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ELLIPTIC OBSTACLE PROBLEMS WITH MEASURABLE NONLINEARITIES IN NON-SMOOTH DOMAINS

Youchan Kim and Seungjin Ryu

Abstract. The Calderón-Zygmund type estimate is proved for elliptic obstacle problems in bounded non-smooth domains. The problems are related to divergence form nonlinear elliptic equation with measurable nonlinearities. Precisely, nonlinearity $\mathbf{a}(\xi, x_1, x')$ is assumed to be only measurable in one spatial variable x_1 and has locally small BMO semi-norm in the other spatial variables x', uniformly in ξ variable. Regarding non-smooth domains, we assume that the boundaries are locally flat in the sense of Reifenberg. We also investigate global regularity in the settings of weighted Orlicz spaces for the weak solutions to the problems considered here.

1. Introduction and main result

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a non-smooth boundary $\partial\Omega$. Given an obstacle $\psi \in H^1(\Omega)$ with $\psi \leq 0$ a.e. on $\partial\Omega$, we define the admissible set for the test functions:

$$\mathcal{A} = \left\{ \phi \in H_0^1(\Omega) : \phi \geq \psi \ \text{ a.e. in } \Omega \right\}.$$

We are interested in functions $u\colon \Omega \to \mathbb{R}$ belonging in $\mathcal{A}$ and satisfying the following variational inequality:

$$\int_\Omega \mathbf{a}(Du, x) \cdot D(\phi - u)\, dx \ \geq \ \int_\Omega F \cdot D(\phi - u)\, dx \tag{1.1}$$

for all $\phi \in \mathcal{A}$, where $F = \{f_i\} \in L^2(\Omega; \mathbb{R}^n)$. Here, the symbol $\cdot$ represents the standard inner product in $\mathbb{R}^n$. Such a function u is called a *weak solution to the variational inequality* (1.1). Throughout this paper, we assume that

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nonlinearity $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in x and differentiable in ξ, and satisfies the following strict monotonicity and uniform boundedness:

$$\begin{cases} \lambda|\xi - \eta|^2 \le [\mathbf{a}(\xi, x) - \mathbf{a}(\eta, x)] \cdot (\xi - \eta), \\ |\mathbf{a}(\xi, x)| + |\xi||D_\xi \mathbf{a}(\xi, x)| \le \Lambda|\xi| \end{cases} \tag{1.2}$$

for all $x, \xi, \eta \in \mathbb{R}^n$ and some constants $0 < \lambda \le 1 \le \Lambda < \infty$. With these basic continuity assumptions, we further assume that for each point and for each scale, nonlinearity $\mathbf{a}(\xi, x_1, x')$ is allowed to be merely measurable in one spatial variable x_1 and has locally small BMO semi-norm in the other spatial variables x' while the boundaries are trapped between two narrow strips.

According to the classical theory of the variational inequalities ([5, 7, 20]), there exists a unique weak solution $u \in \mathcal{A}$ of (1.1), with the estimate

$$\|Du\|_{L^2(\Omega;\mathbb{R}^n)} \le c \left(\|F\|_{L^2(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^2(\Omega;\mathbb{R}^n)} \right),$$

where the positive constant c is depending only on λ, Λ, n, and $|\Omega|$. We further refer the reader to the texts written by Kinderlehrer-Stampacchia [20], Friedman [18] and Rodrigues [28] for further discussions about the classical theory and its application of variational inequalities.

In this paper we consider an optimal Calderón-Zygmund type estimate for the weak solution to the variational inequality (1.1). We are interested in studying how the integrability of F and $D\psi$ is reflected to the integrability of the gradient of solutions, under minimal regularity assumptions on the nonlinearity $\mathbf{a}(\xi, x)$ and the smoothness requirement on the domain Ω. In general, the following $W^{1,p}$ estimates:

$$\|Du\|_{L^p(\Omega;\mathbb{R}^n)} \le c \left(\|F\|_{L^p(\Omega;\mathbb{R}^n)} + \|D\psi\|_{L^p(\Omega;\mathbb{R}^n)} \right) \tag{1.3}$$

holds true only for $p \in (2-\epsilon, 2+\epsilon)$, where ϵ is a small positive constant. Here, the constant c is independent of u, F, and $D\psi$, see [1]. On the contrary, to hold (1.3) for any value of p in the range $[2+\epsilon, \infty)$ requires additional assumptions on both $\mathbf{a}(\xi, x)$ and Ω. In that sense, it provides a natural extension of the previous result [8] which studied divergence form nonlinear elliptic equations without obstacles.

There are many other results in the literature regarding the optimal Calderón -Zygmund type estimate of nonlinear elliptic and parabolic obstacle problems. (See, for instance, [2, 4, 5, 7, 10, 11, 30–33].) In this paper, the nonlinearity $\mathbf{a}(\xi, x_1, x')$ is assumed to be only measurable in x_1 variable and has locally small bounded mean oscillation in x' variables. (See Definition 1.1 for details.) We would like to emphasize that, from the previous results [9, 14–16, 22–24], it can be inferred that this measurable assumption is optimal.

In order to state the additional hypotheses on $\mathbf{a}(\xi, x)$ and $\partial\Omega$ we introduce the following notation:

(1) $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ and $x = (x_1, x'), y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$.
(2) $B'_r(y') = \{x' \in \mathbb{R}^{n-1} : |x' - y'| < r\}$ and $Q_r(y) = (y_1 - r, y_1 + r) \times B'_r(y')$.

(3) $Q_r = Q_r(\mathbf{0})$, $Q_r^+ = Q_r \cap \{x_1 > 0\}$ and $Q_r^- = Q_r \cap \{x_1 < 0\}$.
(4) $\Omega_r(y) = \Omega \cap Q_r(y)$ and $\partial_w \Omega_r(y) = Q_r(y) \cap \partial\Omega$.
(5) $\fint_E g\, dx = \frac{1}{|E|}\int_E g\, dx$, where $g \in L^1(E)$ and E is a measurable subset in $\mathbb{R}^n$ with positive volume $|E|$.
(6) $\bar{g}_{E'}(x_1) = \fint_{E'} g(x_1, x')dx' = \frac{1}{|E'|}\int_{E'} g(x_1, x')dx'$, where E' is a bounded measurable subset of $\mathbb{R}^{n-1}$ and $|E'|$ stands for the $(n-1)$-dimensional Lebesgue measure of E'.

To measure the oscillation of $\mathbf{a}(\xi, x_1, x')$ in x_1-variable on $Q_r(y_1, y')$, uniformly in $\xi \in \mathbb{R}^n \setminus \{0\}$, we consider a function θ defined by

$$\theta(\mathbf{a}, Q_r(y))(x) = \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\mathbf{a}(\xi, x_1, x') - \bar{\mathbf{a}}_{B'_r(y')}(\xi, x_1)|}{|\xi|}, \tag{1.4}$$

where

$$\bar{\mathbf{a}}_{B'_r(y')}(\xi, x_1) = \fint_{B'_r(y')} \mathbf{a}(\xi, x_1, z')\, dz'.$$

We introduce the main assumptions on the nonlinearity $\mathbf{a}$ and the domain Ω.

Definition 1.1. We say that $(\mathbf{a}(\xi, x), \Omega)$ is (δ, R)-*vanishing of codimension* 1 if for every $y \in \Omega$ and for every number $r \in (0, R]$ with

$$\operatorname{dist}(y, \partial\Omega) = \min_{z \in \partial\Omega} \operatorname{dist}(y, z) > \sqrt{2}r,$$

there exists a coordinate system depending on y and r, whose variables are still denoted by $x = (x_1, x')$, so that in this coordinate system y is the origin and

$$\fint_{Q_r} |\theta(\mathbf{a}, Q_r)(x)|^2\, dx \le \delta^2,$$

while, for every $y \in \Omega$ and for every number $r \in (0, R]$ with

$$\operatorname{dist}(y, \partial\Omega) = \min_{z \in \partial\Omega} \operatorname{dist}(y, z) = \operatorname{dist}(y, z_0) \le \sqrt{2}r$$

for some $z_0 \in \partial\Omega$, there exists a coordinate system depending on y and r, whose variables we still denote by $x = (x_1, x')$, so that

$$Q_r^+ \ \subset \ Q_r \cap \Omega \ \subset \ Q_r \cap \{(x_1, x') : x_1 > -2\delta r\} \tag{1.5}$$

and

$$\fint_{Q_r} |\theta(\mathbf{a}, Q_r)(x)|^2\, dx \le \delta^2.$$

We have a few comments about this definition.

Remark 1.2. (1) If $(\mathbf{a}(\xi, x), \Omega)$ is (δ, R)-*vanishing of codimension* 1, then for each point and for each small scale, there is a coordinate system such that $\mathbf{a}(\xi, x_1, x')$ might have big jumps in x_1 variable but $\mathbf{a}(\xi, x_1, x')$ is in the class of BMO in x' variables with small semi-norm.

(2) The geometric condition (1.5) means that Ω satisfies the so-called (δ, R)-*Reifenberg flat condition.* (See [26, 35].) Moreover, it guarantees *the measure density,* that is, there is a constant $c_* = c_*(\delta, n, \delta) > 0$ such that

$$c_* |Q_r(z_0)| \ \leq \ |Q_r(z_0) \cap \Omega| \ \leq \ (1 - c_*)|Q_r(z_0)|$$

for each cylinder $Q_r(z_0)$ with $r \in (0, R)$ and $z_0 \in \partial\Omega$. Note that, in fact, the constant c_* can be uniformly bounded by choosing δ small enough.

(3) Due to the scaling invariance property (see Lemma 2.3 below), one can take for simplicity $R = 1$ or any other constants bigger than 1. On the other hand, the constant δ is invariant under the scaling. It will be determined later to belong to $(0, 1/8)$.

