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# GRADIENT RICCI SOLITON ON $O(n)$-INVARIANT $n$-DIMENSIONAL SUBMANIFOLD IN $S^n(1) \times S^n(1)$

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Abstract. We construct gradient Ricci solitons as $n$-dimensional submanifolds in $S^n \times S^n$ by using solutions of some nonlinear ODE.

## 1. Introduction

A *Ricci soliton* is a natural generalization of an Einstein metric and is defined on a Riemannian manifold $(M, g)$ by

$$\frac{1}{2}\pounds_V g + \mathrm{Ric} - \lambda g = 0, \tag{1.1}$$

where $V$ is a vector field (the potential vector field), $\lambda$ is a constant on $M$. A trivial Ricci soliton is one for which $V$ is zero or Killing, in which case the metric is Einstein. Compact Ricci solitons are the fixed points of the Ricci flow: $\frac{\partial}{\partial t}g = -2\,\mathrm{Ric}$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. The Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$ respectively. Hamilton [11] and Ivey [12] proved that a Ricci soliton on a compact manifold has constant curvature in dimension 2 and 3, respectively. A first non-trivial Ricci soliton on compact manifold was given by Koiso [14]. It is known that on a compact manifold, steady and expanding Ricci solitons are necessarily Einstein. If the vector field $V$ is the gradient of a potential function $F$, then $g$ is called a *gradient Ricci soliton*. Perelman [17] proved that any compact Ricci soliton is the sum of a gradient and a Killing vector field. On non-compact manifolds, there exist non gradient Ricci solitons. A gradient Ricci soliton is called *rigid* if it is isometric to a quotient of $N \times \mathbb{R}^k$, where $N$ is an Einstein manifold, and $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor. Petersen and Wylie [18]

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proved that all homogeneous gradient Ricci solitons are rigid. We refer to [8] for details about Ricci solitons or gradient Ricci solitons.

Locally conformally flat Ricci solitons have been studied intensively in the last years. Derdzinski [9] and Eminenti, La Nave and Mategazza [10] proved that *compact* locally conformally flat Ricci soliton is either the standard sphere $\mathbb{S}^n$ or one of its quotients. According to Ni and Wallach [16], Cao, Wang and Zhang [3], Petersen and Wylie [19] and Zhang [21], complete locally conformally flat gradient shrinking solitons must be $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{R} \times \mathbb{S}^{n-1}$ or one of their quotients. With respect to the classification of complete gradient steady Ricci solitons, Bryant [1] proved that there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci solitons on $\mathbb{R}^n$, together with the trivial Gaussian steady soliton. Cao and Chen [2] proved that these are the only possibilities under the assumption that the steady soliton is locally conformally flat.

On the other hand, from the view point of submanifold geometry, Ricci soliton is very important and interesting subject (cf. [4], [5], [6] and [7]), but not so many results have been obtained. In this paper we study gradient Ricci solitons of $n$-dimensional submanifold in the Riemannian product of unit spheres $S^n(1) \times S^n(1)$, which is invariant under diagonal action of $O(n)$. Note that a lot of interesting results have been obtained for submanifolds in $S^n \times S^n$ (cf. [13], [15] and [20]). We will show that equations of both 'minimality' and 'gradient Ricci soliton' (the potential function being invariant under the action of $O(n)$) are described in terms of systems of nonlinear ordinary differential equations. By computing the Weyl curvature tensors, we can see that they are locally conformally flat. As special cases, we have totally geodesic submanifolds $S^n(1) \times \{\text{point}\}$ and diagonal $S^n(1/2)$, and rigid shrinking Ricci soliton $\mathbb{R} \times S^{n-1}(1)$.

