mathrm{diam}(\Gamma(\mathbb{Z}_n)) = 0$ *if and only if* $n = 4$.

(2) $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 1$ *if and only if* $n = p^2$ *for some prime* $p \geq 3$.

(3) $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 2$ *if and only if* $n = p^r$ *for some prime* p *and some integer* $r \geq 3$, *or* $n = pq$ *for some distinct primes* p *and* q.

(4) $\operatorname{diam}(\Gamma(\mathbb{Z}_n)) = 3$ *if and only if* $n = pqr$ *for some distinct primes* p, q *and some integer* $r \geq 2$.

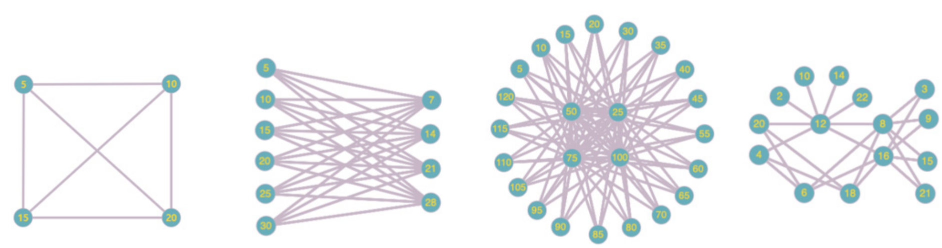
Proof. (1) If $n = 4$, then $\mathrm{Z}(\mathbb{Z}_4) = \{2\}$; so $\operatorname{diam}(\Gamma(\mathbb{Z}_4)) = 0$.

(2) If $n = p^2$ for some prime $p \geq 3$, then $\mathrm{Z}(\mathbb{Z}_{p^2}) = \{p, 2p, \ldots, (p-1)p\}$; so the product of any two elements of $\mathrm{Z}(\mathbb{Z}_{p^2})$ is zero. Hence $\Gamma(\mathbb{Z}_{p^2})$ is the complete graph K_{p-1}. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_{p^2})) = 1$.

(3) If $n = p^r$ for some prime p and some integer $r \geq 3$, then $\mathrm{Z}(\mathbb{Z}_{p^r}) = \{p, 2p, \ldots, (p^{r-1}-1)p\}$; so $ap^{r-1} = 0$ for all $a \in \mathrm{Z}(\mathbb{Z}_{p^r})$. Hence $\operatorname{diam}(\Gamma(\mathbb{Z}_{p^r})) \leq 2$. Note that $p((p^{r-1}-1)p) \neq 0$. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_{p^r})) = 2$.

If $n = pq$ for some distinct primes p and q, then $\mathrm{Z}(\mathbb{Z}_{pq}) = \{p, 2p, \ldots, (q-1)p, q, 2q, \ldots, (p-1)q\}$; so $(ip)(jq) = 0$ for all $i = 1, \ldots, q-1$ and $j = 1, \ldots, p-1$. Note that for any $a, b \in \{p, 2p, \ldots, (q-1)p\}$ and $c, d \in \{q, 2q, \ldots, (p-1)q\}$, $ab \neq 0$ and $cd \neq 0$. Hence $\Gamma(\mathbb{Z}_{pq})$ is the complete bipartite graph $K_{p-1,q-1}$. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_{pq})) = 2$.

(4) Suppose that $n = pqr$ for some distinct primes p, q and some integer $r \geq 2$. Then $p, q \in \mathrm{Z}(\mathbb{Z}_{pqr})$ with $pq \neq 0$; so $\operatorname{diam}(\Gamma(\mathbb{Z}_{pqr})) \geq 2$. If there exists an element $a \in \mathrm{Z}(\mathbb{Z}_{pqr})$ such that $p \sim a \sim q$ is a path, then a is a nonzero multiple of pr and qr; so a is nonzero a multiple of pqr. This is a contradiction. Hence $\operatorname{diam}(\Gamma(\mathbb{Z}_{pqr})) \geq 3$. Thus $\operatorname{diam}(\Gamma(\mathbb{Z}_{pqr})) = 3$ [2, Theorem 2.3]. □



$\operatorname{diam}(\Gamma(\mathbb{Z}_{25})) = 1$ $\operatorname{diam}(\Gamma(\mathbb{Z}_{35})) = 2$ $\operatorname{diam}(\Gamma(\mathbb{Z}_{125})) = 2$ $\operatorname{diam}(\Gamma(\mathbb{Z}_{24})) = 3$

Figure 1: The diameters of some zero-divisor graphs

We next study the girth of $\Gamma(\mathbb{Z}_n)$.

