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SHARPENED FORMS OF ANALYTIC FUNCTIONS CONCERNED WITH HANKEL DETERMINANT

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Abstract. In this paper, we present a Schwarz lemma at the boundary for analytic functions at the unit disc, which generalizes classical Schwarz lemma for bounded analytic functions. For new inequalities, the results of Jack's lemma and Hankel determinant were used. We will get a sharp upper bound for Hankel determinant $H_2(1)$. Also, in a class of analytic functions on the unit disc, assuming the existence of angular limit on the boundary point, the estimations below of the modulus of angular derivative have been obtained.

1. Introduction

The most classical version of the Schwarz Lemma examines the behavior of a bounded, analytic function mapping the origin to the origin in the unit disc $E = \{z : |z| < 1\}$. It is possible to see its effectiveness in the proofs of many important theorems. The Schwarz Lemma, which has broad applications and is the direct application of the maximum modulus principle, is given in the most basic form as follows:

Let E be the unit disc in the complex plane $\mathbb{C}$. Let $f : E \to E$ be an analytic function with $f(0) = 0$. Under these conditions, $|f(z)| \leq |z|$ for all $z \in E$ and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$, then f is a rotation; that is

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$f(z) = ze^{i\theta}$, θ real ([5], p.329). Schwarz lemma has several applications in the field of electrical and electronics engineering. Use of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for analysis of transfer functions in control engineering and multi-notch filter design in signal processing [12, 13].

In order to derive our main results, we have to recall here the following Jack's Lemma [6].

LEMMA 1.1. *Let $f(z)$ be a non-constant anaytic function in E with $f(0) = 0$. If*

$$|f(z_0)| = \max\{|f(z)| : |z| \leq |z_0|\},$$

then there exists a real number $k \geq 1$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ that are analytic in E. Also, let $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$\Re\left(2\frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0. \tag{1.1}$$

In addition, class $\mathcal{M}$ represents the class of convex and starlike functions. The certain analytic functions, which are in the class of $\mathcal{M}$ on the unit disc E, are considered in this paper. The subject of the present paper is to discuss some properties of the function f(z) which belongs to the class of $\mathcal{M}$ by applying the Jack's Lemma.

In this study, we will give the sharp estimates for the Hankel determinant of the first order for the class of the analytic function $f \in \mathcal{A}$ will satisfy the condition (1.1). In particular, the sharp upper bounds on $H_2(1)$ will be obtained for the class $\mathcal{M}$. In addition, the relationship between the coeffcients of the Hankel determinant and the angular derivative of the function f, which provides the class $\mathcal{M}$, will be examined. In this analysis, the coefficients a_2, a_3 and a_4 will be used. Let $f \in \mathcal{A}$. The q^{th} Hankel determinant of f for $n \geq 0$ and $q \geq 1$ is stated by Noonan and Thomas [17] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \ a_1 = 1.$$

From the Hankel determinant for $n = 1$ and $q = 2$, we have

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2.$$

Here, the Hankel determinant $H_2(1) = a_3 - a_2^2$ is well-known as Fekete-Szegö functional [16]. In [18], authors have obtained the upper bounds of the Hankel determinant $|a_2a_4 - a_3^2|$. We will get an upper bound for $H_2(1) = a_3 - a_2^2$ in our study.

Let $f \in \mathcal{M}$ and consider the following function

$$\Theta(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1.$$

It is an analytic function in E and $\Theta(0) = 0$. Now, let us show that $|\Theta(z)| < 1$ in E. From the definition for $\Theta(z)$, we have

$$\frac{2zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 - \frac{z\Theta'(z)}{1 + \Theta(z)}.$$

We suppose that there exists a $z_0 \in E$ such that

$$\max_{|z| \leq |z_0|} |\Theta(z)| = |\Theta(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\Theta(z_0) = e^{i\theta} \text{ and } \frac{z_0\Theta'(z_0)}{\Theta(z_0)} = k.$$

Therefore, we have that

$$\frac{2z_0f'(z_0)}{f(z_0)} - \left(1 + \frac{z_0f''(z_0)}{f'(z_0)}\right) = 1 - \frac{z_0\Theta'(z_0)}{1 + \Theta(z_0)}.$$