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## 2. Preliminaries

Let $(M, g)$ and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds and let $\Phi : M \to \widetilde{M}$ be an isometric immersion. Then for tangent vector fields $X$ and $Y$ on $M$, the Gauss formula is

$$\widetilde{\nabla}_{d\Phi(X)} d\Phi(Y) = d\Phi(\nabla_X Y) + \sigma(X, Y),$$

where $\widetilde{\nabla}$, $\nabla$ and $\sigma$ are Levi-Civita connection of $\widetilde{M}$, $M$ and the second fundamental tensor of $M$ in $\widetilde{M}$, respectively. Also for $X, Y, Z, W \in TM$, the Gauss equation is

$$\begin{aligned} g(R(X,Y)Z, W) &= \tilde{g}(\widetilde{R}(d\Phi(X), d\Phi(Y))d\Phi(Z), d\Phi(W)) \\ &\quad + \tilde{g}(\sigma(Y,Z), \sigma(X,W)) - \tilde{g}(\sigma(X,Z), \sigma(Y,W)), \end{aligned} \tag{2.1}$$

where $\widetilde{R}$ and $R$ denote the curvature tensors of $\widetilde{M}$ and $M$, respectively.

When $\widetilde{M}$ is a real space form of constant sectional curvature $K$, we have

$$\tilde{g}(\widetilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}) = K(\tilde{g}(\tilde{Y},\tilde{Z})\tilde{g}(\tilde{X},\tilde{W}) - \tilde{g}(\tilde{X},\tilde{Z})\tilde{g}(\tilde{Y},\tilde{W})),$$

where $\tilde{X},\tilde{Y},\tilde{Z},\tilde{W} \in TS^n(1)$. Hence if $\widetilde{M}$ is a Riemannian product $S^n(1) \times S^n(1)$ of unit spheres $S^n(1)$ and $S^n(1)$, then we have

$$\begin{aligned}\tilde{g}(\widetilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}) &= \tilde{g}(\tilde{Y}_1,\tilde{Z}_1)\tilde{g}(\tilde{X}_1,\tilde{W}_1) - \tilde{g}(\tilde{X}_1,\tilde{Z}_1)\tilde{g}(\tilde{Y}_1,\tilde{W}_1)\\ &\quad + \tilde{g}(\tilde{Y}_2,\tilde{Z}_2)\tilde{g}(\tilde{X}_2,\tilde{W}_2) - \tilde{g}(\tilde{X}_2,\tilde{Z}_2)\tilde{g}(\tilde{Y}_2,\tilde{W}_2),\end{aligned}\tag{2.2}$$

where

$$\begin{aligned}&\tilde{X} = (\tilde{X}_1,\tilde{X}_2),\ \tilde{Y} = (\tilde{Y}_1,\tilde{Y}_2),\\ &\tilde{Z} = (\tilde{Z}_1,\tilde{Z}_2),\ \tilde{W} = (\tilde{W}_1,\tilde{W}_2) \in T(S^n(1)\times S^n(1)) \cong TS^n(1)\times TS^n(1).\end{aligned}$$

## 3. $O(n)$-invariant $n$-dimensional submanifold in $S^n(1) \times S^n(1)$

Let

$$\begin{aligned}(x,y) : I \to [-\pi/2,\pi/2] \times [-\pi/2,\pi/2] - \{(\pm\pi/2,\pm\pi/2)\},\\ s \mapsto (x(s), y(s))\end{aligned}\tag{3.1}$$

be a smooth curve of unit speed. We may put

$$x'(s) = \cos\alpha(s), \quad y'(s) = \sin\alpha(s)\tag{3.2}$$

for some $\alpha : I \to \mathbb{R}/2\pi\mathbb{Z}$. Let $\Phi : I \times S^{n-1}(1) \to S^n(1) \times S^n(1)$ be a map defined by

$$\Phi(s,p) = ((\cos x(s)p, \sin x(s)), (\cos y(s)p, \sin y(s))).\tag{3.3}$$

Differential of $\Phi$ is given by

$$\begin{aligned}&d\Phi(\partial/\partial s) = (\cos\alpha(s)(-\sin x(s)p, \cos x(s)), \sin\alpha(s)(-\sin y(s)p, \cos y(s))),\\ &d\Phi(e_j) = ((\cos x(s)e_j, 0), (\cos y(s)e_j, 0)) \quad (1 \le j \le n-1),\end{aligned}\tag{3.4}$$