Lemma 2.2. *If* $\mathrm{g}(\Gamma(\mathbb{Z}_n)) = 3$, *then* $\mathrm{g}(\Gamma(\mathbb{Z}_{mn})) = 3$ *for all integers* $m \geq 1$.

Proof. Note that if $a \sim b \sim c \sim a$ is a cycle in $\Gamma(\mathbb{Z}_n)$, then $am \sim bm \sim cm \sim am$ is a cycle in $\Gamma(\mathbb{Z}_{mn})$. Thus $\mathrm{g}(\Gamma(\mathbb{Z}_{mn})) = 3$. □

The next example shows that Lemma 2.2 cannot be extended to the case of girth 4.

Example 2.3.

(1) Note that $g(\Gamma(\mathbb{Z}_{12})) = 4$. In fact, $3 \sim 4 \sim 6 \sim 8 \sim 3$ is a cycle of length 4 in $\Gamma(\mathbb{Z}_{12})$. However, $g(\Gamma(\mathbb{Z}_{24})) = 3$ because $6 \sim 8 \sim 12 \sim 6$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{24})$.

(2) In general, $g(\Gamma(\mathbb{Z}_{4q})) = 4$ but $g(\Gamma(\mathbb{Z}_{2^r q})) = 3$ for all primes $q \geq 3$ and all integers $r \geq 3$. (See Proposition 2.6.)

Proposition 2.4. *If $t \geq 3$ is an integer and $p_1, \ldots, p_t$ are distinct primes, then $g(\Gamma(\mathbb{Z}_{p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}})) = 3$ for all positive integers $r_1, \ldots, r_t$.*

Proof. If $t = 3$, then $p_1^{r_1} p_2^{r_2} \sim p_2^{r_2} p_3^{r_3} \sim p_1^{r_1} p_3^{r_3} \sim p_1^{r_1} p_2^{r_2}$ is a cycle in $\Gamma(\mathbb{Z}_{p_1^{r_1} p_2^{r_2} p_3^{r_3}})$; so $g(\Gamma(\mathbb{Z}_{p_1^{r_1} p_2^{r_2} p_3^{r_3}})) = 3$.

If $t > 3$, then the result follows directly from Lemma 2.2. □

Proposition 2.5. *Let p be a prime and $r \geq 2$ an integer. Then the following assertions hold.*

(1) $g(\Gamma(\mathbb{Z}_{p^r})) = \infty$ *if and only if* $p^r = 4, 8$ *or* 9.

(2) $g(\Gamma(\mathbb{Z}_{p^r})) = 3$ *if and only if each of the following conditions holds.*

(a) $p = 2$ *and* $r \geq 4$.

(b) $p = 3$ *and* $r \geq 3$.

(c) $p \geq 5$ *and* $r \geq 2$.

Proof. (1) It is obvious that $\Gamma(\mathbb{Z}_4)$, $\Gamma(\mathbb{Z}_8)$ and $\Gamma(\mathbb{Z}_9)$ have no cycles. Thus the equivalence follows.

(2) If $p = 2$ and $r \geq 4$, then $2^{r-1}, 2^{r-2}, 3 \cdot 2^{r-2} \in Z(\mathbb{Z}_{2^r})$. Since the product of any two of them is zero, $2^{r-1} \sim 2^{r-2} \sim 3 \cdot 2^{r-2} \sim 2^{r-1}$ is a cycle in $\Gamma(\mathbb{Z}_{2^r})$. Thus $g(\Gamma(\mathbb{Z}_{2^r})) = 3$.

If $p = 3$ and $r \geq 3$, then $3^{r-1}, 2 \cdot 3^{r-1}, 3^{r-2} \in Z(\mathbb{Z}_{3^r})$. Since the product of any two of them is zero, $3^{r-1} \sim 2 \cdot 3^{r-1} \sim 3^{r-2} \sim 3^{r-1}$ is a cycle in $\Gamma(\mathbb{Z}_{3^r})$. Thus $g(\Gamma(\mathbb{Z}_{3^r})) = 3$.

If $p \geq 5$ and $r \geq 2$, then $p^{r-1}, 2p^{r-1}, 3p^{r-1} \in Z(\mathbb{Z}_{p^r})$. Since the product of any two of them is zero, $p^{r-1} \sim 2p^{r-1} \sim 3p^{r-1} \sim p^{r-1}$ is a cycle in $\Gamma(\mathbb{Z}_{p^r})$. Thus $g(\Gamma(\mathbb{Z}_{p^r})) = 3$. □

Proposition 2.6. *Let n be a positive integer which has only two distinct prime divisors. Then the following assertions hold.*

(1) $g(\Gamma(\mathbb{Z}_n)) = \infty$ *if and only if* $n = 2q$ *for some prime* $q \geq 3$.