Our main result is following:

Theorem 1.3. *Let $u \in \mathcal{A}$ be the weak solution to the variational inequality* (1.1)*. For any given $p \in (2, \infty)$, suppose that $F \in L^p(\Omega; \mathbb{R}^n)$ and $D\psi \in L^p(\Omega; \mathbb{R}^n)$. Then there exists a constant $\delta = \delta(\lambda, \Lambda, n, p) \in (0, 1/8)$ such that if $(\mathbf{a}(\xi, x), \Omega)$ is $(\delta, 25R)$-vanishing of codimension* 1*, then $Du \in L^p(\Omega; \mathbb{R}^n)$ with the estimate* (1.3)*. Here, the positive constant c in* (1.3) *depends only on $\lambda, \Lambda, n, p, R,$ and $|\Omega|$.*

The paper is organized as follows. In the next section we present some auxiliary tools which play an important role in the rest of the paper. In Section 3 we give the $W^{1,2}$-approximation. In Section 4 we derive the required $W^{1,p}$-estimate based on the Hardy-Littlewood maximal operator and the Calderón-Zygmund-Krylov-Safonov-type covering lemma. In Section 5 we establish the weighted Orlicz regularity estimates for the variational problems considered here.

2. Preliminaries and auxiliary results

In this section, we summarize some results that will be crucially used in later sections. These results are well-known or otherwise we provide appropriate references. The main ingredients of our approach are the Hardy-Littlewood maximal operator, the Calderón-Zygmund-Krylov-Safonov type covering lemma, and the global Lipschitz regularity for the so-called limiting equation.

We start this section by recalling the Hardy-Littlewood maximal operator $\mathcal{M}$. Given a locally integrable function g defined in $\mathbb{R}^n$, the maximal function $\mathcal{M}g$ of g is

$$(\mathcal{M}g)(y) = \sup_{r>0} \fint_{Q_r(y)} |g(x)| \, dx.$$

If g is defined on a bounded subset of $\mathbb{R}^n$, then

$$\mathcal{M}g \ = \ \mathcal{M}\overline{g},$$

where $\overline{g}$ is the zero extension of g from the bounded set to $\mathbb{R}^n$. The following *weak type* (1, 1) *estimate*

$$|\{x \in \mathbb{R}^n : (\mathcal{M}g)(x) > \lambda\}| \;\leq\; \frac{c(n)}{\lambda} \int_{\mathbb{R}^n} |g(y)|\, dy, \tag{2.1}$$

is well-known for the maximal operator $\mathcal{M}$. Moreover, *strong type* (p, p) *estimate* holds for $p > 1$; that is, for $g \in L^p(\Omega)$ there is a constant $c = c(n, p) > 0$ such that

$$\frac{1}{c}\|g\|_{L^p(\mathbb{R}^n)} \leq \|\mathcal{M}g\|_{L^p(\mathbb{R}^n)} \leq c\|g\|_{L^p(\mathbb{R}^n)}. \tag{2.2}$$

We will need also the following standard measure theory results.

Lemma 2.1. *Given $p \in (1, \infty)$, suppose that g is a nonnegative and measurable function defined on a bounded subset Ω of $\mathbb{R}^n$. Let $\mu > 0$ and $\theta > 1$ be constants. Then*

$$g \in L^p(\Omega) \iff S := \sum_{k \geq 1} \theta^{kp} \left|\left\{x \in \Omega : g(x) > \mu\theta^k\right\}\right| < \infty$$

and

$$\frac{1}{c} S \leq \|g\|^p_{L^p(\Omega)} \leq c\,(|\Omega| + S)$$

with a positive constant c depending only on $\mu, \theta, n,$ and p.

The following Calderón-Zygmund-Krylov-Safonov-type covering lemma plays an important role in this paper. It may be proved in much same way as [8,27].

Lemma 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying (δ, R)-Reifenberg flat condition* (1.5). *Let C and D be measurable sets with $C \subset D \subset \Omega$. Suppose that there exists a small constant $\epsilon \in (0, 1)$ such that*

(1) *for every $y \in \Omega$, $|C \cap Q_R(y)| < \epsilon |Q_R(y)|$,*
(2) *for each $y \in \Omega$ and $r \in (0, R)$ one has that*

$$Q_r(y) \cap \Omega \subset D \quad \text{whenever} \quad |C \cap Q_r(y)| \geq \epsilon |Q_r(y)|.$$

Then

$$|C| \leq \left(\frac{20\sqrt{2}}{1 - \delta}\right)^n \epsilon\, |D|.$$

The obstacle problem considered here is invariant under scaling and normalization, which follows by a direct computations.

Lemma 2.3. *Let $u \in \mathcal{A}$ be the weak solution to the variational inequality* (1.1). *Assume that $(\mathbf{a}(\xi, x), \Omega)$ is (δ, R)-vanishing of codimension 1. Fix $M > 1$ and $0 < \rho < 1$, and define the rescaled maps*

$$\widetilde{\mathbf{a}}(\xi, x) = \frac{\mathbf{a}(M\xi, \rho x)}{M}, \quad \tilde{u}(x) = \frac{u(\rho x)}{M\rho}, \quad \tilde{F}(x) = \frac{F(\rho x)}{M}, \quad \tilde{\psi}(x) = \frac{\psi(\rho x)}{M\rho},$$

and the set $\widetilde{\Omega} = \left\{\frac{1}{\rho} x : x \in \Omega\right\}$.

Then

(1) $\widetilde{\mathbf{a}}(\xi,x)$ *satisfies the basic condition* (1.2) *with the same constants* λ *and* Λ.

(2) $\left(\widetilde{\mathbf{a}}(\xi,x),\widetilde{\Omega}\right)$ *is* $(\delta,\ \frac{R}{\rho})$*-vanishing of codimension* 1.

(3) $\tilde{u}\in\tilde{\mathcal{A}}=\left\{\tilde{\phi}\in H_0^1(\widetilde{\Omega})\colon\tilde{\phi}\geq\tilde{\psi}\text{ a.e. in }\widetilde{\Omega}\right\}$ *is the weak solution to the variational inequality*:

$$\int_{\widetilde{\Omega}}\widetilde{\mathbf{a}}(\xi,x)\cdot D\left(\tilde{\phi}-\tilde{u}\right)dx\geq\int_{\widetilde{\Omega}}\tilde{F}\cdot D\left(\tilde{\phi}-\tilde{u}\right)dx,\quad\forall\tilde{\phi}\in\tilde{\mathcal{A}}.$$

We need the following comparison estimates. The proof can be found in [7, Lemma 3.5].

Lemma 2.4. *Suppose that* $v,\tilde{v}\in H^1(\Omega)$ *satisfy*

$$\begin{cases}-\operatorname{div}\mathbf{a}(Dv,x)\ \leq\ -\operatorname{div}\mathbf{a}(D\tilde{v},x) & \text{in }\ \Omega,\\ \qquad v\leq\tilde{v} & \text{on }\ \partial\Omega.\end{cases}$$

Then we have that $v\leq\tilde{v}$ *a.e. in* Ω.

The next lemma is the Lipschitz regularity of the so-called limiting equations with zero Dirichlet boundary data. It may be driven by [8, Lemma 4.10] with slight modifications. For the further references about the regularity results in obstacle problems, we refer to [3, 12, 13, 19] for the gradients Hölder continuity results and [25] for the gradient continuity result.

Lemma 2.5. *Let* v *be a weak solution of*

$$\begin{cases}-\operatorname{div}\bar{\mathbf{a}}_{B'_{3r}}(Dv,x_1)=0 & \text{in }\ Q_{3r}^+,\\ \qquad v=0 & \text{on }\ Q_{3r}^+\cap\{x_1=0\},\end{cases}$$

where (1.2) *is assumed on* $\bar{\mathbf{a}}_{B'_{3r}}(\xi,x_1)$. *Then we have the following Lipschitz estimate*:

$$\|Dv\|_{L^\infty(Q_{2r}^+)}\leq c\left(\fint_{Q_{3r}^+}|Dv|^2\,dx\right)^{\frac{1}{2}},$$

where the constant c *is depending only on* λ, Λ, *and* n.

From now on, we will use the letter c to denote a constant that can be explicitly computed in terms of known quantities such as λ,Λ,n,p,R, and $|\Omega|$. This constant may vary in different occurrences.

3. $W^{1,2}$-approximation lemmas

We start with an interior approximation lemma.

Lemma 3.1. *Let* u *be the weak solution of the variational inequality* (1.1) *and* $Q_{4r}\subset\Omega$. *Suppose that for a small positive constant* δ

$$\fint_{Q_{4r}}|\theta(\mathbf{a},Q_{4r})|^2\,dx\leq\delta^2,$$

and

$$\fint_{Q_{4r}} |Du|^2\,dx \le 1, \qquad \fint_{Q_{4r}} |F|^2\,dx \le \delta^2, \quad \fint_{Q_{4r}} |D\psi|^2\,dx \le \delta^2$$

hold. Then there exist a constant $n_1 = n_1(n, \lambda, \Lambda) \ge 1$ *and a function* $W \in L^\infty(Q_{2r})$ *such that*

$$\|W\|_{L^\infty(Q_{2r})} \le n_1 \quad \text{and} \quad \fint_{Q_{2r}} |Du - W|^2\,dx \le c\delta^{\frac{\sigma_1}{2+\sigma_1}}$$

for some universal constant $\sigma_1 > 0$.

The proof of this lemma will be omitted as it is very similar to that of Lemma 3.2. We also refer to the proof of Lemmas 4.4 and 4.3 in [7].

We next prove an approximation lemma near the boundary.