On the other hand whereas, we have

$$\Re\left(\frac{2z_0f'(z_0)}{f(z_0)} - \left(1 + \frac{z_0f''(z_0)}{f'(z_0)}\right)\right) = \Re\left(1 - \frac{ke^{i\theta}}{1 + e^{i\theta}}\right), \ k \geq 2.$$

Since

$$\begin{aligned} \frac{e^{i\theta}}{1+e^{i\theta}} &= \frac{1}{1+e^{-i\theta}} = \frac{1}{1+\cos\theta - i\sin\theta} = \frac{1+\cos\theta + i\sin\theta}{(1+\cos\theta)^2 + (\sin\theta)^2} \\ &= \frac{1+\cos\theta + i\sin\theta}{2(1+\cos\theta)} \end{aligned}$$

and

$$\Re\left(\frac{e^{i\theta}}{1+e^{i\theta}}\right) = \frac{1}{2},$$

we obtain

$$\Re\left(\frac{2z_0 f'(z_0)}{f(z_0)} - \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right)\right) = 1 - \frac{k}{2} \le 0, \;\; k \ge 2.$$

This contradicts the fact $f \in \mathcal{M}$. This means that there is no point $z_0 \in E$ such that $\max_{|z| \le |z_0|} |\Theta(z)| = |\Theta(z_0)| = 1$. Hence, we take $|\Theta(z)| < 1$ in E. From the Schwarz lemma, we obtain

$$\begin{aligned} \Theta(z) &= \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \\ &= \left(\frac{z}{z + a_2 z^2 + a_3 z^3 + \ldots}\right)^2 \left(1 + 2a_2 z + 3a_3 z^2 + \ldots\right) - 1 \\ &= \left(a_3 - a_2^2\right) z^2 + \left(2a_4 - 4a_2 a_3 + 2a_2^3\right) z^3 + \ldots, \end{aligned}$$

$$\frac{\Theta(z)}{z^2} = \left(a_3 - a_2^2\right) + \left(2a_4 - 4a_2 a_3 + 2a_2^3\right) z + \ldots$$

and

$$|H_2(1)| = \left|a_3 - a_2^2\right| \le 1.$$

The results is sharp for $f = f_1 \in \mathcal{M}$, where

$$f_1(z) = \frac{z}{1-z^2} = \sum_{n=1}^{\infty} z^{2n+1}.$$

Indeed we have $a_2 = 0$ and $a_3 = 1$. So, we get $|H_2(1)| = 1$.

We thus obtain the following lemma.

Lemma 1.2. *If $f \in \mathcal{M}$, then we have the inequality*

$$|H_2(1)| = \left|a_3 - a_2^2\right| \le 1. \tag{1.2}$$

This result is sharp and the extremal function is

$$f(z) = \frac{z}{1-z^2}.$$

Consider the product

$$B(z) = \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}.$$

The function $B(z)$ is called a finite Blaschke product, where $z_1, z_2, ..., z_n \in E$. Let the function $\Theta(z)$ satisfy the condition of the Schwarz lemma and also have zeros $z_1, z_2, ..., z_n$. Thus, one can see that the inequality (1.2) can be strengthened by standard methods as follows:

$$|H_2(1)| = \left|a_3 - a_2^2\right| \leq \prod_{k=1}^{n} |z_k| \, .$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which is called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows:

If f extends continuously to some boundary point c with $|c| = 1$, and if $|f(c)| = 1$ and $f'(c)$ exists, then $|f'(c)| \geq 1$, which is known as the Schwarz lemma on the boundary. In addition to conditions of the boundary Schwarz Lemma, if f fixes the point zero, that is $f(z) = a_p z^p + a_{p+1} z^{p+1} + ...$, then the inequality

$$|f'(c)| \geq p + \frac{1 - |a_p|}{1 + |a_p|} \geq p \tag{1.3}$$

is obtained [11]. Inequality (1.3) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–4, 7, 9–14]. Mercer [8] prove a version of the Schwarz lemma where the images of two points are known. Also, he considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he obtained a new boundary Schwarz lemma , for analytic functions mapping the unit disk to itself [10].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [15]).