(2) $g(\Gamma(\mathbb{Z}_n)) = 3$ *if and only if one of the following holds.*

(a) $n = 2^r q^s$ *for some prime* $q \geq 3$ *and some integers* $r \geq 1$ *and* $s \geq 2$.

(b) $n = 2^r q$ *for some prime* $q \geq 3$ *and some integer* $r \geq 3$.

(c) $n = 3^r q^s$ *for some prime* $q \geq 5$ *and some integers* $r \geq 1$ *and* $s \geq 2$.

(d) $n = 3^r q$ *for some prime* $q \geq 5$ *and some integer* $r \geq 2$.

(e) $n = p^r q^s$ *for some primes* $q > p \geq 5$ *and some integers* $r, s \geq 1$ *except for* $r = s = 1$.

(3) $\mathrm{g}(\Gamma(\mathbb{Z}_n)) = 4$ *if and only if* $n = pq$ *for some distinct primes* $p, q \geq 3$, *or* $n = 4q$ *for some prime* $q \geq 3$.

Proof. (1) If $n = 2q$ for some prime $q \geq 3$, then $\Gamma(\mathbb{Z}_{2q})$ is a star graph $K_{1,q-1}$ by the proof of Theorem 2.1(3). Hence $\Gamma(\mathbb{Z}_{2q})$ has no cycles, and thus $\mathrm{g}(\Gamma(\mathbb{Z}_{2q})) = \infty$.

(2) Let p and q be the only distinct prime divisors of n. Without loss of generality, we may assume that $p < q$.

Cases (a) and (b). $p = 2$. In this case, $q \sim 2q \sim 4q \sim q$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{2q^2})$; so $\mathrm{g}(\Gamma(\mathbb{Z}_{2q^2})) = 3$. Hence by Lemma 2.2, $\mathrm{g}(\Gamma(\mathbb{Z}_{2^r q^s})) = 3$ for all integers $r \geq 1$ and $s \geq 2$. Also, $4 \sim 2q \sim 4q \sim 4$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{8q})$; so $\mathrm{g}(\Gamma(\mathbb{Z}_{8q})) = 3$. Hence by Lemma 2.2, $\mathrm{g}(\Gamma(\mathbb{Z}_{2^r q})) = 3$ for all integers $r \geq 3$.

Cases (c) and (d). $p = 3$. In this case, $q \sim 3q \sim 6q \sim q$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{3q^2})$; so $\mathrm{g}(\Gamma(\mathbb{Z}_{3q^2})) = 3$. Hence by Lemma 2.2, $\mathrm{g}(\Gamma(\mathbb{Z}_{3^r q^s})) = 3$ for all integers $r \geq 1$ and $s \geq 2$. Also, $3 \sim 3q \sim 6q \sim 3$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{9q})$; so $\mathrm{g}(\Gamma(\mathbb{Z}_{9q})) = 3$. Hence by Lemma 2.2, $\mathrm{g}(\Gamma(\mathbb{Z}_{3^r q})) = 3$ for all integers $r \geq 2$.

Case (e). $p \geq 5$. In this case, $q \geq 7$; so by Proposition 2.5(2), $\mathrm{g}(\Gamma(\mathbb{Z}_{p^r})) = 3 = \mathrm{g}(\Gamma(\mathbb{Z}_{q^s}))$ for all integers $r, s \geq 2$. Hence by Lemma 2.2, $\mathrm{g}(\Gamma(\mathbb{Z}_{p^r q^s})) = 3$ for all integers $r, s \geq 1$ except for $r = s = 1$.

(3) If $n = pq$ for some distinct primes $p, q \geq 3$, then $\Gamma(\mathbb{Z}_{pq})$ is the complete bipartite graph $K_{p-1,q-1}$ by the proof of Theorem 2.1(3). Hence there does not exist a cycle of odd length. Note that $p \sim 2q \sim 2p \sim q \sim p$ is a cycle of length 4. Thus $\mathrm{g}(\Gamma(\mathbb{Z}_{pq})) = 4$.

Let $n = 4q$ for some prime $q \geq 3$, and suppose to the contrary that there exists a cycle $a \sim b \sim c \sim a$ in $\Gamma(\mathbb{Z}_{4q})$. Since ab, bc and ca are divisible by $4q$, q divides at least two of a, b and c. Without loss of generality, we may assume that q divides a and b. If 2 divides a, then $a = 2q$. Since ab is divisible by $4q$, b is divisible by 2; so $b = 2q$. This is absurd. If 2 does not divide a, then $a = q$ or $a = 3q$. Since $4q$ divides ab, b is divisible by 4; so b is a multiple of $4q$. This is a contradiction. Hence there do not exist cycles of length 3 in $\Gamma(\mathbb{Z}_{4q})$. Note that $q \sim 4 \sim 2q \sim 8 \sim q$ is a cycle of length 4. Thus $\mathrm{g}(\Gamma(\mathbb{Z}_{4q})) = 4$. □

In the next remark, we construct a cycle of length 3 in $\Gamma(\mathbb{Z}_n)$ in each case of Proposition 2.6(2).