Lemma 3.2. *Let* u *be the weak solution to the variational inequality* (1.1). *Suppose that for a small positive constant* δ

$$Q_{5r}^+ \subset \Omega_{5r} \subset Q_{5r} \cap \{x_1 > -10\delta r\}, \tag{3.1}$$

$$\fint_{Q_{5r}} |\theta(\mathbf{a}, Q_{5r})|^2\,dx \le \delta^2, \tag{3.2}$$

and

$$\fint_{\Omega_{5r}} |Du|^2\,dx \le 1, \qquad \fint_{\Omega_{5r}} |F|^2\,dx \le \delta^2, \quad \fint_{\Omega_{5r}} |D\psi|^2\,dx \le \delta^2 \tag{3.3}$$

hold. Then there exist a constant $n_2 = n_2(n, \lambda, \Lambda) \ge 1$ *and a function* $V \in L^\infty(\Omega_{2r})$ *such that*

$$\|V\|_{L^\infty(\Omega_{2r})} \le n_2 \quad \text{and} \quad \fint_{\Omega_{2r}} |Du - V|^2\,dx \le c\delta^{\frac{\sigma_2}{2+\sigma_2}}$$

for some universal constant $\sigma_2 > 0$.

Proof. The proof will be divided into several steps. Note that Step 4 might not be required when we prove interior cases.

Step 1. We first consider the following nonlinear elliptic equation.

$$\begin{cases} -\operatorname{div} \mathbf{a}(Dk, x) = -\operatorname{div} \mathbf{a}(D\psi, x) & \text{in} \quad \Omega_{5r}, \\ k = u & \text{on} \quad \partial\Omega_{5r}. \end{cases} \tag{3.4}$$

Since $u \in \mathcal{A}$, we see that $k \ge \psi$ a.e. in Ω_{5r}, by Lemma 2.4. Without loss of generality, we may assume that $k = u\,(\ge \psi)$ in $\Omega \setminus \Omega_{5r}$ and so $k \ge \psi$ a.e. in Ω. It follows from taking $\phi = k$ in (1.1) that

$$\int_{\Omega_{5r}} \mathbf{a}(Du, x) \cdot D\,(u - k)\,dx \;\le\; \int_{\Omega_{5r}} F \cdot D\,(u - k)\,dx. \tag{3.5}$$

By (1.2) and (3.4)-(3.5) we see that

$$\fint_{\Omega_{5r}} |D(u - k)|^2\,dx \le c \fint_{\Omega_{5r}} [\mathbf{a}(Du, x) - \mathbf{a}(Dk, x)] \cdot (Du - Dk)\,dx$$

$$\leq c \fint_{\Omega_{5r}} [F - \mathbf{a}(D\psi, x)] \cdot D(u-k)\, dx$$
$$\leq c \fint_{\Omega_{5r}} \left[|F|^2 + |D\psi|^2\right] dx + \frac{1}{2} \fint_{\Omega_{5r}} |D(u-k)|^2\, dx.$$

The last inequality follows from (1.2) again and *Young's inequality*. Thanks to (3.3), we have that

$$\fint_{\Omega_{5r}} |D(u-k)|^2\, dx \leq c \fint_{\Omega_{5r}} \left[|F|^2 + |D\psi|^2\right]\, dx \leq c\delta^2 \tag{3.6}$$

and then

$$\fint_{\Omega_{5r}} |Dk|^2\, dx \leq c\left(1 + \fint_{\Omega_{5r}} |Du|^2\, dx\right) \leq c. \tag{3.7}$$

Step 2. We next compare k to the unique weak solution w of

$$\begin{cases} -\operatorname{div} \mathbf{a}(Dw, x) = 0 & \text{in} \quad \Omega_{5r}, \\ w = k & \text{on} \quad \partial\Omega_{5r}. \end{cases} \tag{3.8}$$

Then it follows that

$$\fint_{\Omega_{5r}} |D(w-k)|^2\, dx \leq c \fint_{\Omega_{5r}} [\mathbf{a}(Dw, x) - \mathbf{a}(Dk, w)] \cdot D(w-k)\, dx$$
$$\leq c \fint_{\Omega_{5r}} [-\mathbf{a}(Dk, w)] \cdot D(w-k)\, dx.$$

Since $w - k \in H_0^1(\Omega_{5r})$, we take $w - k$ as a test function of (3.4). It follows from (1.2) and *Young's inequality* that

$$\fint_{\Omega_{5r}} [-\mathbf{a}(Dk, x)] \cdot D(w-k)\, dx = \fint_{\Omega_{5r}} [-\mathbf{a}(D\psi, x)] \cdot D(w-k)\, dx$$
$$\leq c \fint_{\Omega_{5r}} |D\psi|^2\, dx + \frac{1}{2} \fint_{\Omega_{5r}} |D(w-k)|^2\, dx.$$

Thus we have that, by (3.3),

$$\fint_{\Omega_{5r}} |D(w-k)|^2\, dx \leq c \fint_{\Omega_{5r}} |D\psi|^2\, dx \leq c\delta^2 \tag{3.9}$$

and that, by (3.7),

$$\fint_{\Omega_{5r}} |Dw|^2\, dx \leq c\left(1 + \fint_{\Omega_{5r}} |Dk|^2\, dx\right) \leq c. \tag{3.10}$$

Step 3. Let h be a unique weak solution of

$$\begin{cases} -\operatorname{div} \bar{\mathbf{a}}_{B'_{4r}}(Dh, x_1) = 0 & \text{in} \quad \Omega_{4r}, \\ h = w & \text{on} \quad \partial\Omega_{4r}. \end{cases} \tag{3.11}$$

Then we have

$$\begin{aligned}&\fint_{\Omega_{4r}} |D(w-h)|^2\,dx\\ &\le c\fint_{\Omega_{4r}} [\bar{\mathbf{a}}_{B'_{4r}}(Dw,x_1) - \bar{\mathbf{a}}_{B'_{4r}}(Dh,x_1)]\cdot D(w-h)\,dx\\ &\le c\fint_{\Omega_{4r}} \bar{\mathbf{a}}_{B'_{4r}}(Dw,x_1)\cdot D(w-h)\,dx.\end{aligned} \tag{3.12}$$

Since $w-h \in H_0^1(\Omega_{4r})$, we may assume that $w-h=0$ in $\Omega\setminus\Omega_{4r}$. Consequently we see that

$$\int_{\Omega_{4r}} \mathbf{a}(Dw,x)\cdot D(w-h)\,dx = \int_{\Omega_{5r}} \mathbf{a}(Dw,x)\cdot D(w-h)\,dx = 0, \tag{3.13}$$

by taking the test function $w-h\in H_0^1(\Omega_{5r})$ in the problem (3.8). It follows from (3.12), (3.13) and (1.4) that

$$\begin{aligned}\fint_{\Omega_{4r}} |D(w-h)|^2\,dx &\le c\fint_{\Omega_{4r}} [\bar{\mathbf{a}}_{B'_{4r}}(Dw,x_1) - \mathbf{a}(Dw,x)]\cdot D(w-h)\,dx\\ &\le c\fint_{\Omega_{4r}} \theta(\mathbf{a},Q_{4r})\,|Dw|\,|D(w-h)|\,dx\\ &\le c\fint_{\Omega_{4r}} |\theta(\mathbf{a},Q_{4r})|^2|Dw|^2dx + \frac{1}{2}\fint_{\Omega_{4r}} |D(w-h)|^2\,dx\end{aligned}$$

and so

$$\fint_{\Omega_{4r}} |D(w-h)|^2\,dx \;\le\; c\fint_{\Omega_{4r}} |\theta(\mathbf{a},Q_{4r})|^2|Dw|^2\,dx. \tag{3.14}$$

Thanks to the Reifenberg flatness condition (3.1), Ω_{4r} satisfies the measure density condition (see Remark 1.2) and thus the local version of Sobolev's inequality and Poincaré's inequality hold true on the Reifenberg flat domain (see [6, 7]). Moreover, the weak solution w of the homogeneous equation (3.8) has the property of self-improving integrability, because $w=0$ on $\partial_w\Omega_{5r}$. Precisely, there exists a small positive constant $\sigma=\sigma(n,\lambda,\Lambda)$ such that

$$\left(\fint_{\Omega_{4r}} |Dw|^{2+\sigma}\,dx\right)^{\frac{1}{2+\sigma}} \;\le\; c\left(\fint_{\Omega_{5r}} |Dw|^2\,dx\right)^{\frac{1}{2}} \tag{3.15}$$

and consequently it follows from (3.10) that

$$\fint_{\Omega_{4r}} |Dw|^{2+\sigma}\,dx \;\le\; c. \tag{3.16}$$

With this self-improving property, we estimate the right hand side of (3.14) as follows:

$$\fint_{\Omega_{4r}} |\theta(\mathbf{a}, Q_{4r})|^2 |Dw|^2\, dx \;\le\; \left(\fint_{\Omega_{4r}} |\theta(\mathbf{a}, Q_{4r})|^{\frac{2(2+\sigma)}{\sigma}}\, dx\right)^{\frac{\sigma}{2+\sigma}} \times \left(\fint_{\Omega_{4r}} |Dw|^{2+\sigma}\, dx\right)^{\frac{2}{2+\sigma}}. \tag{3.17}$$

By (1.2), (1.4) and (3.1)-(3.2), it follows

$$\begin{aligned}\left(\fint_{\Omega_{4r}} |\theta(\mathbf{a}, Q_{4r})|^{\frac{2(2+\sigma)}{\sigma}}\, dx\right)^{\frac{\sigma}{2+\sigma}} &\le c\left(\fint_{Q_{4r}} |\theta(\mathbf{a}, Q_{4r})|^2\, dx\right)^{\frac{\sigma}{2+\sigma}}\\ &\le c\delta^{\frac{2\sigma}{2+\sigma}}.\end{aligned} \tag{3.18}$$

Hence, it follows from (3.14), (3.17), (3.16), and (3.18) that

$$\fint_{\Omega_{4r}} |D(w-h)|^2\, dx \;\le\; c\delta^{\frac{2\sigma}{2+\sigma}}. \tag{3.19}$$

Step 4. Define $\eta = \eta(x_1) \in C^\infty(\mathbb{R})$ with

$$\eta = 0 \;\text{ in }\; (-10\delta r, 0), \quad \eta = 1 \;\text{ in }\; \mathbb{R}\setminus(-12\delta r, 2\delta r), \quad\text{and}\quad |D\eta| \le \frac{c}{\delta r}. \tag{3.20}$$

