J. Appl. Math. & Informatics Vol. **37***(2019), No. 1 - 2, pp. 53 - 63*
https://doi.org/10.14317/jami.2019.053

AN EXISTENCE OF THE SOLUTION TO NEUTRAL STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS UNDER SPECIAL CONDITIONS†

YOUNG-HO KIM

ABSTRACT. In this paper, we show the existence of solution of the neutral stochastic functional differential equations under non-Lipschitz condition, a weakened linear growth condition and a contractive condition. Furthermore, in order to obtain the existence of solution to the equation we used the Picard sequence.

AMS Mathematics Subject Classification : 60H05, 60H10, 60H20.
Key words and phrases : Gronwall's inequality, Bihari's inequality, non-Lipschitz condition, Existence and Uniqueness, Neutral stochastic functional differential equations.

1. Introduction

Together with the development of science systems, usually using the ordinary differential equations to describe the trajectory of systems with phenomenon of time delay is have difference in real measured trajectory. Moreover, we can not ignore the effect of the science systems with time delay.

So, we need an another class of stochastic equations depending on past and present values but that involves derivatives with delays as well as the function itself. Such equations historically have been referred to as *neutral stochastic functional differential equations, or neutral stochastic differential delay equations* (see, [2], [3], [4], [6], [8], [9]). Such equations are more difficult to motivate but often arise in the study of two or more simple electrodynamics or oscillatory systems with some interconnections between them.

For example, In studying the collision problem in electrodynamics, Diver [1] considered the system of neutral type

$$\dot{x}(t) = f_1(x(t), x(\delta(t))) + f_2(x(t), x(\delta(t)))\dot{x}(\delta(t)),$$

Received August 6, 2018. Revised October 5, 2018. Accepted October 10, 2018.
†This research was supported by Changwon National University in 2017-2018

where $\delta(t) \leq t$. Generally, a neutral functional differential equation has the form

$$\frac{d}{dt}[x(t) - D(x_t)] = f(x_t, t).$$

Taking into account stochastic perturbations, we are led to a neutral stochastic functional differential equation

$$d[x(t) - D(x_t)] = f(x_t, t)dt + g(x_t, t)dB(t) \tag{1}$$

Neutral stochastic functional differential equations(NSDEs) are known to model problems from several areas of science and engineering. For instance, in 2007, Mao [8] published the stochastic differential equations and applications, in 2010, Li and Fu [7] considered the stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks, in 2013, Wei *et al.* [10] considered the existence and uniqueness of the solution to following neutral stochastic functional differential equations with infinite delay. Also, Kim [5] considered the solution to following neutral stochastic functional differential equations with infinite delay

$$d[x(t) - G(t, x_t)] = f(t, x_t)dt + g(t, x_t)dB(t), \tag{2}$$

where $x_t = \{x(t + \theta) : -\infty < \theta \leq 0\}$.

Motivated by [5], [10], one of the objectives of this paper is to get one proof to existence theorem for given NSDEs. The other objective of this paper is to estimate on how fast the Picard iterations $x_n(t)$ convergence the unique solution $x(t)$ of the neutral NSDEs.

2. Preliminary

Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\text{trace}(A^T A)}$. And $BC((-\infty, 0]; R^d)$ denote the family of bounded continuous R^d-value functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$. $\mathcal{M}^2((-\infty, T]; R^d)$ denote the family of all $\mathbb{R}^d$-valued measurable $\mathcal{F}_t$-adapted process $\psi(t) = \psi(t, w), t \in (-\infty, T]$ such that $E\int_{-\infty}^{T} |\psi(t)|^2 dt < \infty$.

Let t_0 be a positive constant and $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets).

Let $B(t)$ is a m-dimensional Brownian motion defined on complete probability space, that is $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$.

For $0 \leq t_0 \leq T < \infty$, let $f : [t_0, T] \times BC((-\infty, 0]; R^d) \to R^d$ and $g : [t_0, T] \times BC((-\infty, 0]; R^d) \to R^{d\times m}$ are Borel measurable mapping and $G : [t_0, T] \times BC((-\infty, 0]; R^d) \to R^d$ be continuous mapping.