where $e_1,\dots,e_{n-1}$ is an orthonormal basis of the tangent space $T_pS^{n-1}(1)$ of unit $(n-1)$-sphere at $p$. Hence we have

$$\begin{aligned}&\|d\Phi(\partial/\partial s)\|^2 = 1,\\ &\|d\Phi(e_j)\|^2 = \cos^2 x(s) + \cos^2 y(s) =: A(s),\\ &g(d\Phi(\partial/\partial s), d\Phi(e_j)) = 0 \quad (1 \le j \le n-1).\end{aligned}\tag{3.5}$$

Then $(x(s),y(s)) \neq (\pm\pi/2,\pm\pi/2)$ implies $A(s) > 0$, so $\Phi$ is an immersion. Note that the image $M^n = \Phi(I \times S^{n-1}(1))$ is invariant under the action $O(n)$ on $S^n(1) \times S^n(1)$ by

$$((\cos xp, \sin x), (\cos yp, \sin y)) \mapsto ((\cos xgp, \sin x), (\cos ygp, \sin y)) \quad (g \in O(n)).$$

*Remark.* Sufficient conditions, for which the image $M = \Phi(I \times S^{n-1}(1))$ is a closed submanifold in $S^n(1) \times S^n(1)$, are as follows:

(1) The curve $(x,y)$ given by (3.1) is *closed* and the image $(x,y)(I)$ does not meet the *boundary* of the square $[-\pi/2,\pi/2]\times[-\pi/2,\pi/2]$, i.e., $(x,y)(I)\subset(-\pi/2,\pi/2)\times(-\pi/2,\pi/2)$.
(2) The curve $(x,y)$ starts at a point in the *vertex* $\{(\pm\pi/2,\pm\pi/2)\}$ and ends at a *different point* in the vertex (cf. Example 1 and Example 2).
(3) The curve $(x,y)$ given by (3.1) is *closed* and the image $(x,y)(I)$ does not meet the *vertex* of the square $[-\pi/2,\pi/2]\times[-\pi/2,\pi/2]$, i.e., $(x,y)(I)\cap\{(\pm\pi/2,\pm\pi/2)\}=\emptyset$, and furthermore if $(x,y)(I)$ meets the *edge*

$$((-\pi/2,\pi/2)\times\{\pm\pi/2\})\cup(\{\pm\pi/2\}\times(-\pi/2,\pi/2))$$

at $s=s_0$, then

$$\widetilde{(x,y)}(s):=\begin{cases}(x,y)(s) & (s\le s_0)\\ \text{the image of } (x,y)(s) \text{ under the reflection with respect to} & \\ \text{the line segment of the edge containing } (x(s_0),y(s_0)) & (s\ge s_0)\end{cases}$$

is *smooth* near $(x(s_0),y(s_0))$ in $\mathbb{R}^2$ (cf. Example 3 and Example 4).

If we put

$$E_j:=\frac{1}{\sqrt{A(s)}}e_j\quad(1\le j\le n-1),\tag{3.6}$$

then $\{\partial/\partial s,E_1,\dots,E_{n-1}\}$ is an orthonormal basis of the tangent space $T_{(s,p)}M$ with respect to the induced metric. Also if we denote

$$\begin{aligned}&N_0:=(-\sin\alpha(s)(-\sin x(s)p,\cos x(s)),\cos\alpha(s)(-\sin y(s)p,\cos y(s))),\\ &N_j:=\frac{1}{\sqrt{A(s)}}((-\cos y(s)e_j,0),(\cos x(s)e_j,0))\quad(1\le j\le n-1),\end{aligned}\tag{3.7}$$

then $\{N_0,N_1,\dots,N_{n-1}\}$ is an orthonormal basis of the normal space $T^{\perp}_{(s,p)}M$ of $M$ in $S^n(1)\times S^n(1)$.

