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STABILITY PROPERTIES IN IMPULSIVE DIFFERENTIAL SYSTEMS OF NON-INTEGER ORDER

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Abstract. In this paper we establish some new explicit solutions for impulsive linear fractional differential equations with impulses at fixed times, which provides a handy tool in deriving singular integral-sum inequalities and an impulsive fractional comparison principle. Thus we study the Mittag-Leffler stability of impulsive differential equations with the Caputo fractional derivative by using the impulsive fractional comparison principle and piecewise continuous functions of Lyapunov's method. Also, we give some examples to illustrate our results.

1. Introduction

The impulsive differential equations are suitable mathematical models for the description of evolution processes characterized by the combination of a continuous and jump change of their states. It is now being recognized that the theory of impulsive differential equations is not only richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modelling of many real world phenomena. The qualitative theory of differential equations with impulse effect has been developed by a large number of mathematicians due to the wide applications of these systems to the control theory, biology, electronics, etc. For a detailed theory about impulsive inequalities and some basic concepts of impulsive differential equations, we refer the reader to [1, 2, 18].

Simeonov and Bainov [27] investigated the exponential stability of the solutions for impulsive differential equations by using the comparison method and piecewise continuous auxiliary functions which are analogues to Lyapunov's functions. Also, Kulev and Bainvo [16] introduced the notions of various types of uniform Lipschitz stability for impulsive differential systems and obtained

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sufficient conditions for these notions and their relations. Moreover, Choi et al. [11,12] studied h-stability for the linear impulsive differential equations using the notions of similarity, t_∞-similarity, and impulsive integral inequalities. Many authors [1,2,11,16,18,27,28] have studied the various types of stability of solutions for impulsive differential equations. Choi and Koo [8] showed that the associated variational impulsive system inherits the property of h-stability from the original nonlinear impulsive differential systems. Also, they [10] obtained a converse h-stability theorem for the nonlinear impulsive systems by employing the notion of t_∞-similarity of the associated impulsive variational systems and relations.

Lakshmikantham et al. [17,20] have investigated the basic theory of initial value problems for fractional differential equations involving Riemann-Liouville differential operators of order $0 < q < 1$. They followed the classical approach of the theory of differential equations of integer order, in order to compare and contrast the differences as well as the intricacies that might result in development [19, Vol. I]. Li et al. [23] obtained some results about stability of solutions for fractional-order dynamic systems using fractional Lyapunov direct method and fractional comparison principle. Choi and Koo [7] improved on the monotone property of [20, Lemma 1.7.3] for the case $g(t,u) = \lambda u$ with a nonnegative real number λ. They also investigated Mittag-Leffler stability of solutions of fractional differential equations by using the fractional comparison principle.

Stamova and Stamov [31] investigated the stability for impulsive fractional differential equations by using the comparison principle and the Lyapunov function method. Stamova [28–30] studied the various types of global stability and Mittag-Leffler stability of impulsive fractional differential equations with impulse effect at fixed moments of time by using piecewise continuous functions of the type of Lyapunov's functions and a new fractional comparison principle.

In this paper we present the exact solution of homogeneous linear impulsive fractional differential equations by the help of the Mittag-Leffler functions. Then we develop an impulsive fractional differential inequality and the impulsive fractional comparison principle. Thus we study Mittag-Leffler stability of solutions of impulsive Caputo fractional differential equations via an impulsive fractional integral-sum inequality and piecewise continuous auxiliary functions. Also, we apply the impulsive fractional inequality to study the data dependence of the solution on the initial condition to a certain impulsive Caputo fractional differential equation. Furthermore, we give some examples to illustrate our results.

2. Preliminary notes and definitions

In this section we introduce definitions and preliminary facts which are used throughout this paper. For the basic notions and results concerning fractional calculus, we mainly refer to some books [15,20,25,26].

We recall the notions of Mittag-Leffler functions which was originally introduced by G. M. Mittag-Leffler in 1902 [24] and is a generalization of the exponential function. A function frequently used in the solutions of fractional differential systems is the Mittag-Leffler function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \ \alpha > 0, z \in \mathbb{C}, \tag{2.1}$$

where Γ is the Gamma function. The Mittag-Leffler function with two parameters appears most frequently and has the following form

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{2.2}$$

where $\alpha > 0$ and $\beta > 0$. For $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$ and $E_{1,1}(z) = e^z$ [22, 25].

Let $t_0 \in \mathbb{R}_+ = [0, \infty)$ and $J(t_0) = [t_0, \infty)$.

We recall some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1 ([15, 25])**.** The *Riemann-Liouville fractional integral of order* $\alpha > 0$ with the lower limit t_0 for a function $g : J(t_0) \to \mathbb{R}$ is defined as

$$I^\alpha_{t_0} g(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s)ds, \ t > t_0$$

provided that the right-hand side is pointwisely defined on $J(t_0)$.

Definition 2.2 ([15,25])**.** The *Riemann-Liouville fractional derivative of order* $\alpha > 0$ with the lower limit t_0 for a function $g : J(t_0) \to \mathbb{R}$ is defined by

$$D^\alpha_{t_0} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-\alpha-1} g(s)ds \right), \ t > t_0, n-1 < \alpha < n, n \in \mathbb{N}.$$

If $0 < \alpha < 1$, then the Riemann-Liouville fractional derivative of order α of g reduces to

$$D^\alpha_{t_0} g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-\alpha} g(s)ds.$$

Remark 2.3. The Riemann-Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann-Liouville sense require initial conditions at some point different from $x_0 = t_0$. To overcome this issue, Caputo [4, 1967] defined the fractional derivative in the following way.

Definition 2.4 ([15,25])**.** The *Caputo fractional derivative of order* $\alpha > 0$ with the lower limit t_0 for a function $g : J(t_0) \to \mathbb{R}$ is defined by

$${}^C D^\alpha_{t_0} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} g(s)ds, \ n \in \mathbb{N}.$$

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of g reduces to

$$^C D_{t_0}^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{g'(s)}{(t-s)^{\alpha}} ds.$$

Remark 2.5 ([20]). The main advantage of the Caputo derivative is that the initial conditions for fractional differential equations are the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant c is zero, while the Riemann-Liouville fractional derivative for a constant c is not zero but equals to $D_{t_0}^{\alpha} c = \frac{c(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}$.

We need to mention that there exists a link between Riemnann-Liouville and Caputo fractional derivative of order α. When $0 < \alpha < 1$, we have

$$\begin{aligned} ^C D_{t_0}^{\alpha} g(t) &= D_{t_0}^{\alpha}[g(t) - g(t_0)] \\ &= D_{t_0}^{\alpha} g(t) - \frac{g(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}. \end{aligned}$$

In particular, if $g(t_0) = 0$, then we have

$$^C D_{t_0}^{\alpha} g(t) = D_{t_0}^{\alpha} g(t).$$

Hence, we can see that the Caputo derivative is defined for functions for which the Riemann-Liouville derivative exists. Also, we note that the Mittag-Leffler functions $E_{\alpha}(z)$ and $E_{\alpha,\alpha}(z)$ satisfy the more general differential relation

$$^C D_{t_0}^{\alpha} E_{\alpha}(\lambda(t-t_0)^{\alpha}) = \lambda E_{\alpha}(\lambda(t-t_0)^{\alpha}),$$

$$D_{t_0}^{\alpha}\left((t-t_0)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-t_0)^{\alpha})\right) = \lambda(t-t_0)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-t_0)^{\alpha}),\ \lambda \in \mathbb{R},$$

respectively.