With all the above preparation, consider the following d-dimensional neutral SFDEs:

$$\mathrm{d}[x(t) - G(t, x_t)] = f(t, x_t)\mathrm{d}t + g(t, x_t)\mathrm{d}B(t), \quad t_0 \leq t \leq T, \tag{3}$$

where $x_t = \{x(t+\theta) : -\infty < \theta \le 0\}$ can be considered as a $BC((-\infty, 0]; R^d)$-value stochastic process. The initial value of the system (3)

$$x_{t_0} = \xi = \{\xi(\theta) : -\infty < \theta \le 0\} \tag{4}$$

is an $\mathcal{F}_{t_0}$measurable, $BC((-\infty, 0]; R^d)$-value random variable such that $\xi \in \mathcal{M}^2((-\infty, 0]; R^d)$.

To be more precise, we give the definition of the solution of the equation (3) with initial data (4).

Definition 2.1. [10] R^d-value stochastic process $x(t)$ defined on $-\infty < t \le T$ is called the solution of (3) with initial data (4), if $x(t)$ has the following properties:
(i) $x(t)$ is continuous and $\{x(t)\}_{t_0 \le t \le T}$ is $\mathcal{F}_t$-adapted;
(ii) $\{f(t, x_t)\} \in \mathcal{L}^1([t_0, T]; R^d)$ and $\{g(t, x_t)\} \in \mathcal{L}^2([t_0, T]; R^{d\times m})$;
(iii) $x_{t_0} = \xi$, for each $t_0 \le t \le T$,

$$x(t) = \xi(0) + G(t, x_t) - G(t_0, \xi) + \int_{t_0}^{t} f(s, x_s)ds + \int_{t_0}^{t} g(s, x_s)dB(s) \quad \text{a.s.} \tag{5}$$

The $x(t)$ is called as a unique solution, if any other solution $\overline{x}(t)$ is distinguishable with $x(t)$, that is

$$P\{x(t) = \overline{x}(t), \text{ for any } -\infty < t \le T\} = 1.$$

The following lemmas are known as special name for stochastic integrals which was appear in [8] and will play an important role in next section.

Lemma 2.2. *(Hölder's inequality)* [8] *If* $\frac{1}{p} + \frac{1}{q} = 1$ *for any* $p, q > 1, f \in L^p$, *and* $g \in L^q$, *then* $fg \in L^1$ *and* $\int_a^b fg \ \mathrm{dx} \le (\int_a^b |f|^p \ \mathrm{dx})^{1/p} (\int_a^b |g|^q \ \mathrm{dx})^{1/q}$.

Lemma 2.3. *(Gronwall's inequality)* [8] *Let* $u(t)$ *and* $b(t)$ *be nonnegative continuous functions for* $t \ge \alpha$, *and let* $u(t) \le a + \int_\alpha^t b(s)u(s)\mathrm{d}s$, $t \ge \alpha$, *where* $a \ge 0$ *is a constant. Then*

$$u(t) \le a \exp\biggl(\int_\alpha^t b(s)\mathrm{d}s\biggr), \quad t \ge \alpha.$$

Lemma 2.4. *(Bihari's inequality)* [8] *Let* $T \ge 0$ *and* $u_0 \ge 0$, *let* $u(t)$ *and* $b(t)$ *be continuous functions on* $[0, T]$. *Let* $\kappa(\cdot) : R^+ \to R^+$ *is a concave nondecreasing function such that* $\kappa(0) = 0$, $\kappa(u) > 0$ *for all* $u > 0$. *If* $u(t) \le a + \int_\alpha^t b(s)\kappa(u(s))\mathrm{d}s$, *for all* $0 \le t \le T$. *Then*

$$u(t) \le G^{-1}\biggl(G(u_0) + \int_t^T v(s)\mathrm{d}s\biggr)$$

for all $t \in [0, T]$ *such that* $G(u_0) + \int_t^T v(s)\mathrm{d}s \in \mathrm{Dom}(G^{-1})$, *where* $G(r) = \int_1^r \frac{1}{k(s)}\mathrm{d}s, r \ge 0$ *and* G^{-1} *is the inverse function of* G.