Lemma 1.3 (Julia-Wolff lemma). *Let f be an analytic function in E, $f(0) = 0$ and $f(E) \subset E$. If, in addition, the function f has an angular* *limit $f(c)$ at $c \in \partial E$, $|f(c)| = 1$, then the angular derivative $f'(c)$ exists and $1 \leq |f'(c)| \leq \infty$.*

COROLLARY 1.4. *The analytic function f has a finite angular derivative $f'(c)$ if and only if f' has the finite angular limit $f'(c)$ at $c \in \partial E$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{M}$ class. Assuming the existence of angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. We also show that these estimations are sharp.

THEOREM 2.1. *Let $f \in \mathcal{M}$. Assume that, for some $c \in \partial E$, f has an angular limit $f(c)$ at c, $f'(c) = 0$. Then we have the inequality*

$$|f''(c)| \geq 2\,|f(c)|^2 . \tag{2.1}$$

Moreover, the equality in (2.1) *occurs for the function*

$$f(z) = \frac{z}{1-z^2}.$$

Proof. Let

$$\Theta(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1.$$

$\Theta(z)$ is an analytic function in E, $\Theta(0) = 0$ and $|\Theta(z)| < 1$ for $z \in E$. Also, we take $|\Theta(c)| = 1$ for $c \in \partial E$. Therefore, from (1.3) for $p = 2$, we obtain

$$2 \leq |\Theta'(c)| = \frac{|f''(c)|}{|f(c)|^2}$$

and

$$|f''(c)| \geq 2\,|f(c)|^2 .$$

This result is sharp for $f(z) = \frac{z}{1-z^2} \in \mathcal{M}$ and $c = i \in \partial E$. Indeed, we have

$$|f'(i)| = \frac{1}{2} = 2\,|f(i)|^2 .$$

$\square$

The inequality (2.1) can be strengthened as below by taking into account a_2 and a_3 which is second and third coefficient in the expansion of the function $f(z) = z + a_2 z^2 + a_3 z^3 + ...$

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$|f''(c)| \geq |f(c)|^2 \left(1 + \frac{2}{1 + |H_2(1)|}\right). \tag{2.2}$$

The inequality (2.2) is sharp for the only case $|H_2(1)| = 1$.

Proof. Let $\Theta(z)$ be the same as in the proof of Theorem 2.1. Therefore, from (1.3) for $p = 2$, we obtain

$$2 + \frac{1 - |c_2|}{1 + |c_2|} \leq |\Theta'(c)| = \frac{|f''(c)|}{|f(c)|^2},$$

where $|c_2| = \frac{|\Theta''(0)|}{2!} = |a_3 - a_2^2|$.

Therefore, we take

$$2 + \frac{1 - |a_3 - a_2^2|}{1 + |a_3 - a_2^2|} \leq \frac{|f''(c)|}{|f(c)|^2},$$

$$1 + \frac{2}{1 + |a_3 - a_2^2|} \leq \frac{|f''(c)|}{|f(c)|^2}$$

and

$$|f''(c)| \geq |f(c)|^2 \left(1 + \frac{2}{1 + |a_3 - a_2^2|}\right).$$

Since $|H_2(1)| = |a_3 - a_2^2|$, we obtain

$$|f''(c)| \geq |f(c)|^2 \left(1 + \frac{2}{1 + |H_2(1)|}\right).$$

This result is sharp for $f(z) = \frac{z}{1-z^2} \in \mathcal{M}$ and $c = i \in \partial E$. Indeed, we have

$$|f'(i)| = \frac{1}{2} = |f(i)|^2 \left(1 + \frac{2}{1 + |H_2(1)|}\right),$$

since $|H_2(1)| = 1$. □

In the following theorem, inequality (2.2) has been strenghened by adding the consecutive term a_4 of $f(z)$ function.