Remark 2.6 ([15, 25]). For $\alpha, \beta > 0$ and suitable functions φ, ψ, we have the following properties:

(i) $I_{t_0}^{\alpha} I_{t_0}^{\beta} \varphi(t) = I_{t_0}^{\alpha+\beta} \varphi(t) = I_{t_0}^{\beta} I_{t_0}^{\alpha} \varphi(t)$;
(ii) $I_{t_0}^{\alpha}(\varphi(t) + \psi(t)) = I_{t_0}^{\alpha} \varphi(t) + I_{t_0}^{\alpha} \psi(t)$;
(iii) $I_{t_0}^{\alpha}\, {}^C D_{t_0}^{\alpha} \varphi(t) = \varphi(t) - \varphi(t_0)$ and ${}^C D_{t_0}^{\alpha} I_{t_0}^{\alpha} \varphi(t) = \varphi(t),\ 0 < \alpha < 1$;
(iv) ${}^C D_{t_0}^{\alpha} \varphi(t) = I_{t_0}^{1-\alpha} D\varphi(t) = I_{t_0}^{1-\alpha} \varphi'(t),\ 0 < \alpha < 1$.

3. Impulsive fractional comparison principle

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space with a convenient vector norm $|\cdot|$, and Ω be an open subset of $\mathbb{R}^n$ containing the origin, and $0 < q < 1$. We consider the following impulsive Caputo fractional differential system with impulses at fixed times

$$\begin{cases} {}^C D_{t_0}^{q} x(t) = f(t, x(t)),\ t \in J(t_0),\ t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)),\ k \in \mathbb{N}, \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where ${}^{C}D^{q}_{t_0}$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero and $x_0 \in \Omega$. Assume that the following basic conditions hold:

(A1) A sequence $\{t_k\}$ is unbounded increasing satisfying $0 \le t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

(A2) The function $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is continuous in $(t_{k-1}, t_k] \times \Omega$, $k = 1, 2, \dots$, and $f(t, 0) = 0$ for each $t \in \mathbb{R}_+$.

(A3) For any $x \in \Omega$ and any $k = 1, 2, \dots$, the function f has finite limits as $(t, y) \to (t_k, x)$, $t > t_k$.

(A4) Each function $I_k : \Omega \to \mathbb{R}^n$ is continuous in Ω and there exist nonnegative constants l_k such that

$$|I_k(x) - I_k(y)| \le l_k|x - y|,\ k \in \mathbb{N}, x, y \in \Omega,$$

and $I_k(0) = 0$, $k = 1, 2, \dots$.

(A5) The solution $x(t, t_0, x_0)$ of Eq. (3.1) which satisfies the initial condition $x(t_0^+, t_0, x_0) = x_0$ is defined in the interval (t_0, ∞), and is left continuous.

(A6) At the moments t_k the following relations hold

$$x(t_k^-) = x(t_k),\ \Delta x(t_k) = x(t_k^+) - x(t_k^-),\ k \in \mathbb{N},$$

where $x(t_k^+) = \lim_{\varepsilon\to 0^+} x(t_k + \varepsilon)$ and $x(t_k^-) = \lim_{\varepsilon\to 0^-} x(t_k + \varepsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.

Then it follows from the condition (A5) that the solution $x(t, t_0, x_0)$ of Eq. (3.1) with the initial value (t_0, x_0) is described as the following result.

Lemma 3.1 ([32]). *A function $x \in PC(J(t_0), \mathbb{R}^n)$ is a solution of the fractional integral equation*

$$x(t) = \begin{cases} x(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))ds, & t \in [t_0, t_1], \\ x(t_0) + \sum\limits_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s))ds, & t \in (t_k, t_{k+1}], k \in \mathbb{N}, \\ \vdots \end{cases}$$

if and only if x is a solution of Eq. (3.1).

Let $G_k = \{(t, x) \in \mathbb{R}_+ \times \Omega : t_{k-1} < t < t_k\}, k = 1, 2, \dots$ and $G = \cup_{k=1}^{\infty} G_k$. In the further considerations, we shall use piecewise continuous auxiliary functions.

Definition 3.2 ([18, 27]). We say that a function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ belongs to the *class* υ_0 if

(i) V is continuous in G_k for each $k \in \mathbb{N}$ and $V(t, 0) = 0$ for each $t \in \mathbb{R}_+$;

(ii) V is locally Lipschitz continuous with respect to its second variable on each of the sets G_k and for any $k \in \mathbb{N}$ and $x \in \Omega$ there exist the finite limits

$$V(t_k^-, x) = \lim_{\substack{(t,y)\to(t_k,x) \\ t<t_k}} V(t, y),\ V(t_k^+, x) = \lim_{\substack{(t,y)\to(t_k,x) \\ t>t_k}} V(t, y)$$

and the equality $V(t_k^-, x) = V(t_k, x)$ holds.

We note that if $t \neq t_k$, then $V(t^+, x)$ equals $V(t, x)$.

For the extension of the fractional comparison principle, we need the following result which improves Lemma 1.7.5 in [20].

Lemma 3.3 ([7, Lemma 2.4]). *Let $0 < \alpha < 1$. Consider the Caputo fractional scalar differential equation*

$$^{C}D_{t_0}^{\alpha} u(t) = g(t, u(t)), \ t \geq t_0,$$

where $g(t, u) \geq 0$ and $t_0 \in \mathbb{R}_+$. If the solutions exist and $u(t_0) \geq 0$, then they are nonnegative. Furthermore, if $g(t, u) = \lambda u$ for $\lambda \geq 0$, then the solutions are nondecreasing in t.

Lemma 3.4 ([7, Lemma 2.11]). *Let $0 < \alpha < 1$. Suppose that $w, v \in C(J(t_0), \mathbb{R}_+)$ satisfy*

$$v(t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, v(s))ds < w(t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, w(s))ds,$$

where $g \in C(J(t_0) \times \mathbb{R}, \mathbb{R}_+)$ and $g(t, u)$ is monotone nondecreasing in u for each $t \in J(t_0)$.

If $v(t_0) < w(t_0)$, then we have $v(t) < w(t)$ on $J(t_0)$.

The following result is an impulsive extension of the fractional comparison principle in [7, Lemma 2.11].