Then we consider a unique weak solution v of

$$\begin{cases} -\operatorname{div} \bar{\mathbf{a}}_{B'_{4r}}(Dv, x_1) = 0 & \text{in} \quad Q^+_{3r},\\ v = \eta h & \text{on} \quad \partial Q^+_{3r}.\end{cases} \tag{3.21}$$

It directly follows that

$$\int_{Q^+_{3r}} \bar{\mathbf{a}}_{B'_{4r}}(Dv, x_1)\cdot D(v-\eta h)\, dx = 0$$

and, by (3.20),

$$\int_{\Omega_{3r}} \bar{\mathbf{a}}_{B'_{4r}}(D\bar{v}, x_1)\cdot D(\bar{v}-\eta h)\, dx = 0, \tag{3.22}$$

where $\bar{v}$ is the zero extension of v from Q^+_{3r} to Q_{3r}. Due to (3.11) and $\eta(\bar{v}-\eta h) \in H^1_0(\Omega_{3r})$, we have that

$$\int_{\Omega_{3r}} \bar{\mathbf{a}}_{B'_{4r}}(Dh, x_1)\cdot D\big[\eta(v-\eta h)\big]\, dx = 0. \tag{3.23}$$

Combining (1.2), (3.22), and (3.23) yields

$$\begin{aligned}&\lambda \fint_{\Omega_{3r}} |D(\bar{v}-\eta h)|^2\, dx\\ &\le \fint_{\Omega_{3r}} [\bar{\mathbf{a}}_{B'_{4r}}(D\bar{v}, x_1) - \bar{\mathbf{a}}_{B'_{4r}}(D(\eta h), x_1)]\cdot D(\bar{v}-\eta h)\, dx\\ &= \fint_{\Omega_{3r}} [\bar{\mathbf{a}}_{B'_{4r}}(Dh, x_1) - \bar{\mathbf{a}}_{B'_{4r}}(D(\eta h), x_1)]\cdot D(\bar{v}-\eta h)\, dx\end{aligned}$$

$$
\begin{aligned}
&- \fint_{\Omega_{3r}} \bar{\mathbf{a}}_{B'_{4r}}(Dh, x_1) \cdot D(\bar{v} - \eta h)\, dx \\
&= \fint_{\Omega_{3r}} [\bar{\mathbf{a}}_{B'_{4r}}(Dh, x_1) - \bar{\mathbf{a}}_{B'_{4r}}(D(\eta h), x_1)] \cdot D(\bar{v} - \eta h)\, dx \\
&\qquad + \fint_{\Omega_{3r}} \bar{\mathbf{a}}_{B'_{4r}}(Dh, x_1) \cdot D\eta\, (\bar{v} - \eta h)\, dx \\
&\qquad + \fint_{\Omega_{3r}} \bar{\mathbf{a}}_{B'_{4r}}(Dh, x_1) \cdot D(\bar{v} - \eta h)\, (\eta - 1)\, dx
\end{aligned}
$$

and so, by (1.2) again,

$$
\begin{aligned}
\fint_{\Omega_{3r}} |D(\bar{v} - \eta h)|^2\, dx \le c\bigg(&\fint_{\Omega_{3r}} |Dh - D(\eta h)|\, |D(\bar{v} - \eta h)|\, dx \\
&+ \fint_{\Omega_{3r}} |Dh|\, |D\eta|\, |\bar{v} - \eta h|\, dx \\
&+ \fint_{\Omega_{3r}} |Dh|\, |D(\bar{v} - \eta h)|\, |1 - \eta|\, dx \bigg).
\end{aligned} \tag{3.24}
$$

We first note that, from the self-improving property (see, for example, (3.15)), there exists a constant $\sigma_0 > 0$ such that

$$
\left(\fint_{\Omega_{3r}} |Dh|^{2+\sigma_0} dx \right)^{\frac{1}{2+\sigma_0}} \le c \left(\fint_{\Omega_{4r}} |Dh|^2 dx \right)^{\frac{1}{2}}
$$

and thus, by (3.10) and (3.19),

$$
\left(\fint_{\Omega_{3r}} |Dh|^{2+\sigma_0} dx \right)^{\frac{1}{2+\sigma_0}} \le c. \tag{3.25}
$$

We now estimate the first integral on right-hand side of (3.24). From Hölder's inequality, we see

$$
\begin{aligned}
&\fint_{\Omega_{3r}} |Dh - D(\eta h)|\, |D(\bar{v} - \eta h)|\, dx \\
&\le c \left(\fint_{\Omega_{3r}} |1 - \eta|^2 |Dh|^2 + |D\eta|^2 |h|^2 dx \right)^{\frac{1}{2}} \left(\fint_{\Omega_{3r}} |D(\bar{v} - \eta h)|^2 dx \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.26}
$$

Since it follows from (3.25) that

$$
\begin{aligned}
&\left(\frac{1}{|\Omega_{3r}|} \int_{\Omega_{3r} \cap \{x_1 < 2\delta r\}} |Dh|^2 dx \right)^{\frac{1}{2}} \\
&\le c \left(\fint_{\Omega_{3r}} |Dh|^{2+\sigma_0} dx \right)^{\frac{1}{2+\sigma_0}} \left(\frac{|\Omega_{3r} \cap \{x_1 < 2\delta\}|}{|\Omega_{3r}|} \right)^{\frac{\sigma_0}{2(2+\sigma_0)}} \\
&\le c \delta^{\frac{\sigma_0}{2(2+\sigma_0)}},
\end{aligned} \tag{3.27}
$$

we have that, by (3.20),

$$\left(\fint_{\Omega_{3r}} |1-\eta|^2 |Dh|^2 dx\right)^{\frac{1}{2}} \le \left(\frac{1}{|\Omega_{3r}|}\int_{\Omega_{3r}\cap\{x_1<2\delta r\}} |Dh|^2 dx\right)^{\frac{1}{2}} \le c\delta^{\frac{\sigma_0}{2(2+\sigma_0)}}. \tag{3.28}$$

Let $\bar{h}$ be the zero extension of h from Ω_{3r} to Q_{3r}. Since $h=0$ on $\partial_w\Omega_{3r}$, $\bar{h}\in H^1(Q_{3r})$ and it follows

$$\begin{aligned}
&\int_{\Omega_{3r}\cap\{x_1<2\delta r\}} |h(x)|^2 dx \\
&\le \int_{\Omega_{3r}\cap\{x_1<2\delta r\}} \left(\int_{-12\delta r}^{2\delta r} |D_1\bar{h}(y_1,x')|dy_1\right)^2 dx \\
&\le c\delta r \int_{\Omega_{3r}\cap\{x_1<2\delta r\}} \left(\int_{-12\delta r}^{2\delta r} |D_1\bar{h}(y_1,x')|^2 dy_1\right) dx \\
&\le c(\delta r)^2 \int_{\Omega_{3r}\cap\{x_1<2\delta r\}} |D_1\bar{h}(y_1,x')|^2 dy_1 dx'.
\end{aligned} \tag{3.29}$$

Thus, it follows from (3.20) and (3.29) that

$$\begin{aligned}
\int_{\Omega_{3r}} |D\eta|^2|h|^2 dx &\le \frac{c}{(\delta r)^2}\int_{\Omega_{3r}\cap\{x_1<2\delta r\}} |h|^2 dx \\
&\le c_* \int_{\Omega_{3r}\cap\{x_1<2\delta r\}} |Dh|^2 dx,
\end{aligned} \tag{3.30}$$

where the constant c_* is independent on δ and r. Combining (3.30) and (3.27) implies

$$\left(\fint_{\Omega_{3r}} |D\eta|^2 |h|^2 dx\right)^{\frac{1}{2}} \le c\left(\frac{1}{|\Omega_{3r}|}\int_{\Omega_{3r}\cap\{x_1<2\delta r\}} |Dh|^2 dx\right)^{\frac{1}{2}} \le c\delta^{\frac{\sigma_0}{2(2+\sigma_0)}}. \tag{3.31}$$

Thanks to (3.26), (3.28), and (3.31), thus, we have

$$\fint_{\Omega_{3r}} |Dh-D(\eta h)|\,|D(\bar{v}-\eta h)|\,dx \le c\delta^{\frac{\sigma_0}{2(2+\sigma_0)}}\left(\fint_{\Omega_{3r}} |D(\bar{v}-\eta h)|^2 dx\right)^{\frac{1}{2}}. \tag{3.32}$$

We next estimate the second integral on right-hand side of (3.24). From (3.20) and Hölder's inequality,

$$\begin{aligned}
&\fint_{\Omega_{3r}} |Dh|\,|D\eta|\,|\bar{v}-\eta h|\,dx \\
&\le \frac{c(\delta r)^{-1}}{|\Omega_{3r}|}\int_{\Omega_{3r}\cap\{x_1<2\delta r\}} |Dh|\,|\bar{v}-\eta h|\,dx
\end{aligned} \tag{3.33}$$

$$\leq \frac{c(\delta r)^{-1}}{|\Omega_{3r}|}\left(\int_{\Omega_{3r}\cap\{x_1<2\delta r\}}|Dh|^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega_{3r}\cap\{x_1<2\delta r\}}|\bar{v}-\eta h|^2dx\right)^{\frac{1}{2}}.$$