Theorem 2.3. *Let $f \in \mathcal{M}$. Assume that, for some $c \in \partial E$, f has an angular limit $f(c)$ at c, $f'(c) = 0$. Then we have the inequality*

(2.3)

$$|f''(c)| \geq 2|f(c)|^2 \left(1 + \frac{(1 - |H_2(1)|)^2}{1 - |H_2(1)|^2 + 2|a_4 - a_2(a_2^2 + 2H_2(1))|}\right).$$

Proof. Let $\Theta(z)$ be the same as in the proof of Theorem 2.1 and $B_0(z) = z^2$. By the maximum principle, for each $z \in E$, we have the inequality $|\Theta(z)| \leq |B_0(z)|$. Therefore

$$p(z) = \frac{\Theta(z)}{B_0(z)} \tag{2.4}$$

is analytic function in E and $|p(z)| \leq 1$ for $|z| < 1$. In particular, we have

$$|p(0)| = \left|a_3 - a_2^2\right| = |H_2(1)| \tag{2.5}$$

and

$$|p'(0)| = \left|2a_4 - 4a_2a_3 + 2a_2^3\right|.$$

Furthermore, the geometric meaning of the derivative and the inequality $|\Theta(z)| \leq |B_0(z)|$ imply the inequality

$$\frac{c\Theta'(c)}{\Theta(c)} = |\Theta'(c)| \geq |B_0'(c)| = \frac{cB_0'(c)}{B_0(c)}.$$

That is, since the expression $\frac{c\Theta'(c)}{\Theta(c)}$ is a real number greater or equal to 1 (see [3]) and $f'(c) = 0$ yields $|\Theta(c)| = 1$ and $|B_0(c)| = 1$, we take

$$\frac{c\Theta'(c)}{\Theta(c)} = \left|\frac{c\Theta'(c)}{\Theta(c)}\right| = |\Theta'(c)|$$

and

$$\frac{cB_0'(c)}{B_0(c)} = \left|\frac{cB_0'(c)}{B_0(c)}\right| = |B_0'(c)|.$$

Also, $|\Theta(z)| \leq |B_0(z)|$, we get

$$\frac{1 - |\Theta(z)|}{1 - |z|} \geq \frac{1 - |B_0(z)|}{1 - |z|}.$$

Passing to the angular limit in the last inequality yields $|\Theta'(c)| \geq |B_0'(c)|$. The composite function

$$\Omega(z) = \frac{p(z) - p(0)}{1 - \overline{p(0)}p(z)}$$

is analytic in E, $\Omega(0) = 0$, $|\Omega(z)| < 1$ for $|z| < 1$ and $|\Omega(c)| = 1$ for $c \in \partial E$. For $p = 1$, from (1.3), we obtain

$$\frac{2}{1+|\Omega'(0)|} \leq |\Omega'(c)| = \frac{1-|p(0)|^2}{\left|1-\overline{p(0)}p(c)\right|^2}|p'(c)| \leq \frac{1+|p(0)|}{1-|p(0)|}\left|\frac{\Theta'(c)}{B_0(c)} - \frac{\Theta(c)B_0'(c)}{B_0^2(c)}\right|$$
$$= \frac{1+|p(0)|}{1-|p(0)|}\left|\frac{\Theta(c)}{cB_0(c)}\right|\left|\frac{c\Theta'(c)}{\Theta(c)} - \frac{cB_0'(c)}{B_0(c)}\right| = \frac{1+|p(0)|}{1-|p(0)|}\left\{|\Theta'(c)| - |B_0'(c)|\right\}.$$

Since

$$\Omega'(z) = \frac{1 - |p(0)|^2}{\left(1 - \overline{p(0)}p(z)\right)^2} p'(z)$$

and

$$|\Omega'(0)| = \frac{|p'(0)|}{1 - |p(0)|^2} = \frac{|2a_4 - 4a_2a_3 + 2a_2^3|}{1 - |a_3 - a_2^2|^2},$$

we obtain

$$\frac{2}{1 + \frac{|2a_4 - 4a_2a_3 + 2a_2^3|}{1 - |a_3 - a_2^2|^2}} \leq \frac{1 + |a_3 - a_2^2|}{1 - |a_3 - a_2^2|} \left(\frac{|f''(c)|}{|f(c)|^2} - 2 \right)$$

and

$$|f''(c)| \geq 2\,|f(c)|^2 \left(1 + \frac{(1 - |a_3 - a_2^2|)^2}{1 - |a_3 - a_2^2|^2 + 2\,|a_4 - 2a_2a_3 + a_2^3|} \right).$$