Lemma 3.5. *Let $0 < \alpha < 1$. Suppose that $g \in PC(J(t_0) \times \mathbb{R}, \mathbb{R}_+)$ and $g(t, u)$ is monotone nondecreasing in u for each $t \in \mathbb{R}$. Suppose that $w, v \in PC(J(t_0), \mathbb{R}_+)$ satisfy the following impulsive fractional inequality*

$$v(t) - I_{t_0}^{\alpha} g(t, v) < w(t) - I_{t_0}^{\alpha} g(t, w), \ t \in [t_0, t_1] \tag{3.2}$$

and

$$\begin{aligned} v(t) &- \sum_{i=1}^{k} I_i(v(t_i^-)) - I_{t_0}^{\alpha} g(t, v) \\ &< w(t) - \sum_{i=1}^{k} I_i(w(t_i^-)) - I_{t_0}^{\alpha} g(t, w), \ t \in (t_k, t_{k+1}], \ k \in \mathbb{N}, \end{aligned} \tag{3.3}$$

where $I_{t_0}^{\alpha} g(t, v) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, v(s))ds$, and each function $I_k \in C(\mathbb{R}, \mathbb{R}_+)$ ($k \in \mathbb{N}$) is nondecreasing in $x \in \mathbb{R}$.

Then $v(t_0) < w(t_0)$ implies

$$v(t) < w(t), \ t \in J(t_0). \tag{3.4}$$

Proof. Let $t \in (t_0, t_1]$. Then it follows from [7, Lemma 2.11] that $v(t) < w(t)$, $t \in (t_0, t_1]$.

Suppose that the inequality (3.4) is satisfied for $t \in (t_{k-1}, t_k]$, $k > 1$. We claim that the inequality (3.4) is satisfied for $t \in (t_k, t_{k+1}]$, $k > 1$. Assume that the conclusion is not true. Then there exists a $\tau_k \in (t_k, t_{k+1}]$ such that

$$v(\tau_k) = w(\tau_k),\ v(t) < w(t),\ t_k < t < \tau_k.$$

It follows from (3.3) that

$$\begin{aligned} v(\tau_k) &< w(\tau_k) - \sum_{i=1}^{k} I_i(w(t_i^-)) - I_{t_0}^{\alpha} g(\tau_k, w) + \sum_{i=1}^{k} I_i(v(t_i^-)) + I_{t_0}^{\alpha} g(\tau_k, v) \\ &= w(\tau_k) - [\sum_{i=1}^{k} (I_i(w(t_i^-)) - I_i(v(t_i^-))) + I_{t_0}^{\alpha}(g(\tau_k, w) - g(\tau_k, v))] \\ &\le w(\tau_k), \end{aligned}$$

since each function I_k is nondecreasing and $I_{t_0}^{\alpha} g(\tau_k, w) \ge I_{t_0}^{\alpha} g(\tau_k, v)$. This contradicts the fact that $w(\tau_k) = v(\tau_k)$ at $t = \tau_k$, and hence the inequality (3.4) is valid for $t \in (t_k, t_{k+1}]$. The proof is completed by induction. □

Remark 3.6. If we set $I_k(x) = 0$ for each $k \in \mathbb{N}$ in assumptions of Lemma 3.3, then Lemma 3.3 reduces to lemma 2.11 in [7].

We recall the notions of the Mittag-Leffler stability for Eq. (3.1) which are analogous to the definitions given in [22, 30].

Definition 3.7 ([7, 22])**.** The zero solution $x = 0$ of Eq. (3.1) is said to be

(a) a *Mittag-Leffler system* if

$$|x(t)| \le \{m(x(t_0))E_q(\lambda(t - t_0)^q)\}^b,\ t \ge t_0,$$

where $\lambda \in \mathbb{R}, b > 0, m(0) = 0, m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $B \subseteq \mathbb{R}^n$ with a Lipschitz constant m_0;

(b) *Mittag-Leffler stable* if it is a Mittag-Leffler system with $\lambda \le 0$;

(c) *globally Mittag-Leffler stable* if (b) holds for $\Omega = \mathbb{R}^n$.

The following result is adapted from Theorem 3.4 in [7] and Theorem 2.1 in [21].

Theorem 3.8. *Suppose that the function f in Eq.* (3.1) *satisfies*

$$|f(t, x)| \le g(t, |x|),\ t \ne t_k,$$
$$|I_k(x(t_k))| \le J_k(|x(t_k)|),\ k \in \mathbb{N},$$

where $g \in C(J(t_0) \times \mathbb{R}, \mathbb{R}_+)$ is monotone increasing in u for each $t \in J(t_0)$ and $g(t, 0) = 0$ for each $t \in J(t_0)$. We consider the following impulsive Caputo fractional differential equation

$$\begin{cases} {}^C D_{t_0}^q u(t) = g(t, u(t)),\ t \in J(t_0),\ t \ne t_k, \\ \Delta u(t_k) = J_k(u(t_k)),\ k \in \mathbb{N}, \\ u(t_0) = u_{t_0}, \end{cases} \tag{3.5}$$

where each function $J_k : \mathbb{R}_+ \to \mathbb{R}$, $k \in \mathbb{N}$, is continuous. If the zero solution $u = 0$ of Eq. (3.5) *is a Mittag-Leffler system, then the zero solution $x = 0$ of Eq.* (3.1) *is also a Mittag-Leffler system whenever $u(t_0) > |x(t_0)|$.*

Proof. Note that Eq. (3.1) is equivalent to the following fractional integral equation:

(3.6)
$$x(t)=\begin{cases} x(t_0) + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,x(s))ds, \ t \in [t_0,t_1], \\ x(t_0)+\sum\limits_{i=1}^{k} I_i(x(t_i^-))+\frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,x(s))ds, \ t\in(t_k,t_{k+1}], \ k\in\mathbb{N}, \\ \vdots \end{cases}$$

Then we obtain

$$|x(t)| \le |x(t_0)| + I_{t_0}^q |f(t,x(t))|, \ t \in [t_0,t_1]$$

and

$$\begin{aligned} |x(t)| &\le |x(t_0)| + \sum_{i=1}^{k} |I_i(x(t_i^-))| + I_{t_0}^q |f(t,x(t))| \\ &\le |x(t_0)| + \sum_{i=1}^{k} J_i(|x(t_i^-)|) + I_{t_0}^q g(t,|x(t)|), \ k \in \mathbb{N}, \end{aligned}$$

where $I_{t_0}^\alpha g(t,u) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1} g(s,u(s))ds$. Thus we have

$$\begin{aligned} &|x(t)| - \sum_{i=0}^{k} J_i(|x(t_i^-)|) - I_{t_0}^q g(t,|x(t)|) \\ &\le |x(t_0)| \\ &< u(t_0) \\ &= u(t) - \sum_{i=0}^{k} J_i(u(t_i^-)) - I_{t_0}^q g(t,u(t)), \ t \ge t_0, \ k \in \mathbb{N}, \end{aligned}$$

where $u(t_0) \in \mathbb{R}_+$ and $J_0(u) = 0$ for each $u \in \mathbb{R}$. By Lemma 3.5, we have $|x(t)| < u(t)$ for all $t \ge t_0$. Also we see that

$$\begin{aligned} |x(t)| < u(t) &\le \{m(u(t_0))E_q(\lambda(t-t_0)^q)\}^b \\ &\le \{m_0 u(t_0) E_q(\lambda(t-t_0)^q)\}^b \\ &= \{m_0 d|x(t_0)| E_q(\lambda(t-t_0)^q)\}^b \\ &= \{\hat{m}(|x(t_0)|)E_q(\lambda(t-t_0)^q)\}^b, \ t \ge t_0, \end{aligned}$$

where $\lambda \in \mathbb{R}$, $u(t_0) = |x(t_0)|d$, $d > 1$, and $\hat{m}(x) = m_0 dx$ is locally Lipschtiz with Lipschitz constant $l = m_0 d$. This completes the proof. □