Since $\bar{v}-\eta h=0$ on $Q_{3r}\cap\{x_1=0\}$, on the other hand, it follows

$$|(\bar{v}-\eta h)(x)|\leq\int_0^{2\delta r}|D_1(\bar{v}-\eta h)(y_1,x')|dy_1$$

in $Q^+_{3r}\cap\{x_1<2\delta r\}$ and by Hölder's inequality

$$|(\bar{v}-\eta h)(x)|^2\leq 2\delta r\int_0^{2\delta r}|D_1(\bar{v}-\eta h)(y_1,x')|^2dy_1$$

in $Q^+_{3r}\cap\{x_1<2\delta r\}$. From the fact that $\bar{v}-\eta h=0$ on $\Omega_{3r}\setminus Q^+_{3r}$, we have that

$$\int_{\Omega_{3r}\cap\{x_1<2\delta r\}}|\bar{v}-\eta h|^2dx\leq c(\delta r)^2\int_{\Omega_{3r}\cap\{x_1<2\delta r\}}|D_1(\bar{v}-\eta h)|^2dx$$

and thus

$$\left(\frac{1}{|\Omega_{3r}|}\int_{\Omega_{3r}\cap\{x_1<2\delta r\}}|\bar{v}-\eta h|^2dx\right)^{\frac{1}{2}}\leq c\delta r\left(\fint_{\Omega_{3r}}|D(\bar{v}-\eta h)|^2dx\right)^{\frac{1}{2}}. \tag{3.34}$$

Combining (3.33), (3.27), and (3.34) yields that

$$\fint_{\Omega_{3r}}|Dh|\,|D\eta|\,|\bar{v}-\eta h|\,dx\leq c\delta^{\frac{\sigma_0}{2(2+\sigma_0)}}\left(\fint_{\Omega_{3r}}|D(\bar{v}-\eta h)|^2dx\right)^{\frac{1}{2}}. \tag{3.35}$$

The last integral on right-hand side of (3.24) is estimated as follows:

$$\begin{aligned}&\fint_{\Omega_{3r}}|Dh|\,|D(\bar{v}-\eta h)|\,|1-\eta|\,dx\\ &\leq\left(\frac{1}{|\Omega_{3r}|}\int_{\Omega_{3r}\cap\{x_1<2\delta r\}}|Dh|^2dx\right)^{\frac{1}{2}}\left(\fint_{\Omega_{3r}}|D(\bar{v}-\eta h)|^2dx\right)^{\frac{1}{2}}\\ &\leq c\delta^{\frac{\sigma_0}{2(2+\sigma_0)}}\left(\fint_{\Omega_{3r}}|D(\bar{v}-\eta h)|^2dx\right)^{\frac{1}{2}},\end{aligned} \tag{3.36}$$

where we used Hölder's inequality, (3.20), and (3.31). Therefore, combining (3.24), (3.32), (3.35), and (3.36), we have

$$\fint_{\Omega_{3r}}|D(\bar{v}-\eta h)|^2\,dx\leq c\delta^{\frac{\sigma_0}{2+\sigma_0}}. \tag{3.37}$$

A direct calculation yields that

$$\fint_{\Omega_{3r}}|D(\bar{v}-h)|^2\,dx\leq 2\fint_{\Omega_{3r}}|D(\bar{v}-\eta h)|^2+|D(\eta h)-Dh|^2\,dx$$

and

$$\fint_{\Omega_{3r}}|D(\eta h)-Dh|^2\,dx\leq 2\fint_{\Omega_{3r}}|1-\eta|^2|Dh|^2+|D\eta|^2|h|^2\,dx.$$

Consequently, by (3.37), (3.28), and (3.31), we obtain that

$$\fint_{\Omega_{3r}} |D(\bar{v}-h)|^2\,dx \le c\delta^{\frac{\sigma_0}{2+\sigma_0}} \tag{3.38}$$

and that, by (3.1) and (3.25)

$$\fint_{Q_{3r}^+} |Dv|^2\,dx \le c\fint_{\Omega_{3r}} |D\bar{v}|^2\,dx \le c, \tag{3.39}$$

because $\bar{v}$ is the zero extension of v from Q_{3r}^+ to Q_{3r}.

Step 5. Now, we combine (3.6), (3.9), (3.19), and (3.38) to derive

$$\fint_{\Omega_{2r}} |Du - D\bar{v}|^2\,dx \le c\left(\delta^2 + \delta^2 + \delta^{\frac{2\sigma}{2+\sigma}} + \delta^{\frac{\sigma_0}{2+\sigma_0}}\right) \le c\delta^{\frac{\sigma_2}{2+\sigma_2}}$$

for some $\sigma_2 > 0$. Applying Lemma 2.5 with (3.21) and (3.39) yields that

$$\|D\bar{v}\|_{L^\infty(\Omega_{2r})} \le c\|Dv\|_{L^\infty(Q_{2r}^+)} \le c\left(\fint_{Q_{3r}^+} |Dv|^2 dx\right)^{\frac{1}{2}} \le c.$$

Finally, we complete the proof of this lemma by taking $V = D\bar{v}$. □

4. Global $W^{1,p}$ estimates

In this section, we obtain the optimal $W^{1,p}$ regularity for the weak solution to the variational inequality (1.1) based on Lemma 2.2. So, let $u \in \mathcal{A}$ be the weak solution to (1.1). For a given $p \in (2,\infty)$, assume that F, $D\psi \in L^p(\Omega;\mathbb{R}^n)$.

Now, in order to apply Lemma 2.2 to our situation, we need the following result.

Lemma 4.1. *There exists a constant $N = N(\lambda,\Lambda,n) > 1$ such that for each $\epsilon \in (0,1)$ one can select a small $\delta = \delta(\epsilon,\lambda,\Lambda,n) \in (0,\frac{1}{8})$ such that for such a small δ, if $(\mathbf{a}(\xi,x),\Omega)$ is $(\delta,25R)$-vanishing of codimension 1 and $Q_r(y)$ with $y \in \Omega$ and $r \in (0,R)$ satisfies*

$$\left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^2\right) > N^2\right\} \cap Q_r(y)\right| \ge \epsilon\,|Q_r(y)|\,, \tag{4.1}$$

then we have

$$\begin{aligned}\Omega_r(y) \subset \left\{x \in \Omega\colon \mathcal{M}\left(|Du|^2\right) > 1\right\} &\cup \left\{x \in \Omega\colon \mathcal{M}\left(|F|^2\right) > \delta^2\right\}\\ &\cup \left\{x \in \Omega\colon \mathcal{M}\left(|D\psi|^2\right) > \delta^2\right\}.\end{aligned} \tag{4.2}$$

Proof. We prove the lemma by contradiction. Thus, assume that $Q_r(y)$ satisfies (4.1) and the claim (4.2) is false. Then there exists a point $y_1 \in \Omega_r(y) = Q_r(y) \cap \Omega$ such that for every $\rho > 0$ one has

$$\frac{1}{|Q_\rho(y_1)|}\int_{\Omega_\rho(y_1)} |Du|^2\,dx \le 1,$$

$$\frac{1}{|Q_\rho(y_1)|}\int_{\Omega_\rho(y_1)} |F|^2\,dx \le \delta^2, \quad \text{and} \quad \frac{1}{|Q_\rho(y_1)|}\int_{\Omega_\rho(y_1)} |D\psi|^2\,dx \le \delta^2. \tag{4.3}$$

We first consider the interior case: $Q_{6\sqrt{2}r}(y) \subset \Omega$. Without loss of generality, we may assume that $y = \mathbf{0}$. Then

$$Q_{4\sqrt{2}r} \subset Q_{5\sqrt{2}r}(y_1) \subset Q_{6\sqrt{2}r} \subset \Omega. \tag{4.4}$$

Since $(\mathbf{a}(\xi, x), \Omega)$ is $(\delta, 25R)$-vanishing of codimension 1, there exists a new coordinate system such that

$$Q_{4r} \subset \Omega \quad \text{and} \quad \fint_{Q_{4r}} |\theta(\mathbf{a}, Q_{4r})|^2 \, dx \le \delta^2. \tag{4.5}$$

It follows from (4.3), (4.4), and (4.5) that

$$\fint_{Q_{4r}} |Du|^2 dx \le \frac{|Q_{5\sqrt{2}r}|}{|Q_{4r}|} \fint_{Q_{5\sqrt{2}r}(y_1)} |Du|^2 dx \le c.$$

Similarly, we have that

$$\fint_{Q_{4r}} |F|^2 dx \le c\delta^2 \quad \text{and} \quad \fint_{Q_{4r}} |D\psi|^2 dx \le c\delta^2.$$

Applying Lemma 3.1, one can find $W \in L^\infty(Q_{2r})$ and $n_1 = n_1(n, \lambda, \Lambda) \ge 1$ such that

$$\|W\|_{L^\infty(Q_{2r})} \le n_1 \quad \text{and} \quad \fint_{Q_{2r}} |Du - W|^2 dx \le c\delta^{\frac{\sigma_1}{2+\sigma_1}} \tag{4.6}$$

for some universal constant $\sigma_1 > 0$. Now we set $N_1 = \max\{4n_1^2, 2^n\}$. Since $Q_r \subset \Omega$, we have that

$$\begin{aligned} &\left|\left\{x \in \Omega : \mathcal{M}(|Du|^2) > N_1^2\right\} \cap Q_r\right| \\ \le\ &\left|\left\{x \in Q_r : \mathcal{M}(|Du - W|^2) > n_1^2\right\}\right| + \left|\left\{x \in Q_r : \mathcal{M}(|W|^2) > n_1^2\right\}\right| \\ \le\ &\frac{c}{n_1^2} \int_{Q_r} |Du - W|^2 dx \le c\delta^{\frac{\sigma_1}{2+\sigma_1}} |Q_r| \end{aligned}$$

by (4.6) and *weak type* (1, 1) *estimate* (2.1). Since $y = \mathbf{0}$, we find that, in the original coordinate system, for $N \ge N_1$

$$|\{x \in \Omega : (|Du|^2) > N^2\} \cap Q_r(y)| \le c\delta^{\frac{\sigma_1}{2+\sigma_1}} |Q_r(y)|.$$

This is a contradiction to (4.1) and completes the proof of the interior case.