Remark 2.7.

(1) Let $n = 2^r q^s$ for some prime $q \geq 3$ and some integers $r \geq 1$ and $s \geq 2$. Then $2^r q^{s-1} \sim 2^{r+1} q^{s-1} \sim 2^{r-1} q^s \sim 2^r q^{s-1}$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{2^r q^s})$.

(2) Let $n = 2^r q$ for some prime $q \geq 3$ and some integer $r \geq 3$. Then $2^r \sim 2q \sim 2^{r-1} q \sim 2^r$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{2^r q})$.

(3) Let $n = 3^r q^s$ for some prime $q \geq 5$ and some integers $r \geq 1$ and $s \geq 2$. Then $3^{r-1}q^s \sim 3^r q^{s-1} \sim 2 \cdot 3^r q^{s-1} \sim 3^{r-1}q^s$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{3^r q^s})$.

(4) Let $n = 3^r q$ for some prime $q \geq 5$ and some integer $r \geq 2$. Then $3^r \sim 3^{r-1}q \sim 2 \cdot 3^{r-1}q \sim 3^r$ is a cycle of length 3 in $\Gamma(\mathbb{Z}_{3^r q})$.

(5) Let $n = p^r q^s$ for some primes $q > p \geq 5$ and some integers $r, s \geq 1$ except for $r = s = 1$. If $r \neq 1$ and $s \neq 1$, then $p^r q^{s-1} \sim p^{r-1}q^s \sim 2p^{r-1}q^s \sim p^r q^{s-1}$ and $p^r q^{s-1} \sim 2p^r q^{s-1} \sim 3p^r q^{s-1} \sim p^r q^{s-1}$ are cycles of length 3 in $\Gamma(\mathbb{Z}_{p^r q^s})$, respectively.

By Propositions 2.4, 2.5, and 2.6, we obtain

Theorem 2.8. The following statements hold.

(1) $\mathrm{g}(\Gamma(\mathbb{Z}_n)) = \infty$ if and only if each of the following conditions holds.

  (a) $n = 4, 8, 9$.

  (b) $n = 2q$ for some prime $q \geq 3$.

(2) $\mathrm{g}(\Gamma(\mathbb{Z}_n)) = 4$ if and only if each of the following conditions holds.

  (a) $n = pq$ for some distinct primes $p, q \geq 3$.

  (b) $n = 4q$ for some prime $q \geq 3$.

(3) $\mathrm{g}(\Gamma(\mathbb{Z}_n)) = 3$ in all other cases.

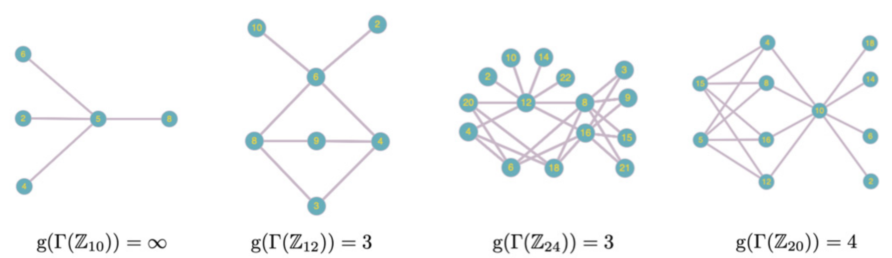


Figure 2: The girth of some zero-divisor graphs

3. The Chromatic Number of $\Gamma(\mathbb{Z}_n)$

In this section, we calculate the chromatic number of $\Gamma(\mathbb{Z}_n)$. Clearly, if there exists a clique in a graph, then the chromatic number of the graph is greater than or equal to the size of the clique; so our method to find the chromatic number of $\Gamma(\mathbb{Z}_n)$ is based on the following three steps:

Step 1. Find a maximal clique C in $\Gamma(\mathbb{Z}_n)$ and color vertices in C.

Step 2. Color vertices in $Z(\mathbb{Z}_n) \setminus C$ by colors used in Step 1.

Step 3. Confirm that there are no adjacent vertices having the same color.