Since $|H_2(1)| = |a_3 - a_2^2|$, we obtain the inequality (2.3). □

If $f(z) - z$ a have zeros different from $z = 0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

THEOREM 2.4. *Let $f \in \mathcal{M}$. Assume that, for some $c \in \partial E$, f has an angular limit $f(c)$ at c, $f'(c) = 0$. Let $z_1, z_2, ..., z_n$ be zeros of the function $f(z) - z$ in E that are different from zero. Then we have the inequality*

$$|f''(c)| \geq |f(c)|^2 \left(2 + \sum_{i=1}^{n} \frac{1 - |z_i|^2}{|c - z_i|^2} \right. \tag{2.6}$$
$$\left. + \frac{2\left(\prod_{i=1}^{n}|z_i| - |H_2(1)|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2 - |H_2(1)|^2 + \prod_{i=1}^{n}|z_i| \left| 2\left(a_4 - a_2\left(a_2^2 + 2H_2(1)\right)\right) + H_2(1)\sum_{i=1}^{n}\frac{1 - |z_i|^2}{z_i} \right|} \right).$$

Proof. Let $\Theta(z)$ be as in the proof of Theorem 1 and $z_1, z_2, ..., z_n$ be zeros of the function $f(z) - z$ in E that are different from zero. Let

$$B(z) = z^2 \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i}z}.$$

$B(z)$ is an analytic function in E and $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in E$, we have $|\Theta(z)| \leq |B(z)|$. Consider the function

$$\begin{aligned} t(z) &= \frac{\Theta(z)}{B(z)} = \left[\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right] \frac{1}{z^2 \prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}} \\ &= \frac{(a_3 - a_2^2)\, z^2 + (2a_4 - 4a_2a_3 + 2a_2^3)\, z^3 + \ldots}{z^2 \prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}, \\ &= \frac{(a_3 - a_2^2) + (2a_4 - 4a_2a_3 + 2a_2^3)\, z + \ldots}{\prod\limits_{i=1}^{n} \frac{z-z_i}{1-\overline{z_i}z}}. \end{aligned}$$

$t(z)$ is analytic in E and $|t(z)| < 1$ for $z \in E$. In particular, we have

$$|t(0)| = \frac{|a_3 - a_2^2|}{\prod\limits_{i=1}^{n} |z_i|} = \frac{|H_2(1)|}{\prod\limits_{i=1}^{n} |z_i|}$$

and

$$|t'(0)| = \frac{\left|2a_4 - 4a_2a_3 + 2a_2^3 + (a_3 - a_2^2) \sum\limits_{i=1}^{n} \frac{1-|z_i|^2}{z_i}\right|}{\prod\limits_{i=1}^{n} |z_i|}.$$

Moreover, with the simple calculations, we get

$$\frac{c\Theta'(c)}{\Theta(c)} = |\Theta'(c)| \geq |B'(c)| = \frac{cB'(c)}{B(c)}$$

and

$$|B'(c)| = 2 + \sum_{i=1}^{n} \frac{1 - |z_i|^2}{|c - z_i|^2}.$$

The auxiliary function

$$\Upsilon(z) = \frac{t(z) - t(0)}{1 - \overline{t(0)}t(z)}$$

is analytic in the unit disc E, $\Upsilon(0)=0$, $|\Upsilon(z)|<1$ for $z\in E$ and $|\Upsilon(c)|=1$ for $c\in\partial E$. From (1.3) for $p=1$, we obtain