Lemma 2.5. *(Moment inequality)* [8] *If* $p \geq 2, g \in \mathcal{M}^2([0,T]; R^{d\times m})$ *such that* $E\int_0^T |g(s)|^p \,\mathrm{d}s < \infty$, *then*

$$E\left(\sup_{0\leq t\leq T}\left|\int_0^t g(s)\,\mathrm{d}B(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |g(s)|^p \,\mathrm{d}s.$$

In order to attain the solution of equation (3) with initial data (4), we propose the following assumptions:

(H1) (non-uniform Lipschitz condition) For any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, we assume that

$$|f(t,\varphi) - f(t,\psi)|^2 \vee |g(t,\varphi) - g(t,\psi)|^2 \leq \kappa(\|\varphi - \psi\|^2),$$

where $\kappa(\cdot) : R^+ \to R^+$ is a concave continuous nondecreasing function such that $\kappa(0) = 0$, $\kappa(u) > 0$ for all $u > 0$ and $\int_{0+} \frac{1}{k(u)}\mathrm{d}u = \infty$.

(H2) (weakened linear growth condition) For any $t \in [t_0, T]$, it follows that $f(t,0), g(t,0) \in L^2$ such that

$$|f(t,0)|^2 \vee |g(t,0)|^2 \leq K,$$

where K is a positive constant.

(H3) (contractive condition) Assuming that there exists a positive number K_0 such that $0 < K_0 < 1$ and for any $\varphi, \psi \in BC((-\infty, 0]; R^d)$ and $t \in [t_0, T]$, it follows that

$$|G(t,\varphi) - G(t,\psi)| \leq K_0(\|\varphi - \psi\|).$$

3. Main results

In order to obtain the existence of solutions to neutral SFDEs, let $x^0_{t_0} = \xi$ and $x^0(t) = \xi(0)$, for $t_0 \leq t \leq T$. For each $n = 1, 2, \ldots$, set $x^n_{t_0} = \xi$ and define the following Picard sequence

$$\begin{aligned} &x^n(t) - \xi(0) \\ &= G(t, x^{n-1}_t) - G(t_0, x^{n-1}_{t_0}) + \int_{t_0}^t f(s, x^{n-1}_s)ds + \int_{t_0}^t g(s, x^{n-1}_s)dB(s). \end{aligned} \quad (6)$$

Now we give the existence theorem to the solution of equation (3) with initial data (4) by approximate solutions by means of Picard sequence.

Theorem 3.1. *Assume that (H1)-(H3) hold. Then, there exists a unique solution to the neutral SFDEs (3) with initial data (4). Moreover, the solution belongs to* $\mathcal{M}^2((-\infty, T]; R^d)$.

We prepare a lemma in order to prove this theorem.

Lemma 3.2. *Let the assumption (H1) and (H3) hold. If* $x(t)$ *is a solution of equation (2.1) with initial data (2.2), then*

$$E\Big(\sup_{-\infty<t\le T}|x(s\wedge\tau_n)|^2\Big)$$
$$\le\left(c_1(\alpha+K)(T-t_0)+\frac{4}{(1-\sqrt{K_0})(1-K_0)}E\|\xi\|^2\right)\exp(c_1\beta(T-t_0)),$$

where α and β are positive constants, $c_1=6(T-t_0+4)/(1-\sqrt{K_0})(1-K_0)$. In particular, $x(t)$ belong to $\mathcal{M}^2((-\infty,\,T];R^d)$.

Proof. For each number $n\ge1$, define the stopping time

$$\tau_n=T\wedge\inf\{t\in[t_0,T]:\|x(t)\|\ge n\}.$$

Obviously, as $n\to\infty,\tau_n\uparrow T$ a.s. Let $x^n(t)=x(t\wedge\tau_n),t\in(-\infty,T]$. Then, for $t_0\le t\le T$, $x^n(t)$ satisfy the following equation

$$x^n(t)=G(t,x^n_t)-G(t_0,x^n_{t_0})+J^n(t),$$

where

$$J^n(t)=\xi(0)+\int_{t_0}^t f(s,x^n_s)I_{[t_0,\tau_n]}(s)\,\mathrm{d}s+\int_{t_0}^t g(s,x^n_s)I_{[t_0,\tau_n]}(s)\,\mathrm{d}B(s).$$