Corollary 3.9. *Suppose that all conditions of Theorem 3.8 hold. If the zero solution $u = 0$ of Eq.* (3.5) *is Mittag-Leffler stable, then the zero solution $x = 0$ of Eq.* (3.1) *is also Mittag-Leffler stable whenever $u_0 > |x_0|$.*

Remark 3.10. Suppose that all conditions of Theorem 3.8 hold. The asymptotic stability of Eq. (3.5) implies the corresponding asymptotic stability of Eq. (3.1).

Lemma 3.11. *If a function $x \in C(J(t_0), \mathbb{R})$ satisfies the linear Caputo fractional scalar differential equation*

$$\begin{cases} {}^C D^q_{t_0} x = \lambda x + h(t), \ t \in J(t_0), \\ x(t_0) = x_{t_0}, \end{cases} \tag{3.7}$$

where $\lambda \in \mathbb{R}$ and $h \in C(J(t_0), \mathbb{R})$. Then a function x also satisfies the fractional integral equation

$$x(t) = x(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} (\lambda x(s) + h(s)) ds, \ t \geq t_0 \tag{3.8}$$

and vice versa. Then we get the unique solution of (3.8) *as*

$$\begin{aligned} x(t) &= x(t_0) E_q(\lambda(t-t_0)) + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) h(s) ds \\ &= x(t_0) E_q(\lambda(t-t_0)) + h(t) * t^{q-1} E_{q,q}(\lambda t^q), \ t \geq t_0, \end{aligned}$$

where $h(t) * t^{q-1} E_{q,q}(\lambda t^q) = \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(\lambda(t-s)^q) h(s) ds$.

Consider the following impulsive Caputo fractional differential equation

$$\begin{cases} {}^C D^q_{t_0} u = \lambda u + d, \ t \in J(t_0), \ t \neq t_k, \\ \Delta u(t_k) = \beta_k u(t_k), \ k \in \mathbb{N}, \\ u(t_0) = u_{t_0}, \end{cases} \tag{3.9}$$

where λ, d and β_k are constants.

Lemma 3.12 ([20])**.** *A function $u \in C(J(t_0), \mathbb{R})$ is a solution of the following linear Caputo fractional differential equation with initial condition*

$$\begin{cases} {}^C D^q_{t_0} u(t) = \lambda u(t), \ t \in J(t_0), \ \lambda \in \mathbb{R}, \\ u(t_0) = u_{t_0}, \end{cases} \tag{3.10}$$

if and only if the solution u of Eq. (3.10) *is given by*

$$u(t) = u(t_0) E_q(\lambda(t-t_0)^q), \ t \geq t_0.$$

Lemma 3.13 ([32])**.** *A function $u \in C(J(t_0), \mathbb{R})$ is a solution of the fractional integral equation*

$$u(t) = u_a - \frac{1}{\Gamma(q)} \int_{t_0}^{a} (a-s)^{q-1} g(s, u(s)) ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, u(s)) ds,$$

if and only if a function u is a solution of the following Caputo fractional differential equation with initial condition

$$\begin{cases} {}^C D_{t_0}^q u = g(t, u(t)), \ t \in J(t_0), \\ u(a) = u_a, \ a > t_0. \end{cases} \tag{3.11}$$

We can obtain the following result on an exact solution of homogeneous linear impulsive fractional differential equations by the help of the Mittag-Leffler functions.

Theorem 3.14 ([9, Theorem 2.4])**.** *If we set $d = 0$ in Eq.* (3.9)*, then the solution $u(t)$ of Eq.* (3.9) *is given by*

(3.12)

$$u(t) = \begin{cases} u(t_0)E_q(\lambda(t-t_0)^q), \ t \in [t_0, t_1], \\ u(t_0)\prod_{i=1}^{k}(1+\beta_i E_q(\lambda(t_i - t_0)^q))E_q(\lambda(t-t_0)^q), \ t \in (t_k, t_{k+1}], k \in \mathbb{N}, \\ \vdots \end{cases}$$

Proof. Let $t \in [t_0, t_1]$. Then we have

$$u(t) = u(t_0)E_q(\lambda(t-t_0)^q), \ t \in [t_0, t_1].$$

Suppose that (3.12) holds for some $k \in \mathbb{N}$. Then we have

$$\begin{aligned} u(t) &= u(t_0)\prod_{i=1}^{k}(1+\beta_i E_q(\lambda(t_i - t_0)^q)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\lambda u(s)ds \\ &= u(t_0)\prod_{i=1}^{k}(1+\beta_i E_q(\lambda(t_i - t_0)^q))E_q(\lambda(t-t_0)^q)), \ t \in (t_k, t_{k+1}]. \end{aligned}$$

Let $t \in (t_{k+1}, t_{k+2}]$. From Lemma 3.13, we obtain

$$\begin{aligned} u(t) &= u(t_{k+1}^+) - \frac{1}{\Gamma(q)}\int_{t_0}^{t_{k+1}}(t_{k+1}-s)^{q-1}\lambda u(s)ds + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\lambda u(s)ds \\ &= (1+\beta_{k+1})u(t_{k+1}^-) - \frac{1}{\Gamma(q)}\int_{t_0}^{t_{k+1}}(t_{k+1}-s)^{q-1}\lambda u(s)ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\lambda u(s)ds \\ &= u(t_0)\prod_{i=1}^{k}(1+\beta_i E_q(\lambda(t_i - t_0)^q)) \\ &\quad + \beta_{k+1}u(t_0)\prod_{i=1}^{k}(1+\beta_i E_q(\lambda(t_i - t_0)^q))E_q(\lambda(t_{k+1}-t_0)^q)) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \lambda u(s) ds \\
&= u(t_0) \prod_{i=1}^{k+1} (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \lambda u(s) ds \\
&= u(t_0) \prod_{i=1}^{k+1} (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) E_q(\lambda(t-t_0)^q),\ t \in (t_{k+1}, t_{k+2}].
\end{aligned}
$$

It follows form induction that

$$u(t) = u(t_0) E_q(\lambda(t-t_0)^q),\ t \in [t_0, t_1]$$

and

$$u(t) = u(t_0) \prod_{i=1}^{k} (1 + \beta_i E_q(\lambda(t_i - t_0)^q)) E_q(\lambda(t-t_0)^q),\ t \in (t_k, t_{k+1}],\ k \in \mathbb{N}.$$

This completes the proof. □

Consider the following impulsive Caputo fractional differential inequality of Gronwall type

$$
\begin{cases}
{}^C D_{t_0}^q u \le \lambda u + d, t \ne t_k, t > t_0, \\
\Delta u(t_k) \le \beta_k u(t_k) + d_k,\ k \in \mathbb{N}, \\
u(t_0) = u_{t_0},
\end{cases}
\tag{3.13}
$$

where λ, d, d_k and $\beta_k, k \in \mathbb{N}$, are constants.