Let $D$ be the Euclidean covariant differentiation of $\mathbb{R}^{2n+2}=\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$. Then we have

$$\begin{aligned}&D_{d\Phi(\partial/\partial s)}d\Phi(\partial/\partial s)\\ &=\alpha'(s)N_0-(\cos^2\alpha(s)(\cos x(s)p,\sin x(s)),\sin^2\alpha(s)(\cos y(s)p,\sin y(s))),\\ &D_{d\Phi(e_j)}d\Phi(\partial/\partial s)\\ &=(\cos\alpha(s)(-\sin x(s)e_j,0),\sin\alpha(s)(-\sin y(s)e_j,0)),\\ &D_{d\Phi(e_k)}d\Phi(e_j)\\ &=((\cos x(s)D_{e_k}e_j,0),(\cos y(s)D_{e_k}e_j,0))\quad(1\le j,k\le n-1),\end{aligned}\tag{3.8}$$

where we also denote the Euclidean connection of $\mathbb{R}^{n+1}$ as $D$. By taking the normal components of $M$ in $S^n(1)\times S^n(1)$, we obtain that second fundamental tensor $\sigma$ is given by

$$\sigma(\partial/\partial s,\partial/\partial s)=\alpha'(s)N_0,$$

$$\sigma(e_j, e_k) = B(s)\delta_{jk} N_0 \quad (1 \le j, k \le n-1), \tag{3.9}$$
$$\sigma(\partial/\partial s, e_j) = \frac{C(s)}{\sqrt{A(s)}} N_j \quad (1 \le j \le n-1),$$

where

$$B(s) = -\sin\alpha(s)\cos x(s)\sin x(s) + \cos\alpha(s)\cos y(s)\sin y(s), \tag{3.10}$$
$$C(s) = \cos\alpha(s)\sin x(s)\cos y(s) - \sin\alpha(s)\cos x(s)\sin y(s). \tag{3.11}$$

Hence the mean curvature vector $H$ of $M^n$ in $S^n(1) \times S^n(1)$ is given by

$$H = \frac{1}{n}\left(\alpha'(s) + (n-1)\frac{B(s)}{A(s)}\right) N_0,$$

and we obtain:

**Theorem 1.** *Let*

$$(x, y) : I \to [-\pi/2, \pi/2] \times [-\pi/2, \pi/2] - \{(\pm\pi/2, \pm\pi/2)\}, s \mapsto (x(s), y(s))$$

*be a smooth curve of unit speed and let* $\Phi : I \times S^{n-1}(1) \to S^n(1) \times S^n(1)$ *be an immersion defined by* (3.3). *Then* $M^n$ *is a minimal submanifold in* $S^n(1) \times S^n(1)$ *if and only if* $x(s)$ *and* $y(s)$ *satisfy the following system of ODE*:

$$x'(s) = \cos\alpha(s), \quad y'(s) = \sin\alpha(s), \quad \alpha'(s) + (n-1)\frac{B(s)}{A(s)} = 0,$$

*where* $A(s)$ *and* $B(s)$ *are defined by* (3.5) *and* (3.10), *respectively.*

We compute curvature tensor $R$ of $M^n$. Using (2.1), (2.2), (3.4) and (3.9), we have

$$\begin{aligned} g(R(e_j, e_k)e_l, e_m) &= \tilde{g}(\widetilde{R}(d\Phi(e_j), d\Phi(e_k))d\Phi(e_l), d\Phi(e_m)) \\ &\quad + \tilde{g}(\sigma(e_k, e_l), \sigma(e_j, e_m)) - \tilde{g}(\sigma(e_j, e_l), \sigma(e_k, e_m)) \\ &= (A_2(s) + B(s)^2)(\delta_{kl}\delta_{jm} - \delta_{jl}\delta_{km}) \quad (1 \le j, k, l, \le n-1), \end{aligned}$$