Applying the elementary inequality $(a+b)^2\le\frac{a^2}{\alpha}+\frac{b^2}{1-\alpha}$ when $a,b>0,0<\alpha<1$, we have

$$\begin{aligned}|x^n(t)|^2&\le\frac{1}{K_0}|G(t,x^n_t)-G(t_0,x^n_{t_0})|^2+\frac{1}{1-K_0}|J^n(t)|^2\\&\le\sqrt{K_0}\|x^n_t\|^2+\frac{K_0}{1-\sqrt{K_0}}\|\xi\|^2+\frac{1}{1-K_0}|J^n(t)|^2,\end{aligned}$$

where condition (H3) has also been used. Taking the expectation on both sides, one sees that

$$E\Big(\sup_{t_0<s\le t}|x^n(s)|^2\Big)$$
$$\le\sqrt{K_0}E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)+\frac{K_0}{1-\sqrt{K_0}}E\|\xi\|^2+\frac{1}{1-K_0}E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big).$$

Noting that $\sup_{-\infty<s\le t}|x^n(s)|^2\le\|\xi\|^2+\sup_{t_0\le s\le t}|x^n(s)|^2$, we get

$$E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)$$
$$\le\sqrt{K_0}E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)+\frac{K_0}{1-\sqrt{K_0}}E\|\xi\|^2+\frac{1}{1-K_0}E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big).$$

Consequently

$$E\Big(\sup_{-\infty<s\le t}|x^n(s)|^2\Big)$$
$$\le\frac{1}{(1-\sqrt{K_0})^2}E\|\xi\|^2+\frac{1}{(1-\sqrt{K_0})(1-K_0)}E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big).\qquad(7)$$

On the other hand, by the elementary inequality $(a+b+c)^2 \le 3a^2+3b^2+3c^2$, one can show that

$$|J^n(s)|^2 \le 3\left[E\|\xi\|^2 + \left|\int_{t_0}^{t} |f(s,x_s^n)|^2 ds\right|^2 + \left|\int_{t_0}^{s} g(r,x_r^n)\,\mathrm{d}B(r)\right|^2\right].$$

By Hölder's inequality and Lemma (2.5), one can show that

$$\begin{aligned} &E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big)\\ &\le 3\left[E\|\xi\|^2 + (T-t_0)\int_{t_0}^{t} E|f(s,x_s^n)|^2 ds + 4\int_{t_0}^{s} E|g(s,x_s^n)|^2\,ds\right]. \end{aligned}$$

By the condition (H2), one can show that

$$E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big) \le 3E\|\xi\|^2 + 6(T-t_0+4)\int_{t_0}^{t} E(\kappa(\|x_s^n\|^2)+K)ds.$$

Since $\kappa(\cdot)$ is concave and $\kappa(0)=0$, we can find a positive constants α and β such that $\kappa(u)\le \alpha+\beta u$ for all $u\ge 0$. Therefore, we have

$$\begin{aligned} &E\Big(\sup_{t_0\le s\le t}|J^n(s)|^2\Big)\\ &\le 3E\|\xi\|^2 + 6(T-t_0+4)(\alpha+K)(T-t_0) + 6\beta(T-t_0+4)\int_{t_0}^{t} E\|x_s^n\|^2 ds. \end{aligned}$$

Substituting this into (7) yields that

$$\begin{aligned} &E\Big(\sup_{-\infty< s\le t}|x^n(s)|^2\Big)\\ &\le c_1(\alpha+K)(T-t_0) + \frac{4}{(1-\sqrt{K_0})(1-K_0)}E\|\xi\|^2 + c_1\beta\int_{t_0}^{t} E\|x_s^n\|^2 ds, \end{aligned}$$

where $c_1 = 6(T-t_0+4)/(1-\sqrt{K_0})(1-K_0)$. Therefore, we have

$$\begin{aligned} &E\Big(\sup_{-\infty< s\le t}|x^n(s)|^2\Big)\\ &\le c_1(\alpha+K)(T-t_0) + \frac{4}{(1-\sqrt{K_0})(1-K_0)}E\|\xi\|^2 \qquad (8)\\ &\qquad + c_1\beta\int_{t_0}^{t}\sup_{-\infty<r\le s} E|x^n(r)|^2 dr. \end{aligned}$$

The Gronwall's inequality then yields that

$$\begin{aligned} &E\Big(\sup_{-\infty< s\le t}|x^n(s)|^2\Big)\\ &\le \left(c_1(\alpha+K)(T-t_0) + \frac{4}{(1-\sqrt{K_0})(1-K_0)}E\|\xi\|^2\right)\exp(c_1\beta(T-t_0)). \end{aligned}$$