We can obtain the following impulsive Caputo fractional differential inequality of Gronwall type.

Lemma 3.15. *Suppose that a function $m \in PC(J(t_0), \mathbb{R})$ satisfies*

$$
\begin{cases}
{}^C D_{t_0}^q m(t) \le \lambda m(t),\ t \in J(t_0),\ t \ne t_k, \\
m(t_k^+) \le (1 + \beta_k) m(t_k^-),\ k \in \mathbb{N}, \\
m(t_0^+) = m_{t_0},
\end{cases}
\tag{3.14}
$$

where $\lambda, \beta_k, k \in \mathbb{N}$, are constants. Then we have

$$
m(t) \le \begin{cases}
m(t_0) E_q(\lambda(t-t_0)^q),\ t \in [t_0, t_1], \\
m(t_0) \prod_{i=1}^{k} [1 + \beta_i E_q(\lambda(t_i - t_0)^q)] E_q(\lambda(t-t_0)^q),\ t \in (t_k, t_{k+1}],\ k \in \mathbb{N}.
\end{cases}
$$

Proof. Let $t \in [t_0, t_1]$. From Lemma 3.4 in [5], we have

$$m(t) \le m(t_0) E_q(\lambda(t-t_0)^q),\ t \in [t_0, t_1].$$

Let $t \in (t_1, t_2]$. There exist a nonnegative function $n(t)$ and nonnegative constants d_k satisfying

$$\begin{cases} {}^C D_{t_0}^q m(t) = \lambda m(t) - n(t), \ t \neq t_k, \\ m(t_k^+) = (1 + \beta_k) m(t_k^-) - d_k, \ k \in \mathbb{N}. \end{cases}$$

From Lemma 3.13, we obtain

$$\begin{aligned} m(t) &= m(t_1^+) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} (\lambda m(s) - n(s)) ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} (\lambda m(s) - n(s)) ds \\ &= (1 + \beta_1) m(t_1^-) - d_1 - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} (\lambda m(s) - n(s)) ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} (\lambda m(s) - n(s)) ds \\ &= m(t_0)(1 + \beta_1 E_q(\lambda (t_1 - t_0)^q)) - d_1 \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} (\lambda m(s) - n(s)) ds, \ t \in (t_1, t_2]. \end{aligned}$$

It follows from Lemma 3.11 that

$$\begin{aligned} m(t) = {} & [m(t_0)(1 + \beta_1 E_q(\lambda (t_1 - t_0)^q)) - d_1] E_q(\lambda (t - t_0)^q) \\ & - n(t) * t^{q-1} E_{q,q}(\lambda t^q), \ t \in (t_1, t_2], \end{aligned}$$

where $*$ denotes the convolution operator of nonnegative functions $n(t)$ and $t^{q-1} E_{q,q}(\lambda t^q)$. Since $n(t) * t^{q-1} E_{q,q}(\lambda t^q)$ and $d_1 E_q(\lambda (t - t_0)^q)$ are nonnegative for each $t \geq t_0$, then we have

$$m(t) \leq m(t_0)[1 + \beta_1 E_q(\lambda (t_1 - t_0)^q)] E_q(\lambda (t - t_0)^q), \ t \in (t_1, t_2],$$

where λ and β_1 are constants. From induction, we obtain

$$m(t) \leq m(t_0) \prod_{i=1}^{k} [1 + \beta_i E_q(\lambda (t_i - t_0)^q)] E_q(\lambda (t - t_0)^q), \ t \in (t_k, t_{k+1}], \ k \in \mathbb{N},$$

where λ and $\beta_i, i \in \mathbb{N}$, are constants. This completes the proof. □

We obtain the following impulsive fractional integral inequality by induction as in Lemma 2.2 in [14].

Lemma 3.16 ([14, Lemma 2.2]). *Let a function $u \in PC(J(t_0), \mathbb{R})$ satisfies the following integral-sum inequality*

$$u(t) \leq c + \lambda \int_{t_0}^{t} (t - s)^{q-1} u(s) ds + \sum_{t_0 < t_k < t} \beta_k u(t_k^-), \ k \in \mathbb{N}, \tag{3.15}$$

where $c, \lambda,$ *and* $\beta_k, k \in \mathbb{N}$, *are nonnegative constants. Then*

(3.16)

$$u(t) \le \begin{cases} cE_q(\Gamma(q)\lambda(t-t_0)^q),\ t \in (t_0, t_1], \\ c\prod_{i=1}^{k}(1+\beta_i E_q(\Gamma(q)\lambda(t_i-t_0)^q))E_q(\Gamma(q)\lambda(t-t_0)^q),\ t \in (t_k, t_{k+1}],\ k\in\mathbb{N}, \\ \vdots \end{cases}$$

4. Mittag-Leffler stability

In this paper we investigate the Mittag-Leffler stability of solutions of impulsive Caputo fractional differential equations via a new impulsive fractional comparison principle and piecewise continuous auxiliary functions of the type of Lyapunov's functions.

Theorem 4.1. *Let* $\alpha_3 \in \mathbb{R}$. *Suppose that there is a function* $V \in v_0$ *such that*

$$\alpha_1|x|^a \le V(t,x) \le \alpha_2|x|^a,\ (t,x) \in \mathbb{R}_+ \times \Omega, \tag{4.1}$$

$$^C D_{t_0}^q V(t,x) \le \alpha_3|x|^a,\ (t,x) \in G_k, \tag{4.2}$$

$$V(t_k^+, x + I_k(x)) \le V(t_k, x),\ x \in \Omega,\ k \in \mathbb{N}, \tag{4.3}$$

where $\Omega \subset \mathbb{R}^n$ *is a domain containing the origin,* $q \in (0,1), \alpha_1, \alpha_2$ *and* a *are arbitrary positive constants. Then the zero solution* $x = 0$ *of Eq.* (3.1) *is a Mittag-Leffler system.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of Eq. (3.1). Then it follows from (4.1) and (4.2) that

$$^C D_{t_0}^q V(t,x) \le \alpha_3|x|^a \tag{4.4}$$

$$\le \begin{cases} \frac{\alpha_3}{\alpha_1}V(t,x),\ (t,x) \in G_k \text{ if } \alpha_3 \ge 0, \\ \frac{\alpha_3}{\alpha_2}V(t,x),\ (t,x) \in G_k \text{ if } \alpha_3 < 0, \end{cases} \tag{4.5}$$

$$= \lambda V(t,x),\ (t,x) \in G_k, \tag{4.6}$$

$$V(t_k^+, x(t_k^+)) = V(t_k^+, x + I_k(x)) \le V(t_k, x),\ x \in \Omega,\ k \in \mathbb{N}, \tag{4.7}$$

where

$$\lambda = \begin{cases} \frac{\alpha_3}{\alpha_1},\ \text{if } \alpha_3 \ge 0, \\ \frac{\alpha_3}{\alpha_2},\ \text{if } \alpha_3 < 0. \end{cases} \tag{4.8}$$