where

$$A_2(s) = \cos^4 x(s) + \cos^4 y(s). \tag{3.12}$$

Hence with respect to unit vectors $E_j$ $(1 \le j \le n-1)$ defined by (3.6), we obtain

$$g(R(E_j, E_k)E_l, E_m) = K(s)(\delta_{kl}\delta_{jm} - \delta_{jl}\delta_{km}), \tag{3.13}$$

where

$$K(s) = \frac{A_2(s) + B(s)^2}{A(s)^2}. \tag{3.14}$$

Let $L_s := \{\Phi(s, p) \,|\, p \in S^{n-1}\}$ be a level set in $M$. Then the shape operator $A^s$ of $L_s$ in $M$ is

$$g(A^s e_j, e_k) = \tilde{g}(D_{d\Phi(e_k)} d\Phi(e_j), d\Phi(\partial/\partial s)) = E(s)\delta_{jk},$$

where

$$E(s) := \cos\alpha(s)\cos x(s)\sin x(s) + \sin\alpha(s)\cos y(s)\sin y(s). \tag{3.15}$$

Hence each $L_s$ has constant sectional curvature $K(s) + E(s)^2/A(s)^2$.

Similarly we obtain

$$\begin{aligned} g(R(\partial/\partial s, E_j)E_k, E_l) &= 0, \\ g(R(\partial/\partial s, E_j)E_k, \partial/\partial s) &= T(s)\delta_{jk}, \end{aligned} \tag{3.16}$$

where

$$T(s) = \frac{1}{A(s)}\left(\alpha'(s)B(s) - \frac{C(s)^2}{A(s)} + D(s)\right), \tag{3.17}$$

and

$$D(s) = \cos^2\alpha(s)\cos^2 x(s) + \sin^2\alpha(s)\cos^2 y(s). \tag{3.18}$$

Hence the Ricci tensor of $M$ is given by

$$\begin{aligned} \mathrm{Ric}(\partial/\partial s, \partial/\partial s) &= (n-1)T(s), \\ \mathrm{Ric}(\partial/\partial s, E_j) &= 0, \\ \mathrm{Ric}(E_j, E_k) &= (T(s) + (n-2)K(s))\delta_{jk}. \end{aligned} \tag{3.19}$$

Next we calculate Hessian $H^f$ of a function $f = f(s)$ on $M$. We put $F(s) := f'(s)$. Then the gradient of $f(s)$ is $F(s)\partial/\partial s$. Hence using (3.4) and (3.8), we obtain

$$\begin{aligned} H^f(\partial/\partial s, \partial/\partial s) &= g(\nabla_{\partial/\partial s}(F(s)\partial/\partial s), \partial/\partial s) = F'(s), \\ H^f(\partial/\partial s, E_j) &= g(\nabla_{\partial/\partial s}(F(s)\partial/\partial s), e_j)/\sqrt{A(s)} = 0, \\ H^f(E_j, E_k) &= g(\nabla_{e_j}(F(s)\partial/\partial s), e_k)/A(s) = -\frac{E(s)F(s)}{A(s)}. \end{aligned} \tag{3.20}$$

Hence the gradient Ricci soliton equation $H^f + Ric - \lambda g = 0$ ($\lambda \in \mathbb{R}$) of $M$ is equivalent to

$$F'(s) + (n-1)T(s) = -\frac{E(s)F(s)}{A(s)} + T(s) + (n-2)K(s) = \lambda. \tag{3.21}$$

Consequently we obtain:

**Theorem 2.** *Let*

$$(x, y) : I \to [-\pi/2, \pi/2] \times [-\pi/2, \pi/2] - \{(\pm\pi/2, \pm\pi/2)\}, s \mapsto (x(s), y(s))$$