$$\begin{aligned}
\frac{2}{1+|\Upsilon'(0)|} &\leq |\Upsilon'(c)|=\frac{1-|t(0)|^2}{\left|1-\overline{t(0)}t(c)\right|^2}|t'(c)| \\
&\leq \frac{1+|t(0)|}{1-|t(0)|}\left|\frac{\Theta'(c)}{B(c)}-\frac{\Theta(c)B'(c)}{B^2(c)}\right| \\
&= \frac{1+|t(0)|}{1-|t(0)|}\left|\frac{\Theta(c)}{cB(c)}\right|\left|\frac{c\Theta'(c)}{\Theta(c)}-\frac{cB'(c)}{B(c)}\right| \\
&= \frac{1+|t(0)|}{1-|t(0)|}\left\{|\Theta'(c)|-|B'(c)|\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
|\Upsilon'(0)| &= \frac{|t'(0)|}{1-|t(0)|^2}=\frac{\dfrac{\left|2a_4-4a_2a_3+2a_2^3+\left(a_3-a_2^2\right)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\prod\limits_{i=1}^{n}|z_i|}}{1-\left(\dfrac{\left|a_3-a_2^2\right|}{\prod\limits_{i=1}^{n}|z_i|}\right)^2} \\
&= \prod_{i=1}^{n}|z_i|\frac{\left|2a_4-4a_2a_3+2a_2^3+(a_3-a_2^2)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-|a_3-a_2^2|^2},
\end{aligned}$$

we get

$$\frac{2}{1+\prod\limits_{i=1}^{n}|z_i|\dfrac{\left|2a_4-4a_2a_3+2a_2^3+\left(a_3-a_2^2\right)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-\left|a_3-a_2^2\right|^2}}\leq$$

$$\frac{1+\dfrac{\left|a_3-a_2^2\right|}{\prod\limits_{i=1}^{n}|z_i|}}{1-\dfrac{\left|a_3-a_2^2\right|}{\prod\limits_{i=1}^{n}|z_i|}}\left\{\frac{|f''(c)|}{|f(c)|^2}-2-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right\},$$

$$\frac{2\left(\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-\left|a_3-a_2^2\right|^2\right)}{\left(\prod\limits_{i=1}^{n}|z_i|\right)^2-\left|a_3-a_2^2\right|^2+\prod\limits_{i=1}^{n}|z_i|\left|2a_4-4a_2a_3+2a_2^3+\left(a_3-a_2^2\right)\sum\limits_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\le \frac{\prod_{i=1}^{n}|z_i|+\left|a_3-a_2^2\right|}{\prod_{i=1}^{n}|z_i|-\left|a_3-a_2^2\right|}\left\{\frac{|f''(c)|}{|f(c)|^2}-2-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right\},$$

and

$$|f''(c)| \ge |f(c)|^2\left(2+\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right.$$
$$\left.+\frac{2\left(\prod_{i=1}^{n}|z_i|-\left|a_3-a_2^2\right|\right)^2}{\left(\prod_{i=1}^{n}|z_i|\right)^2-\left|a_3-a_2^2\right|^2+\prod_{i=1}^{n}|z_i|\left|2a_4-4a_2a_3+2a_2^3+\left(a_3-a_2^2\right)\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}\right).$$

Since $|H_2(1)|=|a_3-a_2^2|$, we take the inequality (2.6). □

If $f(z)-z$ has no zeros different from $z=0$ in Theorem 3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

THEOREM 2.5. *Let $f\in\mathcal{M}$ and $a_3>a_2^2$ $(a_2>0, a_3>0)$. Also, $f(z)-z$ has no zeros in E except $z=0$. Further assume that, for some $c\in\partial E$, f has an angular limit $f(c)$ at c, $f'(c)=0$. Then we have the inequality*

(2.7)
$$|f''(c)| \ge 2\,|f(c)|^2\left(1-\frac{H_2(1)\ln^2(H_2(1))}{2H_2(1)\ln(H_2(1))-2\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}\right)$$

and

$$\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right| \le \left|H_2(1)\ln(H_2(1))\right|. \tag{2.8}$$