Put $m(t) = V(t, x(t))$ and $\beta_k = 0$, $k \in \mathbb{N}$, in the assumption of Lemma 3.15. It follows from Lemma 3.15 that

$$V(t, x(t)) \le \begin{cases} V(t_0, x(t_0))E_q(\lambda(t-t_0)^q),\ t \in [t_0, t_1], \\ V(t_0, x(t_0))E_q(\lambda(t-t_0)^q),\ t \in (t_k, t_{k+1}],\ k \in \mathbb{N}. \end{cases} \tag{4.9}$$

In view of (4.1), we have

$$|x(t)| \le \{\frac{\alpha_2}{\alpha_1}|x(t_0)|^a E_q(\lambda(t-t_0)^q)\}^{\frac{1}{a}}$$

$$= \{m(|x(t_0)|)E_q(\lambda(t-t_0)^q)\}^{\frac{1}{a}},\ t \geq t_0,$$

where $m(x) = \frac{\alpha_2}{\alpha_1}x^a$ is locally Lipschitz on Ω and λ is given by

$$\lambda = \begin{cases} \frac{\alpha_3}{\alpha_1}, \text{ if } \alpha_3 \geq 0, \\ \frac{\alpha_3}{\alpha_2}, \text{ if } \alpha_3 < 0. \end{cases}$$

Hence the zero solution $x = 0$ of Eq. (3.1) is a Mittag-Leffler system. This complete the proof. □

We obtain the following results as the special cases of Theorem 4.1.

Corollary 4.2. *Suppose that all conditions of Theorem 4.1 hold and α_3 is a nonpositive constant. Then the zero solution $x = 0$ of Eq.* (3.1) *is Mittag-Leffler stable. Furthermore, if α_3 is a negative constant, then all solutions $x(t)$ of Eq.* (3.1) *tend monotonically zero as $t \to \infty$.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of Eq. (3.1). Then it follows from Theorem 4.1 that

$$|x(t)| \leq \{\frac{\alpha_2}{\alpha_1}|x(t_0)|^a E_q(\frac{\alpha_3}{\alpha_2}(t-t_0)^q)\}^{\frac{1}{a}},\ t \geq t_0. \tag{4.10}$$

Since $\frac{\alpha_3}{\alpha_2}$ is negative in (4.10), it follows from Lemma 5 in [6] that $E_q(\frac{\alpha_3}{\alpha_2}(t-t_0)^q)$ tends monotonically zero as $t \to \infty$. Hence, all solutions $x(t)$ of Eq. (3.1) tend monotonically zero as $t \to \infty$. This completes the proof. □

We also obtain the boundedness of solutions of impulsive fractional differential systems via the fractional Lyapunov method.

Remark 4.3. In addition to the assumptions of Theorem 4.1, suppose that α_3 is a nonpositive constant. Then all solutions of Eq. (3.1) are bounded on $\mathbb{R}_+$.

By using Lyapunov function method and fractional comparison principle, Stamova [30, Theorems 4.1-4.3] studied Mittag-Leffler stability of the solutions of impulsive differential equations of fractional order.

In order to study Mittag-Leffler stability of Eq. (3.1), we revise the assumption (4.2) of [30, Theorems 4.1] to the condition (4.13) with impulse effects at fixed times. We will apply the new impulsive fractional differential inequality of Gronwall type to the proof of the following result.

Theorem 4.4. *Suppose that there exists a function $V \in v_0$ such that*

$$\alpha_1|x|^a \leq V(t,x) \leq \alpha_2|x|^{ab},\ (t,x) \in \mathbb{R}_+ \times \Omega, \tag{4.11}$$

$$^C D^q_{t_0} V(t,x) \leq -\alpha_3|x|^{ab},\ (t,x) \in G_k, \tag{4.12}$$

$$V(t_k^+, x + I_k(x)) \leq (1+e_k)V(t_k, x),\ x \in \Omega \subset \mathbb{R}^n,\ k \in \mathbb{N}, \tag{4.13}$$

where each e_k, $k \in \mathbb{N}$, is a constant with $\prod_{k=1}^{\infty}(1+|e_k|) < \infty$, and $q \in (0,1)$, $\alpha_1, \alpha_2, \alpha_3, a$ and b are arbitrary positive constants. Then the zero solution $x = 0$ of Eq. (3.1) *is Mittag-Leffler stable.*

Proof. Let $x(t) = x(t,t_0,x_0)$ be any solution of Eq. (3.1). From (4.11) and (4.12) it follows that

$$(4.14)\qquad {}^{C}D_{t_0}^{q}V(t,x) \le -\alpha_3|x|^{ab} \le -\frac{\alpha_3}{\alpha_2}V(t,x),\ (t,x)\in G_k$$

$$(4.15)\qquad V(t_k^+,x(t_k^+)) = V(t_k^+,x+I_k(x)) \le (1+e_k)V(t_k,x),\ x\in\Omega,\ k\in\mathbb{N}.$$

Putting $m(t) = V(t,x(t))$ and application of Lemma 3.15 yield

(4.16)

$$V(t,x(t)) \le \begin{cases} V(t_0,x(t_0))E_q(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q),\ t\in[t_0,t_1], \\ V(t_0,x(t_0))\prod_{i=1}^{k}[1+e_iE_q(-\dfrac{\alpha_3}{\alpha_2}(t_i-t_0)^q)]E_q(-\dfrac{\alpha_3}{\alpha_2}(t-t_0)^q), \\ t\in(t_k,t_{k+1}],\ k\in\mathbb{N}. \end{cases}$$

From monotonicity of $E_q(-\frac{\alpha_3}{\alpha_2}(t_i-t_0)^q)$ for each $t_i \ge t_0$ and (4.11), we have

$$\begin{aligned}|x(t)| &\le \{\frac{\alpha_2}{\alpha_1}|x(t_0)|^{ab}\prod_{i=1}^{k}(1+|e_i|)E_q(-\frac{\alpha_3}{\alpha_2}(t-t_0)^q)\}^{\frac{1}{a}} \\ &\le \{m(|x(t_0)|)E_q(-\lambda(t-t_0)^q)\}^{\frac{1}{a}},\ t\ge t_0,\end{aligned}$$

where $\lambda = \frac{\alpha_3}{\alpha_2} > 0$ and $m(x) = \frac{\alpha_2}{\alpha_1}\prod_{i=1}^{\infty}(1+|e_i|)x^{ab}$ is locally Lipschitz on Ω. Hence the zero solution $x = 0$ of Eq. (3.1) is Mittag-Leffler stable. This completes the proof. □

Remark 4.5. We obtain the following results as the special cases of Theorems 4.1 and 4.4.