*be a smooth curve of unit speed and let $\Phi : I \times S^{n-1}(1) \to S^n(1) \times S^n(1)$ be an immersion defined by (3.3). Then the induced metric of $M^n = \Phi(I \times S^{n-1}(1))$ satisfies the gradient Ricci soliton equation $H^f + Ric - \lambda g = 0$ ($\lambda \in \mathbb{R}$) if and only if $f(s)$, $x(s)$ and $y(s)$ satisfy the following system of ODE*:

$$f'(s) = F(s), \quad x'(s) = \cos\alpha(s), \quad y'(s) = \sin\alpha(s),$$

$$F'(s) = -(n-1)\frac{E(s)F(s)}{A(s)} + (n-1)(n-2)K(s) - (n-2)\lambda,$$

$$B(s)\alpha'(s) = \frac{C(s)^2}{A(s)} - D(s) + E(s)F(s) - (n-2)A(s)K(s) + \lambda A(s),$$

*where* $A(s)$, $B(s)$, $C(s)$, $D(s)$, $E(s)$ *and* $K(s)$ *are given by* (3.5), (3.10), (3.11), (3.18), (3.15), (3.14), *respectively.*

*Remark.* By computing the Weyl curvature tensors, we can see that submanifolds obtained in this section are locally conformally flat.

## 4. Examples

In this section we give some examples.

**Example 1.** We consider a curve

$$(4.1) \qquad (x,y) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \quad (x(s), y(s)) = \left(s, \frac{\pi}{2}\right).$$

Then by (3.3), we have $M = \Phi(I \times S^{n-1})$ is $S^n(1) \times \{(0, \ldots, 0, 1)\}$ in $S^n(1) \times S^n(1)$. $x'(s) = 1$ and $y'(s) = 0$ imply

$$\alpha(s) = B(s) = C(s) = 0,$$
$$A(s) = D(s) = \cos^2 s, \ A_2(s) = \cos^4 s,$$
$$K(s) = T(s) = 1, \quad E(s) = \cos s \sin s.$$

Hence (3.9) yield $\sigma = 0$, i.e., $M$ is totally geodesic in $S^n(1) \times S^n(1)$. Also using (3.19), we see that $\mathrm{Ric} = (n-1)g$ and $M$ is Einstein of constant sectional curvature 1.

**Example 2.** Let

$$(x,y) : \left[-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$(4.2) \qquad (x(s), y(s)) = \left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right).$$

Then by (3.3), $\Phi : S^n(1) \to S^n(1) \times S^n(1)$ is the diagonal embedding. $x'(s) = y'(s) = 1/\sqrt{2}$ implies

$$\alpha(s) = \frac{\pi}{4}, \quad B(s) = C(s) = 0, \quad A(s) = 2\cos^2\frac{s}{\sqrt{2}},$$
$$D(s) = \cos^2\frac{s}{\sqrt{2}}, \quad A_2(s) = 2\cos^4\frac{s}{\sqrt{2}},$$
$$K(s) = T(s) = \frac{1}{2}, \quad E(s) = \sqrt{2}\cos\frac{s}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}.$$

Hence (3.9) yields $\sigma = 0$, i.e., $M$ is totally geodesic in $S^n(1) \times S^n(1)$. Also using (3.19), we see that $\mathrm{Ric} = ((n-1)/2)g$ and $M$ is Einstein of constant sectional curvature 1/2.

**Example 3.** Let

$$(x,y):\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\quad (x(s),y(s))=(s,0)\,. \tag{4.3}$$

Then we have $x'(s)=1$, $y'(s)=0$ and

$$\alpha(s)=0,\quad A(s)=\cos^2 s+1,\quad B(s)=0,\quad C(s)=\sin s.$$

(3.9) yields

$$\begin{aligned}\sigma(\partial/\partial s,\partial/\partial s)&=\sigma(e_j,e_k)=0\quad(1\le j,k\le n-1),\\ \sigma(\partial/\partial s,e_j)&=\frac{\sin s}{\sqrt{\cos^2 s+1}}N_j\quad(1\le j\le n-1).\end{aligned}$$

Hence $M^n$ is non-totally geodesic and minimal submanifold in $S^n(1)\times S^n(1)$.