We next consider the boundary case: $Q_{6\sqrt{2}r}(y) \not\subset \Omega$. Since $(\mathbf{a}(\xi, x), \Omega)$ is $(\delta, 25R)$-vanishing of codimension 1, there exists a coordinate system such that

$$Q^+_{25r} \subset \Omega_{25r} \subset Q_{25r} \cap \{x_1 > -50\delta r\} \quad \text{and} \quad \fint_{Q_{25r}} |\theta(\mathbf{a}, Q_{25r})|^2 dx \le \delta^2. \tag{4.7}$$

Since $\Omega_{25r} \subset Q_{25\sqrt{2}r} \subset Q_{31\sqrt{2}r}(y) \subset Q_{32\sqrt{2}r}(y_1)$,

$$\fint_{\Omega_{25r}} |Du|^2 dx \le \frac{|Q_{32\sqrt{2}r}(y_1)|}{|\Omega_{25r}|} \fint_{Q_{32\sqrt{2}r}(y_1)} |Du|^2 dx \le c \tag{4.8}$$

by (4.3). Similarly, we have

$$\fint_{\Omega_{25r}} |F|^2 dx \le c\delta^2 \quad \text{and} \quad \fint_{\Omega_{25r}} |D\psi|^2 dx \le c\delta^2. \tag{4.9}$$

It follows from Lemma 3.2 with (4.7), (4.8), and (4.9) that there exist $V \in L^\infty(\Omega_{10r})$ and $n_2 = n_2(n, \lambda, \Lambda) \ge 1$ such that

$$\|V\|_{L^\infty(\Omega_{10r})} \le n_2 \qquad \text{and} \qquad \fint_{\Omega_{10r}} |Du - V|^2 dx \le c\delta^{\frac{\sigma_2}{2+\sigma_2}}$$

for some universal constant $\sigma_2 > 0$. Now, letting $N_2 = \max\{4n_2^2, 4^n\}$, we estimate as follows:

$$\begin{aligned} &\left|\left\{x \in \Omega : \mathcal{M}(|Du|^2) > N_2^2\right\} \cap Q_{10r}\right| \\ \le\ & \left|\left\{x \in \Omega_{10r} : \mathcal{M}(|Du - V|^2) > n_2^2\right\}\right| + \left|\left\{x \in \Omega_{10r} : \mathcal{M}(|V|^2) > n_2^2\right\}\right| \\ \le\ & c \int_{\Omega_{10r}} |Du - V|^2 dx \le c\delta^{\frac{\sigma_2}{2+\sigma_2}} |\Omega_{10r}|. \end{aligned}$$

Owing to that $Q_r(y) \subset Q_{10r}$, in the original coordinate system, for $N \ge N_2$

$$\begin{aligned} &\left|\left\{x \in \Omega : \mathcal{M}(|Du|^2) > N^2\right\} \cap Q_r(y)\right| \\ \le\ & \left|\left\{x \in \Omega_{10r} : \mathcal{M}(|Du - V|^2) > n_2^2\right\}\right| \\ \le\ & c\delta^{\frac{\sigma_2}{2+\sigma_2}} |\Omega_{10r}| \ \le\ c\delta^{\frac{\sigma_2}{2+\sigma_2}} |Q_r(y)|. \end{aligned}$$

This is a contradiction to (4.1) and completes the proof of the boundary case. □

Now fix $\epsilon > 0$ and take δ and N as given in Lemma 4.1. Based on Lemma 2.2, we have the following power decay estimates.

Lemma 4.2. *Let u be the weak solution to the variational inequality* (1.1). *Suppose that $(\mathbf{a}(\xi, x), \Omega)$ is $(\delta, 25R)$-vanishing of codimension* 1 *and suppose that for every $y \in \Omega$*

$$\left|\left\{x \in \Omega : \mathcal{M}(|Du|^2) > N^2\right\} \cap Q_R(y)\right| < \epsilon |Q_R|. \tag{4.10}$$

Finally, set $\epsilon_* = \left(\frac{20\sqrt{2}}{1-\delta}\right)^n \epsilon$. Then for each positive integer k, we have

$$\begin{aligned} &\left|\left\{x \in \Omega \colon \mathcal{M}(|Du|^2) > N^{2k}\right\}\right| \\ \le\ & \epsilon_*^k \left|\left\{x \in \Omega \colon \mathcal{M}(|Du|^2) > 1\right\}\right| + \sum_{i=1}^k \epsilon_*^i \left|\left\{x \in \Omega \colon \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right| \\ &\quad + \sum_{i=1}^k \epsilon_*^i \left|\left\{x \in \Omega \colon \mathcal{M}(|D\psi|^2) > \delta^2 N^{2(k-i)}\right\}\right|. \end{aligned}$$

Proof. Define

$$C = \left\{x \in \Omega \colon \mathcal{M}(|Du|^2) > N^2\right\}$$

and

$$D = \{x \in \Omega\colon \mathcal{M}(|Du|^2) > 1\} \cup \{x \in \Omega\colon \mathcal{M}(|F|^2) > \delta^2\} \cup \{x \in \Omega\colon \mathcal{M}(|D\psi|^2) > \delta^2\}.$$

Clearly $C \subset D \subset \Omega$ and the first condition of Lemma 2.2 is the assumption (4.10). Moreover, the second one of Lemma 2.2 is a direct consequence of Lemma 4.1. So, applying Lemma 2.2 yields the claim in the case $k = 1$ and the proof is completed by iteration. □

Finally, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. We first take the universal constant $\epsilon \in (0,1)$ so that

$$N^p 80^n \epsilon < \frac{1}{2} \tag{4.11}$$

and then find a corresponding $\delta = \delta(\epsilon, \lambda, \Lambda, n) \in (0, \frac{1}{8})$ from Lemma 4.1.

We consider the renormalized maps:

$$\tilde{u} = \frac{\delta |Q_R|^{1/2}\, u}{\|F\|_{L^2(\Omega)} + \|D\psi\|_{L^2(\Omega)}}, \tag{4.12}$$

and

$$\tilde{F} = \frac{\delta |Q_R|^{1/2}\, F}{\|F\|_{L^2(\Omega)} + \|D\psi\|_{L^2(\Omega)}}, \quad \text{and} \quad \tilde{\psi} = \frac{\delta |Q_R|^{1/2}\, \psi}{\|F\|_{L^2(\Omega)} + \|D\psi\|_{L^2(\Omega)}}. \tag{4.13}$$

Then, thanks to Lemma 2.3 with $M = \frac{\|F\|_{L^2(\Omega)} + \|D\psi\|_{L^2(\Omega)}}{\delta |Q_R|^{1/2}}$ and L^2 estimates, we have that

$$\int_\Omega |D\tilde{u}|^2 dx \le c \int_\Omega \left[|\tilde{F}|^2 + |D\tilde{\psi}|^2\right] dx \le c\delta^2 |Q_R| \tag{4.14}$$

and so

$$\left|\{x \in \Omega : \mathcal{M}(|D\tilde{u}|^2) > N^2\}\right| \le c \int_\Omega |D\tilde{u}|^2 dx \le c\delta^2 |Q_R| < \epsilon |Q_R|, \tag{4.15}$$

where the last inequality will be held by further selecting a smaller δ depending

$$\delta = \delta(\epsilon, \lambda, \Lambda, n, p) \in (0, 1/8).$$

On the other hand, due to Lemma 2.1 with $g = \mathcal{M}(\delta^{-2}|\tilde{F}|^2)$, $\theta = N$ and $\mu = 1$, we have that

$$\sum_{k=i}^{\infty} N^{p(k-i)} |\{x \in \Omega : \mathcal{M}(|\tilde{F}|^2) > \delta^2 N^{2(k-i)}\}| \le c \left\|\mathcal{M}(\delta^{-2}|\tilde{F}|^2)\right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}}$$

and that, by *strong type* (p, p) *estimate* (2.2) and (4.14),

$$\left\|\mathcal{M}(\delta^{-2}|\tilde{F}|^2)\right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \le c \left\|\delta^{-2}|\tilde{F}|^2\right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{p}{2}} \le c \left\|\delta^{-1}\tilde{F}\right\|_{L^p(\Omega)}^{p}.$$

Thus it follows that

$$\sum_{k=i}^{\infty} N^{p(k-i)} \left|\left\{x \in \Omega : \mathcal{M}(|\tilde{F}|^2) > \delta^2 N^{2(k-i)}\right\}\right| \leq c_1 \left\|\delta^{-1} \tilde{F}\right\|^p_{L^p(\Omega)}. \tag{4.16}$$

Similarly, we have

$$\sum_{k=i}^{\infty} N^{p(k-i)} \left|\left\{x \in \Omega : \mathcal{M}(|D\tilde{\psi}|^2) > \delta^2 N^{2(k-i)}\right\}\right| \leq c_2 \left\|\delta^{-1} D\tilde{\psi}\right\|^p_{L^p(\Omega)}. \tag{4.17}$$

Note that the universal constants c_1 and c_2 are independent of δ.