Lemma 3.1. *If $r \geq 2$ is an integer, $n = p_1 \cdots p_r$ for distinct primes $p_1, \ldots, p_r$, and $C = \{\frac{n}{p_i} \,|\, i = 1, \ldots, r\}$, then C is a maximal clique of $\Gamma(\mathbb{Z}_n)$.*

Proof. Note that the product of any two distinct members of C is a multiple of n; so C is a clique. Suppose that there exists an element $a \in Z(\mathbb{Z}_n) \setminus C$ such that ca is a multiple of n for all $c \in C$. Then for all $i = 1, \ldots, r$, p_i divides a; so n divides a. This is a contradiction. Thus C is a maximal clique of $\Gamma(\mathbb{Z}_n)$. □

Theorem 3.2. *If $r \geq 2$ is an integer and $n = p_1 \cdots p_r$ for distinct primes $p_1, \ldots, p_r$, then $\chi(\Gamma(\mathbb{Z}_n)) = r$.*

Proof. Let $C = \{\frac{n}{p_i} \,|\, i = 1, \ldots, r\}$. Then by Lemma 3.1, C is a maximal clique of $\Gamma(\mathbb{Z}_n)$; so the chromatic number of the induced subgraph of $\Gamma(\mathbb{Z}_n)$ induced by C is r. For each $i = 1, \ldots, r$, let $\bar{i}$ be the color of $\frac{n}{p_i}$. Clearly, $Z(\mathbb{Z}_n) \setminus C$ is nonempty. For each $a \in Z(\mathbb{Z}_n) \setminus C$, let $S_a = \{c \in C \,|\, a \text{ and } c \text{ are not adjacent}\}$. Note that by Lemma 3.1, C is a maximal clique; so S_a is a nonempty set. Hence we can find the smallest integer $k \in \{1, \ldots, r\}$ such that a and $\frac{n}{p_k}$ are not adjacent. In this case, we color a with $\bar{k}$.

To complete the proof, we need to check that any two elements in $Z(\mathbb{Z}_n)$ with the same color cannot be adjacent. Let a and b be distinct elements in $Z(\mathbb{Z}_n)$ with the same color $\bar{k}$. Since C is a clique, a and b cannot belong to C at the same time. Suppose that $a \in C$ and $b \in Z(\mathbb{Z}_n) \setminus C$. Then $a = \frac{n}{p_k}$; so by the coloring of b, a and b are not adjacent. Suppose that $a, b \in Z(\mathbb{Z}_n) \setminus C$. Then $\frac{n}{p_k}a$ and $\frac{n}{p_k}b$ are not divisible by n; so neither a nor b is divisible by p_k. Therefore ab is not divisible by n, and hence a cannot be adjacent to b. Thus $\chi(\Gamma(\mathbb{Z}_n)) = r$. □

Example 3.3. Consider $\Gamma(\mathbb{Z}_{15})$. Let $C = \{3, 5\}$. Then by Theorem 3.2, we color 5 and 3 with $\bar{1}$ and $\bar{2}$, respectively. Let $R = \{6, 9, 12\}$ and let $v \in R$. Then v is not adjacent to 3; so we color v with $\bar{2}$. Note that 10 is not adjacent to 5; so we color 5 with $\bar{1}$.

For the detail, see Figure 3. Note that in Figure 3, $\bar{1}$ and $\bar{2}$ are represented by blue and red, respectively.

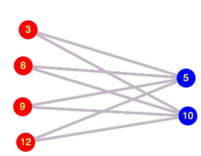


Figure 3: The coloring of $\Gamma(\mathbb{Z}_{15})$

Lemma 3.4. *Let* $n = p_1^{2a_1} \cdots p_r^{2a_r}$ *for distinct primes* $p_1, \ldots, p_r$ *and positive integers* $a_1, \ldots, a_r$*, and let* $C = \{k\sqrt{n} \,|\, k = 1, \ldots, \sqrt{n} - 1\}$*. Then* C *is a clique of* $\Gamma(\mathbb{Z}_n)$*.*

Proof. Note that the product of any two distinct elements of C is a multiple of n. Thus C is a clique. □

Theorem 3.5. *Let* $n = p_1^{2a_1} \cdots p_r^{2a_r}$ *for distinct primes* $p_1, \ldots, p_r$ *and positive integers* $a_1, \ldots, a_r$*. Then* $\chi(\Gamma(\mathbb{Z}_n)) = \sqrt{n} - 1$*.*

Proof. Let $C = \{k\sqrt{n} \,|\, k = 1, \ldots, \sqrt{n} - 1\}$. Then by Lemma 3.4, C is a clique of $\Gamma(\mathbb{Z}_n)$ with $\sqrt{n} - 1$ elements. For each $k \in \{1, \ldots, \sqrt{n} - 1\}$, let $\widehat{k}$ denote the color of $k\sqrt{n}$.