(1) In case $q = 1$ in the assumptions of Theorem 4.1, the Mittag-Leffler stability of impulsive fractional differential equations implies the exponential stability of differential equations with impulse effect in [8, Corollary 3.14].

(2) If one sets $e_k = 0$ for each $k\in\mathbb{N}$ in the assumptions of Theorem 4.4 , the assumption (4.13) of Theorem 4.4 reduces to the assumption (4.2) of Theorem 4.1 in [30].

(3) Furthermore, if we sets $V(t_k^+,x+I_k(x)) = V(t_k,x)$ for each $k\in\mathbb{N}$ in the assumptions of Theorem 4.4, then Theorem 4.4 reduces to Corollary 18 in [6].

Lemma 4.6 ([22])**.** *Let $\alpha\ge 0$, and $f(t,x)$ be continuous on $\mathbb{R}_+\times\Omega$. Then*

$$||{}^{C}D_{t_0}^{-\alpha}f(t,x)|| \le {}^{C}D_{t_0}^{-\alpha}||f(t,x)||,\ (t,x)\in\mathbb{R}_+\times\Omega.$$

We obtain the following result adapted from Theorem 4.3 in [30] by using the impulsive fractional differential inequality of Gronwall type.

Theorem 4.7. *Assume that the following conditions hold.*

(1) *The function $f(t,x)$ is Lipschitz continuous with respect to $x \in \Omega$ with Lipschitz constant $l > 0$.*
(2) *There exists a function $V \in v_0$ such that*

$$\begin{aligned} \alpha_1|x|^a &\le V(t,x) \le \alpha_2|x|, \ (t,x) \in \mathbb{R}_+ \times \Omega, \\ \dot{V}_{(3.1)}(t,x) &\le -\alpha_3|x|^a, \ (t,x) \in G_k, \\ V(t_k^+, x + I_k(x)) &\le (1+e_k)V(t_k,x), \ x \in \Omega \subset \mathbb{R}^n, \ k \in \mathbb{N}, \end{aligned}$$

where each e_k is a constant with $\prod_{k=1}^{\infty}(1+|e_k|) < \infty$, and $\dot{V}_{(3.1)}(t,x) = \frac{dV(t,x)}{dt}$, and $\alpha_1, \alpha_2, \alpha_3$ and a are arbitrary positive constants.

Then the zero solution $x = 0$ of Eq. (3.1) *is Mittag-Leffler stable.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of Eq. (3.1). Following the same proof as for Theorem 4.3 in [30] and the assumptions of Theorem 4.4 yield

$$\begin{aligned} {}^C D_{t_0}^q V(t,x(t)) &= {}^C D_{t_0}^{1-\alpha} V(t,x(t)) \\ &= D_{t_0}^{-\alpha}\dot{V}(t,x(t)) \\ &\le -\alpha_3 D_{t_0}^{-\alpha}|x(t)| \le -\frac{\alpha_3}{l} D_{t_0}^{-\alpha}|f(t,x)| \le -\frac{\alpha_3}{l}|x(t)| \\ &\le -\frac{\alpha_3}{\alpha_2 l} V(t,x), \ (t,x) \in G_k, \ k \in \mathbb{N}, \end{aligned}$$

where $q = 1 - \alpha$ and $D_{t_0}^{-\alpha}x(t_0) = 0$. From similar argument in the proof of Lemma 3.15, we obtain

$$\begin{aligned} |x(t)| &\le \{\frac{\alpha_2}{\alpha_1}|x(t_0)|\prod_{i=1}^{k}(1+|e_i|)E_q(-\frac{\alpha_3}{\alpha_2 l}(t-t_0)^q)\}^{\frac{1}{a}} \\ &\le \{m(|x(t_0)|)E_q(-\lambda(t-t_0)^q)\}^{\frac{1}{a}}, \ t \ge t_0, \end{aligned}$$

where $\lambda = \frac{\alpha_3}{\alpha_2 l} > 0$ and $m(x) = \frac{\alpha_2}{\alpha_1}\prod_{i=1}^{\infty}(1+|e_i|)x$ is locally Lipschitz on Ω. Hence the zero solution $x = 0$ of Eq. (3.1) is Mittag-Leffler stable. This completes the proof. □

5. An application and examples

In this section we apply our results on impulsive fractional inequality to study the data dependence of the solution on the initial condition to a certain fractional differential equation involving the Caputo fractional derivative. Furthermore, we give some examples to illustrate our results.

We consider the following nonlinear impulsive fractional scalar differential equation with initial value

$$\begin{cases} {}^C D_{t_0}^q x(t) = f(t,x(t)), \ t \in J(t_0), \ t \ne t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \ k \in \mathbb{N}, \\ x(t_0) = x_0 \in \mathbb{R}. \end{cases} \tag{5.1}$$

Furthermore, assume that the following basic conditions hold:

(A1) The function $f : J(t_0) \times \mathbb{R} \to \mathbb{R}$ is jointly continuous. There exists a positive constant L such that

$$|f(t,u) - f(t,v)| \leq L|u-v| \text{ for all } t \in J(t_0), \text{ and all } u,v \in \mathbb{R}.$$

(A2) Each function $I_k : \mathbb{R} \to \mathbb{R}$ is continuous and there exist positive constants $K_k, k \in \mathbb{N}$, such that

$$|I_k(t,u) - I_k(t,v)| \leq K_k|u-v|, \;\; u,v \in \mathbb{R}, \; k \in \mathbb{N},$$

where each constant K_k is nonnegative for each $k \in \mathbb{N}$.

(A3) The solution $x(t,t_0,x_0)$ of Eq. (5.1) which satisfies the initial condition $x(t_0^+, t_0, x_0) = x_0$ is defined in the interval (t_0, ∞), and is left continuous. Then the solution of Eq. (5.1) satisfies the fractional integral equation:

$$x(t,t_0,x_0) = \begin{cases} x(t_0) + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,x(s))ds, \; t \in [t_0,t_1], \\ x(t_0) + \sum\limits_{i=1}^{k} I_i(x(t_i^-)) + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1} f(s,x(s))ds, \; t \in (t_k,t_{k+1}], \; k \in \mathbb{N}, \\ \vdots \end{cases}$$

The existence and uniqueness of solutions of Eq. (5.1) have been investigated in [32, Theorem 3.10]. Then we obtain the following data dependence result.