Set $\epsilon_* = \left(\frac{20\sqrt{2}}{1-\delta}\right)^n \epsilon$. It follows from (4.11) that

$$N^p \epsilon_* \leq N^p 80^n \epsilon \leq \frac{1}{2}. \tag{4.18}$$

Combining Lemma 4.2 with (4.15), (4.16)-(4.17), and (4.18) yields that

$$\begin{aligned}
&\sum_{k=1}^{\infty} N^{kp} \left|\left\{x \in \Omega \colon \mathcal{M}(|Du|^2) > N^{2k}\right\}\right| \\
&\leq \sum_{k=1}^{\infty} N^{kp} \epsilon_*^k \left|\left\{x \in \Omega : \mathcal{M}(|Du|^2) > 1\right\}\right| \\
&\qquad + \sum_{k=1}^{\infty} N^{kp} \sum_{i=1}^{k} \epsilon_*^i \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right| \\
&\qquad + \sum_{k=1}^{\infty} N^{kp} \sum_{i=1}^{k} \epsilon_*^i \left|\left\{x \in \Omega : \mathcal{M}(|D\psi|^2) > \delta^2 N^{2(k-i)}\right\}\right| \\
&\leq \sum_{k=1}^{\infty} (N^p \epsilon_*)^k |\Omega| \\
&\qquad + \sum_{i=1}^{\infty} (N^p \epsilon_*)^i \left(\sum_{k=i}^{\infty} N^{(k-i)p} \left|\left\{x \in \Omega : \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right|\right) \\
&\qquad + \sum_{i=1}^{\infty} (N^p \epsilon_*)^i \left(\sum_{k=i}^{\infty} N^{(k-i)p} \left|\left\{x \in \Omega : \mathcal{M}(|D\psi|^2) > \delta^2 N^{2(k-i)}\right\}\right|\right) \\
&\leq c \left(|\Omega| + \left\|\delta^{-1}\tilde{F}\right\|^p_{L^p(\Omega)} + \left\|\delta^{-1} D\tilde{\psi}\right\|^p_{L^p(\Omega)}\right) \sum_{k=1}^{\infty} (N^p \epsilon_*)^k \\
&\leq c \left(|\Omega| + \left\|\delta^{-1}\tilde{F}\right\|^p_{L^p(\Omega)} + \left\|\delta^{-1} D\tilde{\psi}\right\|^p_{L^p(\Omega)}\right).
\end{aligned}$$

Therefore, we have proved that combining *strong type* (p, p) *estimate* (2.2) and Lemma 2.1, $D\tilde{u} \in L^p(\Omega; \mathbb{R}^n)$ with the estimate

$$\|D\tilde{u}\|^p_{L^p(\Omega;\mathbb{R}^n)} \leq c \left(|\Omega| + \left\|\delta^{-1}\tilde{F}\right\|^p_{L^p(\Omega)} + \left\|\delta^{-1} D\tilde{\psi}\right\|^p_{L^p(\Omega)}\right).$$

We recall that $\epsilon = \epsilon(\lambda, \Lambda, n, p) \in (0,1)$ is a constant taken in (4.18), and the corresponding constant $\delta = \delta(\epsilon, \lambda, \Lambda, n, p) \in (0, 1/8)$ is chosen by Lemma 4.1 and (4.15). Therefore, by the definitions of $\tilde{u}$, $\tilde{F}$, and $\tilde{\psi}$ and Hölder's inequality, we have that

$$\|Du\|^p_{L^p(\Omega;\mathbb{R}^n)} \le c\left((|\Omega|/|Q_R|)^{\frac{p}{2}} + 1\right)\left(\|F\|^p_{L^p(\Omega;\mathbb{R}^n)} + \|D\psi\|^p_{L^p(\Omega;\mathbb{R}^n)}\right),$$

where the constant c depends only on λ, Λ, n, and p. This completes the proof. □

5. Weighted Orlicz regularity estimates

In this section, we derive regularity estimates for the weak solution to the variational inequality (1.1) in the weighted Orlicz spaces. We first recall the definition of the *Muckenhoupt classes* A_p of weights. A positive locally integrable function w on $\mathbb{R}^n$ is said to be a weight. For a given $1 < p < \infty$, the weight $w = w(x)$ belongs to the Muckenhoupt class A_p if

$$[w]_p := \sup \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{\frac{-1}{p-1}}\,dx\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Note that A_p class can be defined in another way; that is, $w \in A_p$ if and only if

$$\left(\frac{1}{|Q|}\int_Q f(x)dx\right)^p \le \frac{c}{w(Q)}\int_Q \big(f(x)\big)^p w(x)dx \tag{5.1}$$

holds for all positive f and all cubes Q. The smallest constant c for which (5.1) is valid equals the constant $[w]_p$. As a direct consequence of (5.1), we have

$$\frac{1}{[w]_p}\left(\frac{|E|}{|Q|}\right)^p \le \frac{w(E)}{w(Q)}$$

whenever $w \in A_p$ for some $p \in (1,\infty)$ and E is a measurable subset of Q. Here, we used the notation $w(E) = \int_E w(x)\,dx$. In addition, the reverse Hölder inequality which is an essential property of A_p-weight yields that

$$\frac{w(E)}{w(Q)} \le \mu\left(\frac{|E|}{|Q|}\right)^\tau \tag{5.2}$$

for some constants $\mu > 1$ and $\tau \in (0,1)$. Note that these μ and τ depend only on n, p, and $[w]_p$. We refer to [34, 36] for more details about A_p-weight.

We now introduce Orlicz spaces. The function $\Phi : [0,\infty) \to [0,\infty)$ is said to be a *Young function* if Φ is increasing, convex, and satisfies

$$\Phi(0) = 0,\ \Phi(\infty) = \lim_{\rho\to+\infty}\Phi(\rho) = +\infty,\ \lim_{\rho\to 0+}\frac{\Phi(\rho)}{\rho} = 0,\ \lim_{\rho\to+\infty}\frac{\Phi(\rho)}{\rho} = +\infty.$$

To define reflexive Banach spaces, we further assume that the Young function Φ satisfies the so-called Δ_2 and ∇_2 conditions, denoted by $\Phi \in \Delta_2 \cap \nabla_2$,

$$\Phi(2\rho) \le \nu_1\Phi(\rho) \text{ for some } \nu_1 > 1 \text{ and all } \rho > 0 \qquad (\Phi \in \Delta_2)$$

and

$$2\nu_2\Phi(\rho) \le \Phi(\nu_2\rho) \text{ for some } \nu_2 > 1 \text{ and all } \rho > 0 \qquad (\Phi \in \nabla_2).$$

Note that for $\Phi \in \Delta_2 \cap \nabla_2$ one can find two constants τ_1 and τ_2 with $1 < \tau_1 \le \tau_2 < \infty$ such that

$$\frac{1}{c}\min\{\lambda^{\tau_1}, \lambda^{\tau_2}\}\Phi(\rho) \le \Phi(\lambda\rho) \le c\max\{\lambda^{\tau_1}, \lambda^{\tau_2}\}\Phi(\rho), \qquad \lambda, \rho \ge 0, \tag{5.3}$$

where the constant c is independent of λ and ρ. We next define the lower index of Φ, denoted by $i(\Phi)$, by

$$i(\Phi) = \lim_{\lambda \to 0+} \frac{\log(h_\Phi(\lambda))}{\log \lambda} = \sup_{0<\lambda<1} \frac{\log(h_\Phi(\lambda))}{\log \lambda},$$

where

$$h_\Phi(\lambda) = \sup_{\rho>0} \frac{\Phi(\lambda\rho)}{\Phi(\rho)} \qquad (\lambda > 0).$$

In fact, the lower index number $i(\Phi)$ equals the supremum of τ_1 satisfying (5.3) and $1 < i(\Phi) < \infty$ due to $\Phi \in \Delta_2 \cap \nabla_2$.

We finally ready to define the weighted Orlicz space considered here. For a Young function $\Phi \in \Delta \cap \nabla_2$ and a weight $w \in A_{i(\Phi)}$, the weighted Orlicz space $L^\Phi_w(\Omega)$ is the class of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \Phi(|g(x)|)w(x)dx \ < \ +\infty.$$

This weighted Orlicz space $L^\Phi_w(\Omega)$ can be equipped with the weighted Luxemburg norm;

$$\|g\|_{L^\Phi_w(\Omega)} = \inf\left\{\kappa > 0 : \int_\Omega \Phi\left(\frac{|g(x)|}{\kappa}\right) w(x)\,dx \le 1\right\}.$$

Owing to (5.3) and the convexity of Φ, we have that

$$\begin{aligned}\frac{1}{c}\min\left\{\|g\|^{\tau_1}_{L^\Phi_w(\Omega)}, \|g\|^{\tau_2}_{L^\Phi_w(\Omega)}\right\} &\le \int_\Omega \Phi(|g(x)|)w(x)dx \\ &\le c\max\left\{\|g\|^{\tau_1}_{L^\Phi_w(\Omega)}, \|g\|^{\tau_2}_{L^\Phi_w(\Omega)}\right\}.\end{aligned} \tag{5.4}$$

In addition, it is well-known that the Hardy-Littlewood maximal operator is bounded from weighted Lebesgue space $L^p_w(\mathbb{R}^n)$ to itself. Similarly, for a given Young function $\Phi \in \Delta_2 \cap \nabla_2$ and a weight $w \in A_{i(\Phi)}$, there exists $c = c(n, \Phi, w)$ such that

$$\int_{\mathbb{R}^n} \Phi\Big(\mathcal{M}g(x)\Big)w(x)dx \le c\int_{\mathbb{R}^n} \Phi\Big(|g(x)|\Big)w(x)dx \tag{5.5}$$

for all $g \in L^\Phi_w(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$. We finally refer to [9,17,21,29] for a more discussion on $A_{i(\Phi)}$-weight and weighted Orlicz spaces.

We are now ready to state the main result.

Theorem 5.1. *Let $u \in \mathcal{A}$ be the weak solution to the variational inequality (1.1). Suppose that $w \in A_{i(\Phi)}$ with $\Phi \in \Delta_2 \cap \nabla_2$ and further that $|F|^2$, $|D\psi|^2 \in L^{\Phi}_w(\Omega)$. Then there exists a constant $\delta = \delta(\lambda, \Lambda, n, \Phi, w) \in (0, 1/8)$ such that if $(\mathbf{a}(\xi, x), \Omega)$ is $(\delta, 25R)$-vanishing of codimension 1, then $|Du|^2 \in L^{\Phi}_w(\Omega)$ with the estimate*

$$\||Du|^2\|_{L^{\Phi}_w(\Omega)} \leq c\left(\||F|^2\|_{L^{\Phi}_w(\Omega)} + \||D\psi|^2\|_{L^{\Phi}_w(\Omega)}\right), \tag{5.6}$$

where c is a positive constant depending only on $\lambda, \Lambda, n, R, \Phi, w,$ and Ω.