For all $n = 0, 1, 2, \dots$, we deduce that

$$E\Big(\sup_{-\infty<s\leq t}|x(s\wedge\tau_n)|^2\Big)$$
$$\leq \bigg(c_1(\alpha+K)(T-t_0)+\frac{4}{(1-\sqrt{K_0})(1-K_0)}E\|\xi\|^2\bigg)\exp(c_1\beta(T-t_0)).$$

Consequently the required inequality follows by letting $n\to\infty$. □

Proof of Theorem 3.1. To check the uniqueness, let $x(t)$ and $\overline{x}(t)$ be any two solutions of (3) with initial data (4). By Lemma 3.2, $x(t), \overline{x}(t) \in \mathcal{M}^2((-\infty, T]; R^d)$. Note that

$$x(t)-\overline{x}(t) = G(t,x_t)-G(t,\overline{x}_t)+J(t),$$

where $J(t) = \int_{t_0}^t [f(s,x_s)-f(s,\overline{x}_s)]ds + \int_{t_0}^t [g(s,x_s)-g(s,\overline{x}_s)]dB(s)$. One then gets

$$|x(t)-\overline{x}(t)|^2 \leq \frac{1}{K_0}|G(t,x_t)-G(t,\overline{x}_t)|^2+\frac{1}{1-K_0}|J(t)|^2,$$

where $0<K_0<1$. We derive that

$$|x(t)-\overline{x}(t)|^2 \leq K_0\|x_t-\overline{x}_t\|^2+\frac{1}{1-K_0}|J(s)|^2.$$

Therefore

$$E\Big(\sup_{t_0\leq s\leq t}|x(s)-\overline{x}(s)|^2\Big) \leq K_0E\Big(\sup_{t_0\leq s\leq t}|x(s)-\overline{x}(s)|^2\Big) + \tfrac{1}{(1-K_0)}E\Big(\sup_{t_0\leq s\leq t}|J(s)|^2\Big).$$

Consequently

$$E\Big(\sup_{t_0\leq s\leq t}|x(t)-\overline{x}(t)|^2\Big) \leq \frac{1}{(1-K_0)^2}E\Big(\sup_{t_0\leq s\leq t}|J(s)|^2\Big).$$

On the other hand, one can show that

$$E\Big(\sup_{t_0\leq s\leq t}|J(s)|^2\Big)$$
$$\leq 2\bigg[(T-t_0)E\int_{t_0}^t|f(s,x_s)-f(s,\overline{x}_s)|^2ds+4E\int_{t_0}^t|g(s,x_s)-g(s,\overline{x}_s)|^2\,\mathrm{d}s\bigg]$$
$$\leq 2(T-t_0+4)\int_{t_0}^t E\kappa(\|x_s-\overline{x}_s\|^2)\,ds.$$

For any $\varepsilon>0$, by the Jensen Inequality of the continuous function κ, this yields that

$$E\Big(\sup_{t_0\leq s\leq t}|J(s)|^2\Big) \leq \varepsilon + 2(T-t_0+4)\int_{t_0}^t \kappa(E\sup_{t_0<r\leq s}|x(r)-\overline{x}(r)|^2)\,ds.$$

By the Bihari's inequality, this yields that for sufficiently small $\varepsilon > 0$

$$E\Big(\sup_{t_0 \le s \le t} |J(s)|^2\Big) \le G^{-1}(G(\varepsilon) + 2(T - t_0 + 4)(T - t_0)),$$

where $G(r) = \int_0^r \frac{1}{\kappa(u)}\, du$ on $r > 0$ and $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$. By assumption $\int_{0^+} \frac{1}{\kappa(u)}\, du = \infty$ and the definition of $\kappa(\cdot)$ on see that $\lim_{\epsilon \downarrow 0} G(\epsilon) = \infty$ and then

$$\lim_{\epsilon \downarrow 0} G^{-1}(G(\varepsilon) + 2(T - t_0 + 4)(T - t_0)) = 0.$$

Therefore, by letting $\epsilon \to 0$, we have $E \sup_{t_0 \le s \le t} |J(s)|^2 = 0$. This implies that

$$E\Big(\sup_{t_0 < s \le t} |x(t) - \overline{x}(t)|^2\Big) = 0.$$

Hence, we get $x(t) = \overline{x}(t)$ for $t_0 \le t \le T$ a.s. The uniqueness has been proved.