Case 1. $n = p_1^2$. In this case, $\Gamma(\mathbb{Z}_n)$ is a complete graph by Theorem 2.1. Hence $\mathrm{Z}(\mathbb{Z}_n) = C$. Thus $\chi(\Gamma(\mathbb{Z}_n)) = \sqrt{n} - 1$.

Case 2. $n \neq p_1^2$. In this case, $\Gamma(\mathbb{Z}_n)$ is not a complete graph by Theorem 2.1; so $\mathrm{Z}(\mathbb{Z}_n) \setminus C \neq \emptyset$. Let $v \in \mathrm{Z}(\mathbb{Z}_n) \setminus C$. Then there exists an element $m \in \{1, \ldots, r\}$ such that $p_m^{a_m}$ does not divide v. Take the positive integer $s \leq a_m$ such that p_m^{s-1} divides v but p_m^s does not divide v. Then $v((\sqrt{n}/p_m^s)\sqrt{n})$ is not a multiple of n; so v and $(\sqrt{n}/p_m^s)\sqrt{n}$ are not adjacent. We color v with $\widehat{\sqrt{n}/p_m^s}$.

Now, it remains to check that any two vertices with the same color cannot be adjacent. Let v_1 and v_2 be distinct elements of $\mathrm{Z}(\mathbb{Z}_n)$ which have the same color. Since C is a clique, v_1 and v_2 cannot both belong to C. Suppose that $v_1 \in C$ and $v_2 \in \mathrm{Z}(\mathbb{Z}_n) \setminus C$. Let $\widehat{\sqrt{n}/p_m^s}$ be the color of v_2 for some $m \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, a_m\}$. Then v_2 is not divisible by p_m^s by the coloring of v_2. Since $v_1 \in C$, $v_1 = (\sqrt{n}/p_m^s)\sqrt{n}$. Therefore $p_m^{2a_m}$ does not divide $v_1 v_2$. Hence v_1 is not adjacent to v_2. Suppose that $v_1, v_2 \in \mathrm{Z}(\mathbb{Z}_n) \setminus C$, and let $\widehat{\sqrt{n}/p_m^s}$ be the color of v_1 and v_2 for some $m \in \{1, \ldots, r\}$ and $s \in \{1, \ldots, a_m\}$. Then by the coloring of v_1 and v_2, p_m^s divides neither v_1 nor v_2; so p_m^{2s} does not divide $v_1 v_2$. Since $s \leq a_m$, $p_m^{2a_m}$ does not divide $v_1 v_2$. Hence $v_1 v_2$ is not a multiple of n, which means that v_1 and v_2 are not adjacent.

In either case, $\Gamma(\mathbb{Z}_n)$ is $(\sqrt{n}-1)$-colorable. Note that by Lemma 3.4, $\chi(\Gamma(\mathbb{Z}_n)) \geq \sqrt{n} - 1$. Thus $\chi(\Gamma(\mathbb{Z}_n)) = \sqrt{n} - 1$. □

Example 3.6. Consider $\Gamma(\mathbb{Z}_{36})$. Let $C = \{6, 12, 18, 24, 30\}$. Then by Theorem 3.5, we color $6, 12, 18, 24$ and 30 with $\widehat{1}, \widehat{2}, \widehat{3}, \widehat{4}$ and $\widehat{5}$, respectively. Let $R = \{2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 34\}$ and let $v \in R$. Then v is not divisible by 3; so we color v with $\widehat{2}$. Let $B = \{3, 9, 15, 21, 27, 33\}$ and let $v \in B$. Then v is not divisible by 2; so we color v with $\widehat{3}$.

For the detail, see Figure 4. Note that in Figure 4, $\widehat{1}, \widehat{2}, \widehat{3}, \widehat{4}$ and $\widehat{5}$ are represented by green, red, blue, pink and brown, respectively.

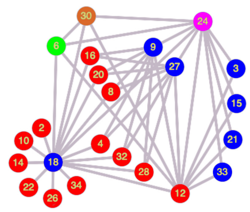


Figure 4: The coloring of $\Gamma(\mathbb{Z}_{36})$

Lemma 3.7. *Let $p_1, \ldots, p_r, q_1, \ldots, q_s$ be distinct primes, $a_1, \ldots, a_r, b_1, \ldots, b_s$ nonnegative integers, not all zero, and let $n = p_1^{2a_1} \cdots p_r^{2a_r} q_1^{2b_1+1} \cdots q_s^{2b_s+1}$. If $C_1 = \{kp_1^{a_1} \cdots p_r^{a_r} q_1^{b_1+1} \cdots q_s^{b_s+1} \mid k = 1, \ldots, p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s} - 1\}$ and $C_2 = \{p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1+1} \cdots q_s^{b_s+1}/q_i \mid i = 1, \ldots, s\}$, then $C_1 \cup C_2$ is a maximal clique of $\Gamma(\mathbb{Z}_n)$.*