We remark that the variational inequality (1.1) has a unique weak solution $u \in \mathcal{A}$, under the assumptions $|F|^2$, $|D\psi|^2 \in L^{\Phi}_w(\Omega)$ with $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_{i(\Phi)}$. In fact, it follows $L^{\Phi}_w(\Omega) \subset L^1(\Omega)$ and precisely for $|F|^2 \in L^{\Phi}_w(\Omega)$ the following estimate is obtained:

$$\int_\Omega |F(x)|^2\, dx \leq c\left[\left(⨍_\Omega \Phi(|F|^2) w(x) dx\right)^{\frac{1}{\tau_1}} + \left(⨍_\Omega \Phi(|F|^2) w(x) dx\right)^{\frac{1}{\tau_2}}\right]. \tag{5.7}$$

(Note that $⨍_\Omega \cdots$ means $\frac{1}{w(\Omega)}\int_\Omega \cdots$ on the right hand side of (5.7) and the constant c depends only on n, $[w]_{i(\Phi)}$, and $diam(\Omega)$.) Thus, the existence and uniqueness of weak solution in $\mathcal{A}$ to (1.1) is obtained by the classical theory. The inequality (5.7) is followed by the reverse Hölder property of $A_{i(\Phi)}$-weight; that is, $w \in A_{i(\Phi)-\epsilon_0}$ with $[w]_{i(\Phi)-\epsilon_0} \leq c_{n,i(\Phi)}[w]_{i(\Phi)}$. By (5.3),

$$\lambda^{i(\Phi)-\epsilon_0}\Phi(t) \leq c\,\Phi(\lambda t), \qquad \lambda \geq 1,\ t \geq 0,$$

and specially

$$|g(x)|^{i(\Phi)-\epsilon_0} \leq \frac{c}{\Phi(1)}\Phi(|g(x)|) \qquad \text{if} \quad |g(x)| \geq 1.$$

Consequently we have that

$$\int_{\{\Omega:|g(x)|\geq 1\}} |g(x)|\, dx \leq c\left(\int_\Omega \Phi(|g(x)|) w(x) dx\right)^{\frac{1}{i(\Phi)-\epsilon_0}}$$

for some $c = c(w, \Phi, diam(\Omega)) > 0$. For more details, we refer to [9, 17, 29].

Lemma 5.2. *Given a Young function $\Phi \in \Delta_2 \cap \nabla_2$, let $w \in A_{i(\Phi)}$. Assume that g is a nonnegative and measurable function defined on a bounded domain Ω in $\mathbb{R}^n$. Let $\theta > 0$ and $\lambda > 1$ be constants. Then*

$$g \in L^{\Phi}_w(\Omega) \quad \Longleftrightarrow \quad S = \sum_{k\geq 1} \Phi(\lambda^k)\, w\Big(\{x \in \Omega : g(x) > \theta\lambda^k\}\Big) < \infty$$

and

$$\frac{1}{c}S \leq \int_\Omega \Phi\Big(g(x)\Big) w(x)\, dx \leq c\big(w(\Omega) + S\big),$$

the positive constant c depending only on θ, λ, Φ, and w.

The weighted measure version of the Calderon-Zygmund-Krylov-Safonov-type covering lemma is used to prove the main theorem. The following lemma can be found in [27, 29] with slight modifications.

Lemma 5.3. *Given a Young function $\Phi \in \Delta_2 \cap \nabla_2$, let $w \in A_{i(\Phi)}$. Let Ω be a bounded (δ, R)-Reifenberg flat domain for some small $\delta > 0$ and let C and D be measurable sets with $C \subset D \subset \Omega$. Suppose that there exists small $\epsilon > 0$ such that*

(1) *for any $y \in \Omega$, $w(C \cap Q_R(y)) < \epsilon w(Q_R(y))$,*
(2) *for each $y \in \Omega$ and $r \in (0, R)$,*

$$\text{if } \ w(C \cap Q_r(y)) \geq \epsilon w(Q_r(y)), \quad \text{then } \ Q_r(y) \cap \Omega \subset D.$$

Then

$$w(C) \leq c_* \, \epsilon \, w(D),$$

where the constant c_* is depending only on n, Φ, w, and $\frac{1}{1-\delta}$.

Now, we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. In view of (5.4), we have that

$$\| |Du|^2 \|^{\alpha}_{L^{\Phi}_w(\Omega)} \leq c \int_{\Omega} \Phi\Big(|Du|^2\Big) w(x)dx \leq c \int_{\Omega} \Phi\Big(\mathcal{M}(|Du|^2)\Big) w(x)dx$$

for some $\alpha > 0$. Thanks to Lemma 5.2, it suffices to show that

$$S = \sum_{k \geq 1} \Phi\big(N^{2k}\big) \, w\Big(\{x \in \Omega : \mathcal{M}(|Du|^2) > N^{2k}\}\Big) < \infty,$$

under the assumption $\| |F|^2 \|_{L^{\Phi}_w(\Omega)} + \| |D\psi|^2 \|_{L^{\Phi}_w(\Omega)} \leq \delta^2$. Then the desired estimate (5.6) is follows from Lemma 2.3. Note that, by (5.4) and (5.7), we have that

$$\int_{\Omega} \big(|F|^2 + |D\psi|^2\big) \, dx \ \leq \ c \left(\| |F|^2 \|_{L^{\Phi}_w(\Omega)} + \| |D\psi|^2 \|_{L^{\Phi}_w(\Omega)} \right)^{\tau_3} \leq c\delta^{2\tau_3} \tag{5.8}$$

for some $\tau_3 = \tau_3(\tau_1, \tau_2) > 0$.

Fix ϵ and then take δ and N as given in Lemma 4.1 with $\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tau}}$, in place of ϵ, where μ and τ are given by (5.2). Define the sets

$$C = \left\{ x \in \Omega \colon \mathcal{M}(|Du|^2) > N^2 \right\}$$

and

$$D = \left\{ x \in \Omega \colon \mathcal{M}(|Du|^2) > 1 \right\} \cup \left\{ x \in \Omega \colon \mathcal{M}(|F|^2) > \delta^2 \right\} \cup \left\{ x \in \Omega \colon \mathcal{M}(|D\psi|^2) > \delta^2 \right\}.$$

To apply Lemma 5.3, check its hypotheses. Clearly $C \subset D \subset \Omega$ and for each $y \in \Omega$ it follows from (5.2), weak type (1,1) estimate (2.1), the standard

L^2-estimate, (5.8) that

$$\frac{w(C\cap Q_R(y))}{w(Q_R(y))} \le \mu\left(\frac{|C\cap Q_R(y)|}{|Q_R(y)|}\right)^{\tau_1} \le c\,|C|^{\tau} \le c\left(\int_\Omega |Du|^2\,dx\right)^{\tau}$$
$$\le c\left(\int_\Omega \left(|F|^2+|D\psi|^2\right)\,dx\right)^{\tau} \le c\delta^{2\tau\tau_3} < \epsilon,$$

where the last inequality will be held by further selecting a small δ depending

$$\delta = \delta(\epsilon, n, \lambda, \Lambda, \Phi, w, R, diam(\Omega)) \in (0, 1/8).$$

To check the second condition of Lemma 5.3, we suppose that $w(C\cap Q_r(y)) \ge \epsilon w(Q_r(y))$. It follows from (5.2) that

$$\epsilon \le \frac{w(C\cap Q_r(y))}{w(Q_r(y))} \le \mu\left(\frac{|C\cap Q_r(y)|}{|Q_r(y)|}\right)^{\tau}$$

and so

$$\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tau}}|Q_r(y)| \le |C\cap Q_R(y)| = \left|\left\{x\in\Omega\colon \mathcal{M}(|Du|^2) > N^2\right\}\cap Q_R(y)\right|.$$

By the choice of δ, one may apply Lemma 4.1 for the constant $\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tau}}$ instead of ϵ to find that δ, $Q_r(y)\cap\Omega\subset D$. Thus we have Lemma 5.3 and that

$$\begin{aligned} w\left(\left\{x\in\Omega\colon \mathcal{M}(|Du|^2) > N^2\right\}\right) \le{}& \epsilon_* w\left(\left\{x\in\Omega\colon \mathcal{M}(|Du|^2) > 1\right\}\right)\\ &+ \epsilon_* w\left(\left\{x\in\Omega\colon \mathcal{M}(|F|^2) > \delta^2\right\}\right)\\ &+ \epsilon_* w\left(\left\{x\in\Omega\colon \mathcal{M}(|D\psi|^2) > \delta^2\right\}\right), \end{aligned}$$

where $\epsilon_* = c_*\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tau}}$. Moreover, we have the following power decay estimates due to its iteration argument: for $k = 1, 2, 3, \ldots$,

$$\begin{aligned} &w\left(\left\{x\in\Omega\colon \mathcal{M}(|Du|^2) > N^{2k}\right\}\right)\\ &\le \epsilon_*^k w\left(\left\{x\in\Omega\colon \mathcal{M}(|Du|^2) > 1\right\}\right)\\ &\quad+\sum_{i=1}^{k}\epsilon_*^i w\left(\left\{x\in\Omega\colon \mathcal{M}(|F|^2) > \delta^2 N^{2(k-i)}\right\}\right)\\ &\quad+\sum_{i=1}^{k}\epsilon_*^i w\left(\left\{x\in\Omega\colon \mathcal{M}(|D\psi|^2) > \delta^2 N^{2(k-i)}\right\}\right). \end{aligned}$$

The rest of the proof is the same as the proof of Theorem 1.3 in Section 4, but we have to use (5.5) and Lemma 5.2 instead of using (2.2) and Lemma 2.1, respectively. For more details, we refer to [29]. This completes the proof. □

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Youchan Kim
Department of Mathematics
University of Seoul
Seoul 02504, Korea
Email address: youchankim@uos.ac.kr

Seungjin Ryu
Department of Mathematics
University of Seoul
Seoul 02504, Korea
Email address: seungjinryu@uos.ac.kr