Now we check the existence of the solution using the Picard sequence (6). Obviously, from the Picard iterations, we have $x^0(t) \in \mathcal{M}^2([t_0, T] : R^d)$. Moreover, one can show the boundedness of the sequence $\{x^n(t), n \ge 0\}$ that $x^n(t) \in \mathcal{M}^2((-\infty, T] : R^d)$, in fact

$$x^n(t) = G(t, x_t^{n-1}) - G(t_0, x_{t_0}^{n-1}) + J^{n-1}(t),$$

where

$$J^{n-1}(t) = \xi(0) + \int_{t_0}^t f(s, x_s^{n-1}) ds + \int_{t_0}^t g(s, x_s^{n-1}) dB(s).$$

Applying the elementary inequality $(a+b)^2 \le \frac{a^2}{\alpha} + \frac{b^2}{1-\alpha}$ when $a, b > 0, 0 < \alpha < 1$, we have

$$\begin{aligned} |x^n(t)|^2 &\le \frac{1}{K_0} |G(t, x_t^{n-1}) - G(t_0, \xi)|^2 + \frac{1}{1 - K_0} |J^{n-1}(t)|^2 \\ &\le \sqrt{K_0} \|x_t^{n-1}\|^2 + \frac{K_0}{1 - \sqrt{K_0}} \|\xi\|^2 + \frac{1}{1 - K_0} |J^{n-1}(t)|^2, \end{aligned}$$

where condition (H3) has also been used. Taking the expectation on both sides, one sees that

$$\begin{aligned} &E\Big(\sup_{t_0 \le s \le t} |x^n(s)|^2\Big) - \sqrt{K_0} E \sup_{t_0 \le s \le t} |x^{n-1}(s)|^2 \\ &\le \frac{K_0}{1 - \sqrt{K_0}} E\|\xi\|^2 + \frac{1}{1 - K_0} E\Big(\sup_{t_0 \le s \le t} |J^{n-1}(s)|^2\Big). \end{aligned} \tag{9}$$

On the other hand, by elementary inequality, Hölder's inequality and moment inequality, one can show that

$$\begin{aligned} &E\Big(\sup_{t_0 \le s \le t} |J^{n-1}(s)|^2\Big) \\ &\le 3\Big[E\|\xi\|^2 + (T - t_0) E \int_{t_0}^t |f(s, x_s^{n-1})|^2 ds + 4E \int_{t_0}^t |g(s, x_s^{n-1})|^2 \,\mathrm{d}s\Big] \end{aligned}$$

$$\leq 3E\|\xi\|^2 + 6(T - t_0 + 4)E\int_{t_0}^{t}(\kappa(\|x_s^{n-1}\|^2) + K)\,ds.$$

Since $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a positive constants α and β such that $\kappa(u) \leq \alpha + \beta u$ for all $u \geq 0$. Therefore, we have

$$E\Big(\sup_{t_0\leq s\leq t}|J^{n-1}(s)|^2\Big) \leq 3E\|\xi\|^2 + \gamma_1 + 6\beta(T - t_0 + 4)\int_{t_0}^{t}E\|x_s^{n-1}\|^2\,ds,$$

where $\gamma_1 = 6(T - t_0 + 4)(T - t_0)(\alpha + K)$. Substituting this into (9) yields that

$$E\sup_{t_0\leq s\leq t}|x^n(s)|^2$$
$$\leq c_2 + \sqrt{K_0}E\sup_{t_0\leq s\leq t}|x^{n-1}(s)|^2 + \frac{6\beta(T - t_0 + 4)}{1 - K_0}\int_{t_0}^{t}E\sup_{t_0\leq r\leq s}|x^{n-1}(r)|^2 dr,$$

where $c_2 = \frac{\gamma_1}{1-K_0} + \frac{4+6\beta(T-t_0+4)(T-t_0)}{(1-\sqrt{K_0})(1-K_0)}E\|\xi\|^2$. It also follows note that for any $k \geq 1$,

$$\max_{1\leq n\leq k}E\Big(\sup|x^{n-1}(u)|^2\Big)$$
$$= \max\Big\{E\|\xi\|^2, E(\sup|x^1(u)|^2), \ldots, E(\sup|x^{k-1}(u)|^2)\Big\}$$
$$\leq \max\Big\{E\|\xi\|^2, E(\sup|x^1(u)|^2), \ldots, E(\sup|x^{k-1}(u)|^2), E(\sup|x^k(u)|^2)\Big\}$$
$$\leq E\|\xi\|^2 + \max_{1\leq n\leq k}E(\sup|x^n(u)|^2).$$