Proof. We first note that $C_1 \cap C_2 = \emptyset$. Let $C = C_1 \cup C_2$. Then for any distinct elements $\alpha, \beta \in C$, $\alpha\beta$ is a multiple of n; so C is a clique. Suppose that C is not a maximal clique. Then there exists an element $m \in Z(\mathbb{Z}_n) \setminus C$ such that $mc = 0$ for all $c \in C$. Therefore m is a multiple of $p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_{i-1}^{b_{i-1}} q_i^{b_i+1} q_{i+1}^{b_{i+1}} \cdots q_s^{b_s}$ for all $i = 1, \ldots, s$. Hence m is a multiple of $p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1+1} \cdots q_s^{b_s+1}$, which implies that $m \in C_1$. This contradicts the choice of m. Thus C is a maximal clique of $\Gamma(\mathbb{Z}_n)$. □

Theorem 3.8. *Let $p_1, \ldots, p_r, q_1, \ldots, q_s$ be distinct primes and $a_1, \ldots, a_r, b_1, \ldots, b_s$ nonnegative integers, not all zero. If $n = p_1^{2a_1} \cdots p_r^{2a_r} q_1^{2b_1+1} \cdots q_s^{2b_s+1}$, then $\chi(\Gamma(\mathbb{Z}_n)) = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s} - 1 + s$.*

Proof. Let $x = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s}$ and $y = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1+1} \cdots q_s^{b_s+1}$. Then $n = xy$. Let $C_1 = \{ky \mid k = 1, \ldots, x-1\}$, $C_2 = \{y/q_i \mid i = 1, \ldots, s\}$, and $C = C_1 \cup C_2$. Then by Lemma 3.7, C is a maximal clique of $\Gamma(\mathbb{Z}_n)$. For each $k = 1, \ldots, x-1$, let $\widehat{k}$ be the color of ky and for each $i = 1, \ldots, s$, let $\bar{i}$ be the color of y/q_i. Note that by Theorem 2.1, $\Gamma(\mathbb{Z}_n)$ is not a complete graph. Let $v \in Z(\mathbb{Z}_n) \setminus C$.

Case 1. There exists an element $c \in C_1$ which is not adjacent to v. In this case, v is not divisible by x. If $q_1^{b_1} \cdots q_s^{b_s}$ divides v, then $p_m^{a_m}$ does not divide v for some $m \in \{1, \ldots, r\}$; so we can take the positive integer $\alpha \leq a_m$ such that $p_m^{\alpha-1}$ divides v but p_m^{α} does not divide v. Hence v is not adjacent to $(x/p_m^{\alpha})y$. We color v with $\widehat{x/p_m^{\alpha}}$. If $q_t^{b_t}$ does not divide v for some $t \in \{1, \ldots, s\}$, then we can find the positive integer $\beta \leq b_t$ such that $q_t^{\beta-1}$ divides v but q_t^{β} does not divide v. Hence v is not adjacent to $(x/q_t^{\beta})y$. We color v with $\widehat{x/q_t^{\beta}}$.

Case 2. v is adjacent to c for all $c \in C_1$. In this case, v is a multiple of x. If $q_1^{b_1+1} \cdots q_s^{b_s+1}$ divides v, then $v \in C_1$, which is a contradiction to the choice of v. Therefore we can find an element $i \in \{1, \ldots, s\}$ such that $q_i^{b_i}$ divides v but $q_i^{b_i+1}$

does not divide v. Clearly, $v(y/q_i)$ is not a multiple of n. Hence v and y/q_i are not adjacent. We color v with $\bar{i}$.