Now if we extend the curve $(x,y)$ of (4.3) as

$$\begin{aligned}(x,y)&:\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\\ (x(s),y(s))&=\begin{cases}(s,0) & \left(-\frac{\pi}{2}\le s\le\frac{\pi}{2}\right),\\ (-s+\pi,0) & \left(\frac{\pi}{2}\le s\le\frac{3\pi}{2}\right),\end{cases}\end{aligned}$$

then the corresponding submanifold $M^n$ in $S^n(1)\times S^n(1)$ is closed.

**Example 4.** Let

$$\begin{aligned}(x,y)&:\left[-\frac{\pi}{\sqrt{8}},\frac{\pi}{\sqrt{8}}\right]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\\ (x(s),y(s))&=\left(\frac{s}{\sqrt{2}}+\frac{\pi}{4},\frac{s}{\sqrt{2}}-\frac{\pi}{4}\right).\end{aligned} \tag{4.4}$$

Then we have $A(s)=\cos^2 x(s)+\cos^2 y(s)=1$. Hence the induced metric on $M^n$ is the Riemannian product metric on $I\times S^{n-1}(1)$. Furthermore we obtain:

$$\begin{aligned}x'(s)&=y'(s)=\frac{1}{\sqrt{2}},\\ \alpha(s)&=\frac{\pi}{4},\quad B(s)=-\frac{1}{\sqrt{2}}\cos(\sqrt{2}s),\\ C(s)&=\frac{1}{\sqrt{2}},\quad D(s)=\frac{1}{2},\quad A_2(s)=\frac{1}{2}(1+\sin^2(\sqrt{2}s)),\\ K(s)&=1,\quad T(s)=0,\quad E(s)=0,\\ \mathrm{Ric}(\partial/\partial s,\partial/\partial s)&=\mathrm{Ric}(\partial/\partial s,E_j)=0,\\ \mathrm{Ric}(E_j,E_k)&=(n-2)\delta_{jk}.\end{aligned}$$

Then the gradient Ricci soliton equation (3.21) is $F'(s)=n-2=\lambda$. Consequently the potential function $f(s)$ is a quadratic function and Ricci soliton is shrinking and rigid.

If we extend the curve $(x, y)$ of (4.4) as

$$(x,y):\left[-\frac{\pi}{\sqrt{8}},\frac{7\pi}{\sqrt{8}}\right]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right],$$

$$(x(s),y(s))=\begin{cases}\left(\frac{s}{\sqrt{2}}+\frac{\pi}{4},\frac{s}{\sqrt{2}}-\frac{\pi}{4}\right) & \left(-\frac{\pi}{\sqrt{8}}\leq s\leq\frac{\pi}{\sqrt{8}}\right),\\ \left(-\frac{s}{\sqrt{2}}+\frac{3\pi}{4},\frac{s}{\sqrt{2}}-\frac{\pi}{4}\right) & \left(\frac{\pi}{\sqrt{8}}\leq s\leq\frac{3\pi}{\sqrt{8}}\right),\\ \left(-\frac{s}{\sqrt{2}}+\frac{3\pi}{4},-\frac{s}{\sqrt{2}}+\frac{5\pi}{4}\right) & \left(\frac{3\pi}{\sqrt{8}}\leq s\leq\frac{5\pi}{\sqrt{8}}\right),\\ \left(\frac{s}{\sqrt{2}}-\frac{7\pi}{4},-\frac{s}{\sqrt{2}}+\frac{5\pi}{4}\right) & \left(\frac{5\pi}{\sqrt{8}}\leq s\leq\frac{7\pi}{\sqrt{8}}\right),\end{cases}$$

then the corresponding submanifold $M^n$ is $S^1\times S^{n-1}$ in $S^n(1)\times S^n(1)$, and rigid Ricci soliton is nothing but the universal covering $\mathbb{R}^1\times S^{n-1}$ of $S^1\times S^{n-1}$.

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