Therefore, one can derive that

$$\max_{1\leq n\leq k}E\Big(\sup_{t_0\leq s\leq t}|x^n(s)|^2\Big)$$
$$\leq c_3 + \frac{6\beta(T - t_0 + 4)}{(1 - \sqrt{K_0})(1 - K_0)}\int_{t_0}^{t}\max_{1\leq n\leq k}E\Big(\sup_{t_0\leq r\leq s}|x^n(r)|^2\Big)\,dr,$$

where $c_3 = \frac{c_2}{1-\sqrt{K_0}} + \frac{1+6\beta(T-t_0+4)(T-t_0)}{(1-\sqrt{K_0})(1-K_0)}E\|\xi\|^2$. By Gronwall's inequality, we have

$$\max_{1\leq n\leq k}E\Big(\sup_{t_0\leq s\leq t}|x^n(s)|^2\Big) \leq c_3\exp\Big(\frac{6\beta(T - t_0 + 4)(T - t_0)}{(1 - \sqrt{K_0})(1 - K_0)}\Big).$$

Since k is arbitrary, for all $n = 0, 1, 2, \ldots$, we deduce that

$$E\Big(\sup_{t_0\leq s\leq t}|x^n(s)|^2\Big) \leq c_3\exp\Big(\frac{6\beta(T - t_0 + 4)(T - t_0)}{(1 - \sqrt{K_0})(1 - K_0)}\Big),$$

which shows the boundedness of the sequence $\{x^n(t), n \geq 0\}$.

Next, we check that the sequence $\{x^n(t)\}$ is Cauchy sequence. For all $n \geq 0$ and $t_0 \leq t \leq T$, we have

$$x^{n+1}(t) - x^n(t) = G(t, x_t^n) - G(t, x_t^{n-1})$$

$$+\int_{t_0}^{t}[f(s,x_s^n)-f(s,x_s^{n-1})]ds+\int_{t_0}^{t}[g(s,x_s^n)-g(s,x_s^{n-1})]dB(s).$$

Using an elementary inequality $(u+v)^2 \le \frac{1}{\alpha}u^2+\frac{1}{1-\alpha}v^2$ and the condition (H3), we derive that

$$E\Big(\sup_{t_0<s\le t}|x^{n+1}(s)-x^n(s)|^2\Big)\le K_0E\Big(\sup_{t_0<s\le t}|x^n(s)-x^{n-1}(s)|^2\Big)$$
$$+\frac{2(T-t_0+4)}{1-K_0}E\int_{t_0}^{t}\kappa\Big(\sup_{t_0\le r\le s}|x^n(r)-x^{n-1}(r)|^2\Big)\,ds.$$

This yields that

$$\max_{1\le n\le k}E\Big(\sup_{t_0<s\le t}|x^{n+1}(s)-x^n(s)|^2\Big)$$
$$\le\frac{2(T-t_0+4)}{(1-K_0)^2}\int_{t_0}^{t}\kappa\Big(\max_{1\le n\le k}E\Big(\sup_{t_0\le u\le s}|x^{n+1}(u)-x^n(u)|^2\Big)\Big)\,ds.$$

Let $Z(t)=\limsup_{n,m\to\infty}\max_{1\le n\le k}E\Big(\sup_{t_0\le s\le t}|x^{n+1}(s)-x^n(s)|^2\Big)$, we get

$$Z(t)\le\epsilon+\frac{2(T-t_0+4)}{(1-K_0)^2}\int_{t_0}^{t}\kappa(Z(s))\,ds.$$

By Bihari's inequality, we get $Z(t)=0$. This shows the sequence $\{x^n(t),n\ge 0\}$ is a Cauchy sequence in L^2. Hence, as $n\to\infty, x^n(t)\to x(t)$, that is $E|x^n(t)-x(t)|^2\to 0$. Therefore, we obtain that $x(t)\in\mathcal{M}^2((-\infty,\,T];R^d)$. Now to show that $x(t)$ satisfy (5).