It remains to show that there are no adjacent vertices with the same color. Let $v_1, v_2 \in Z(\mathbb{Z}_n)$ have the same color. Since C is a clique, at least one of v_1 and v_2 does not belong to C. Suppose that $v_1 \in C$ but $v_2 \in Z(\mathbb{Z}_n) \setminus C$. If the color of v_1 and v_2 is $\widehat{x/p_m^\alpha}$ for some $m \in \{1, \ldots, r\}$ and $\alpha \in \{1, \ldots, a_m\}$, then by the coloring of v_1 and v_2, $v_1 = (x/p_m^\alpha)y$ and p_m^α does not divide v_2; so v_1v_2 is not a multiple of n. Hence v_1 is not adjacent to v_2. If the color of v_1 and v_2 is $\widehat{x/q_t^\beta}$ for some $t \in \{1, \ldots, s\}$ and $\beta \in \{1, \ldots, b_t\}$, then by the coloring of v_1 and v_2, $v_1 = (x/q_t^\beta)y$ and q_t^β does not divide v_2; so v_1v_2 is not a multiple of n. Hence v_1 is not adjacent to v_2. If $\bar{i}$ is the color of v_1 and v_2 for some $i \in \{1, \ldots, s\}$, then $v_1 = y/q_i$ and so by the coloring of v_2, v_1 and v_2 are not adjacent. We next suppose that $v_1, v_2 \in Z(\mathbb{Z}_n) \setminus C$. If $\widehat{x/p_m^\alpha}$ is the color of v_1 and v_2 for some $m \in \{1, \ldots, r\}$ and $\alpha \in \{1, \ldots, a_m\}$, then by Case 1, p_m^α divides neither v_1 nor v_2; so $p_m^{2a_m}$ does not divide v_1v_2. Hence v_1 and v_2 are not adjacent. If $\widehat{x/q_t^\beta}$ is the color of v_1 and v_2 for some $t \in \{1, \ldots, s\}$ and $\beta \in \{1, \ldots, b_t\}$, then by Case 1, q_t^β divides neither v_1 nor v_2; so $q_t^{2b_t}$ does not divide v_1v_2. Hence v_1 and v_2 are not adjacent. If $\bar{i}$ is the color of v_1 and v_2 for some $i \in \{1, \ldots, s\}$, then by Case 2, $q_i^{b_i+1}$ divides neither v_1 nor v_2. Hence $q_i^{2b_i+1}$ cannot divide v_1v_2, which means that v_1 and v_2 are not adjacent. Consequently, $\Gamma(\mathbb{Z}_n)$ is $(p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s} - 1 + s)$-colorable. Note that by Lemma 3.7, $\chi(\Gamma(\mathbb{Z}_n)) \geq p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s} - 1 + s$. Thus $\chi(\Gamma(\mathbb{Z}_n)) = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s} - 1 + s$. □

Example 3.9. Consider $\Gamma(\mathbb{Z}_{18})$. Let $C_1 = \{6, 12\}$ and $C_2 = \{3\}$. Then by Theorem 3.8, we color $6, 12$ and 3 with $\hat{1}, \hat{2}$ and $\bar{1}$, respectively. Let $R = \{2, 4, 8, 10, 14, 16\}$ and let $v \in R$. Then v is not adjacent to 6. Note that v is not divisible by 3; so we color v with $\hat{1}$. Let $B = \{9, 15\}$ and let $v \in B$. Then v is adjacent to all elements in C_1. Note that v is not divisible by 2; so we color v with $\bar{1}$.

For the detail, see Figure 5. Note that in Figure 5, $\hat{1}, \hat{2}$ and $\bar{1}$ are represented by red, green and blue, respectively.

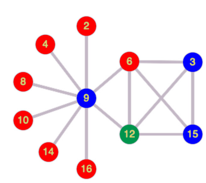


Figure 5: The coloring of $\Gamma(\mathbb{Z}_{18})$

Example 3.10. Consider $\Gamma(\mathbb{Z}_{72})$. Let $C_1 = \{12, 24, 36, 48, 60\}$ and $C_2 = \{6\}$. Then by Theorem 3.6, we color $6, 12, 24, 36, 48$ and 60 with $\bar{1}, \hat{1}, \hat{2}, \hat{3}, \hat{4}$ and $\hat{5}$, respectively.

Let $P = \{2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 34, 38, 40, 44, 46, 50, 52, 56, 58, 62, 64, 68, 70\}$ and let $v \in P$. Then v is not adjacent to 12. Note that v is a multiple of 2 but not a multiple of 3; so we color v with $\widehat{2}$. Let $B = \{3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69\}$ and let $v \in B$. Then v is not adjacent to 12. Note that v is not divisible by 2; so we color v with $\widehat{3}$. Let $Y = \{18, 30, 42, 54, 66\}$ and let $v \in Y$. Then v is adjacent to all elements in C_1. Since v and 6 are not adjacent, we color v with $\overline{1}$.

For the detail, see Figure 6. Note that in Figure 6, $\overline{1}, \widehat{1}, \widehat{2}, \widehat{3}, \widehat{4}$ and $\widehat{5}$ are represented by yellow, red, pink, blue, sky-blue and green, respectively.

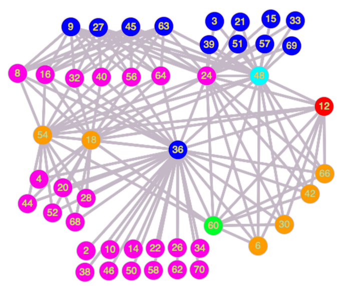


Figure 6: The coloring of $\Gamma(\mathbb{Z}_{72})$

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