Theorem 5.1. *Assume that conditions* (A1)-(A3) *hold. Let $x, y : J(t_0) \to \mathbb{R}$ be the solutions of Eq.* (5.1) *with initial values x_0 and y_0, respectively. Then we have*

$$|x(t,t_0,x_0) - y(t,t_0,y_0)| \leq \begin{cases} |x_0 - y_0| E_q(L(t-t_0)^q), \; t \in (t_0,t_1], \\ |x_0 - y_0| \prod\limits_{i=1}^{k} [1 + K_i E_q(L(t_i - t_0)^q)] E_q(L(t-t_0)^q), \; t \in (t_k,t_{k+1}], \; k \in \mathbb{N}, \\ \vdots \end{cases}$$

Proof. Let $t \in J(t_0)$. From conditions (A1)-(A3), we obtain

$$|x(t) - y(t)| \leq |x_0 - y_0| + \sum_{t_0 < t_k < t} K_i |x(t_i^-) - y(t_i^-)| + \frac{L}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} |x(s) - y(s)| ds, \; t \in (t_k,t_{k+1}], \; k \in \mathbb{N}.$$

Letting $u(t) = |x(t) - y(t)|$ in Lemma 3.16 yields

$$|x(t) - y(t)| \leq |x_0 - y_0| E_q(L(t-t_0)^q), \; t \in [t_0,t_1]$$

and

$$|x(t) - y(t)|$$
$$\leq |x_0 - y_0|[\prod_{i=1}^{k} 1 + K_i E_q(L(t_i - t_0)^q)]E_q(L(t - t_0)^q),\ t \in (t_k, t_{k+1}],\ k \in \mathbb{N}.$$

This completes the proof. □

We give an example to illustrate Theorem 5.1.

Example 5.2 ([3, 13])**.** Let $0 < q \leq 1$, $t_0 = 0$ and $t_k = \frac{k}{2}$, $k \in \mathbb{N}$. Let $J = [0, T] \subset \mathbb{R}_+$. Consider the following impulsive fractional differential equation with initial value

$$(5.2)\qquad \begin{cases} {}^C D_0^q x(t) = \frac{|x(t)|}{(1+99e^t)(1+|x(t)|)},\ t \in (0, T],\ t \neq t_k = \frac{k}{2}, \\ \Delta x(\frac{k}{2}) = \frac{|x(\frac{k}{2})|}{100(1+|x(\frac{k}{2})|)},\ k \in \mathbb{N}, \\ x(0) = x_0 \in \mathbb{R}_+, \end{cases}$$

where $f(t, x) = \frac{|x|}{(1+99e^t)(1+|x|)}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $I_k(x) = \frac{|x|}{100(1+|x|)}$, $x \in \mathbb{R}_+$. Let $x, y \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$. Then we have

$$\begin{aligned} |f(t, x) - f(t, y)| &= \frac{1}{(1 + 99e^t)}|\frac{|x|}{1 + |x|} - \frac{|y|}{1 + |y|}| \\ &= \frac{|x - y|}{(1 + 99e^t)(1 + x)(1 + y)} \\ &\leq \frac{1}{100}|x - y|,\ x, y \in \mathbb{R}_+. \end{aligned}$$

Thus the condition (A1) holds with $L = \frac{1}{100}$. Also, we have

$$|I_k(x) - I_k(y)| \leq \frac{1}{100}|x - y|,\ x, y \in \mathbb{R}_+,\ k \in \mathbb{N}.$$

It follows from Theorem 5.1 that

$$\begin{aligned} &|x(t, 0, x_0) - y(t, 0, y_0)| \\ &\leq \begin{cases} |x_0 - y_0|E_q(L(t)^q),\ t \in [0, \frac{1}{2}] \\ |x_0 - y_0| \prod_{i=1}^{k} [1 + K_i E_q(Lt_i)^q)]E_q(L(t)^q),\ t \in (\frac{k}{2}, \frac{k+1}{2}],\ k \in \mathbb{N}, \end{cases} \\ &= \begin{cases} |x_0 - y_0|E_q(\frac{1}{100}(t)^q), t \in [0, \frac{1}{2}] \\ |x_0 - y_0| \prod_{i=1}^{k} [1 + \frac{1}{100}E_q(\frac{1}{100}(\frac{i}{2})^q)]E_q(\frac{1}{100}t^q),\ t \in (\frac{k}{2}, \frac{k+1}{2}],\ k \in \mathbb{N}, \end{cases} \end{aligned}$$

where $x(t)$ and $y(t)$ are solutions of Eq. (5.1) with initial values x_0 and y_0, respectively.

Remark 5.3. In particular, we suppose that $t_0 = 0$, $T = 1$ and $t_k = t_1 = \frac{1}{2}$ in assumptions of Example 5.2. Furthermore, suppose that q satisfies the following condition

$$\frac{T^q L(k+1)}{\Gamma(q+1)} + kK_k < 1 \Leftrightarrow \Gamma(q+1) > \frac{2}{99}. \tag{5.3}$$

Then Eq. (5.2) has a unique solution on $J = [0, 1]$.

Next, we give two examples to illustrate Theorems 4.1 and 4.4.

Example 5.4. Let $\lambda \in \mathbb{R}$ and $t_0 = 0$. Consider the following impulsive fractional differential equation

$$\begin{cases} {}^C D_0^q |x(t)| = \lambda \frac{|x(t)|}{(1+x^2(t))}, \ t > 0, \ t \neq t_k, \\ \Delta x(t_k) = -\frac{x(t_k)}{1+x^2(t_k)}, \ k \in \mathbb{N}, \\ x(0^+) = x_0 \in \mathbb{R}, \end{cases} \tag{5.4}$$

where $I_k(x(t_k)) = -\frac{x(t_k)}{1+x^2(t_k)}$, $k \in \mathbb{N}$. Let $V(t, x) = |x|$ for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then we have

$$\begin{aligned} {}^C D_0^q V(t, x) = {}^C D_0^q |x(t)| &= \lambda \frac{|x(t)|}{(1 + x^2(t))} \\ &\le \lambda |x(t)| = \lambda V(t, x), \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \ t \neq t_k \end{aligned}$$

and

$$\begin{aligned} V(t_k^+, x + I_k(x)) = |x(t_k) - \frac{x(t_k)}{1 + x^2(t_k)}| &\le (1 - \frac{1}{1 + x^2(t_k)})|x(t_k)| \\ &\le V(t_k, x), \ k \in \mathbb{N}. \end{aligned}$$

Thus the zero solution $x = 0$ of Eq. (5.4) is a Mittag Leffler system by Theorem 4.1.

Example 5.5 ([28])**.** Consider the following impulsive fractional differential equation

$$\begin{cases} {}^C D_0^q |x(t)| = -c|x(t)|(1 + x^2(t)), \ t > 0, \ t \neq t_k, \\ \Delta x(t_k) = e_k x(t_k), \ k \in \mathbb{N}, \\ x(0^+) = x_0, \end{cases} \tag{5.5}$$

where $x_0 \in \mathbb{R}$ and $-1 < e_k < 1$, $k \in \mathbb{N}$. Let $V(t, x) = |x|$ for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. Then we have

$$\begin{aligned} {}^C D_0^q V(t, x) = {}^C D_0^q |x(t)| &= -c|x(t)|(1 + x^2(t)) \le -c|x(t)| \\ &= -cV(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \ t \neq t_k \end{aligned}$$

and

$$V(t_k^+, x + e_k x) = |(1 + e_k)x(t_k)| \le (1 + |e_k|)V(t_k, x(t_k)), \ k \in \mathbb{N},$$

where $\prod_{k=1}^{\infty}(1 + |e_k|) < \infty$. Thus the zero solution $x = 0$ of Eq. (5.5) is Mittag-Leffler stable by Theorem 4.4.

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