$$E\Big|\int_{t_0}^{t}[f(s,x_s^n)-f(s,x_s)]ds+\int_{t_0}^{t}[g(s,x_s^n)-g(s,x_s)]\,\mathrm{d}B(s)\Big|^2$$
$$\le 2\Big[(T-t_0)E\int_{t_0}^{t}|f(s,x_s^n)-f(s,x_s)|^2ds+4E\int_{t_0}^{t}|g(s,x_s^n)-g(s,x_s)|^2\,\mathrm{ds}\Big]$$
$$\le 2(T-t_0+4)\int_{t_0}^{t}\kappa\Big(E\Big(\sup_{t_0\le u\le s}|x^n(u)-x(u)|^2\Big)\Big)\,\mathrm{ds}.$$

Noting that sequence $x^n(t)$ is uniformly converge on $(-\infty,T]$, it means that

$$E\Big(\sup_{t_0\le u\le s}|x^n(u)-x(u)|^2\Big)\to 0$$

as $n\to\infty$, further $\kappa(E(\sup_{t_0\le u\le s}|x^n(u)-x(u)|^2))\to 0$ as $n\to\infty$. Hence, taking limits on both sides in the Picard sequence, we obtain that

$$x(t)=\xi(0)+G(t,x_t)-G(t_0,x_{t_0})+\int_{t_0}^{t}f(s,x_s)ds+\int_{t_0}^{t}g(s,x_s)dB(s).$$

The above expression demonstrates that $x(t)$ is a solution of equation (3) satisfying the initial condition (4). So far, the existence of theorem is complete. □

Remark 3.1. In the proof of Theorem 3.1 we have shown that the Picard iterations $x^n(t)$ converge to the unique solution $x(t)$ of equation (3). In the next study, we should gives an estimate on the difference between $x^n(t)$ and $x(t)$ under some special condition, and it clearly shows that one can use the Picard iteration procedure to obtain the approximate solutions to equations (3).

References

1. R.D. Driver, *A functional differential system of neutral type arising in a two-body problem of classical electrodynamics, in "Nonlinear Differential Equation and Nonlinear Mechanics"*, Academic Press (1963), 474-484.
2. T.E. Govindan, *Stability of mild solution of stochastic evolution equations with variable delay*, Stochastic Anal. Appl. **21** (2003), 1059-1077.
3. N. Halidias, *Remarks and corrections on "An existence theorem for stochastic functional differential equations with dealys under weak assumptions, Statistics and Probability Letters 78, 2008" by N. Halidias and Y. Ren*, Stochastics and Probability Letters **79** (2009), 2220-2222.
4. Y.-H. Kim, *An estimate on the solutions for stochastic functional differential equations* , J. Appl. Math. and Informatics **29** (2011) no.5-5, 1549-1556.
5. Y.-H. Kim, *A note on the solutions of neutral SFDEs with infinite delay*, J. inequalities and Applications **181** (2013) no.1, 1-11.
6. K. Liu, *Lyapunov functionals and asymptotic of stochastic delay evolution equations*, Stochastics and Stochastic Rep. **63** (1998), 1-26.
7. X. Li and X. Fu, *Stability analysis of stochastic functional differential equations with infinite delay and its application to recurrent neural networks*, J. Comput. Appl. Math. **234** (2010), 407-417.
8. X. Mao, *Stochastic Differential Equations and Applications,* Horwood Publication Chichester, UK (2007).
9. X. Mao, Y. Shen, and C. Yuan, *Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching*, Stochastic process. Appl. **118** (2008), 1385-1406.
10. F. Wei and Y. Cai, *Existence, uniqueness and stability of the solution to neutral stochastic functional differential equations with infinite delay under non-Lipschitz conditions*, Advances in Difference Equations **151** (2013), 1-12.

Young-Ho Kim received M.Sc. from Chung-Ang University and Ph.D at Chung-Ang University. Since 1997 he has been at Changwon National University. His research interests include Stochastic differential Equations and Stochastic Analysis.

Department of Mathematics, Changwon National University, Changwon Gyeongsangnam-do 51140, Korea.
e-mail: yhkim@changwon.ac.kr