Proof. Let $a_3>a_2^2$ and $\Theta(z)$, $p(z)$ be as in the proof of Theorem 2.3. Having in mind inequality (2.4), and also inequality (2.5), we denote by $\ln p(z)$ the analytic branch of the logarithm normed by the condition

$$\ln p(0)=\ln\left(a_3-a_2^2\right)=\ln H_2(1)<0.$$

The function

$$T(z)=\frac{\ln p(z)-\ln p(0)}{\ln p(z)+\ln p(0)}$$

is analytic in the unit disc E, $|T(z)|<1$ for $z\in E$, $T(0)=0$ and $|T(c)|=1$ for $c\in\partial E$. From (1.3) for $p=1$, we obtain

$$\begin{aligned}\frac{2}{1+|T'(0)|} &\leq |T'(c)|=\frac{|2\ln p(0)|}{|\ln p(c)+\ln p(0)|^2}\left|\frac{p'(c)}{p(c)}\right| \\ &= \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(c)}\left|\frac{\Theta'(c)}{B_0(c)}-\frac{\Theta(c)B_0'(c)}{B_0^2(c)}\right| \\ &= \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(c)}\left|\frac{\Theta(c)}{cB_0(c)}\right|\left|\frac{c\Theta'(c)}{\Theta(c)}-\frac{cB_0'(c)}{B_0(c)}\right| \\ &= \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(c)}\left\{|\Theta'(c)|-|B_0'(c)|\right\}.\end{aligned}$$

Since

$$\begin{aligned}|T'(0)| &= \frac{1}{|2\ln p(0)|}\left|\frac{p'(0)}{p(0)}\right|=\frac{-1}{2\ln H_2(1)}\frac{|2a_4-4a_2a_3+2a_2^3|}{H_2(1)} \\ &= \frac{-1}{2\ln H_2(1)}\frac{2\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}{H_2(1)} \\ &= \frac{-1}{\ln H_2(1)}\frac{\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}{H_2(1)},\end{aligned}$$

we take

$$\frac{1}{1-\frac{\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}{H_2(1)\ln H_2(1)}}\leq\frac{-\ln p(0)}{\ln^2 p(0)+\arg^2 p(c)}\left(\frac{|f''(c)|}{|f(c)|^2}-2\right).$$

Replacing $\arg^2 p(c)$ by zero, we take

$$\frac{1}{1-\frac{\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}{H_2(1)\ln H_2(1)}}\leq\frac{-1}{\ln p(0)}\left(\frac{|f''(c)|}{|f(c)|^2}-2\right)=\frac{-1}{\ln H_2(1)}\left(\frac{|f''(c)|}{|f(c)|^2}-2\right),$$

$$2-\frac{H_2(1)\ln^2 H_2(1)}{H_2(1)\ln H_2(1)-\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}\leq\frac{|f''(c)|}{|f(c)|^2}$$

and

$$|f''(c)|\geq 2|f(c)|^2\left(1-\frac{1}{2}\frac{H_2(1)\ln^2 H_2(1)}{H_2(1)\ln H_2(1)-\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}\right).$$

Similarly, the function $T(z)$ satisfies the assumptions of the Schwarz lemma, we obtain

$$\begin{aligned} 1 \geq |T'(0)| &= \frac{|2\ln p(0)|}{|\ln p(0)+\ln p(0)|^2}\left|\frac{p'(0)}{p(0)}\right| = \frac{-1}{2\ln p(0)}\left|\frac{p'(0)}{p(0)}\right| \\ &= \frac{-1}{2\ln H_2(1)}\frac{2\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right|}{H_2(1)} \end{aligned}$$

and

$$\left|a_4-a_2\left(a_2^2+2H_2(1)\right)\right| \leq \left|H_2(1)\ln H_2(1)\right|.$$

□

THEOREM 2.6. *Under hypotheses of Theorem 2.5, we have*

$$|f''(c)| \geq 2|f(c)|^2\left(1-\frac{1}{4}\ln H_2(1)\right). \tag{2.9}$$

Proof. From proof of Theorem 2.5, using the inequality (1.3) for the function $T(z)$, we obtain

$$1 \leq \frac{|2\ln p(0)|}{|\ln p(c)+\ln p(0)|^2}\left|\frac{p'(c)}{p(c)}\right| = \frac{-2}{\ln H_2(1)}\left(\frac{|f''(c)|}{|f(c)|^2}-2\right)$$

and

$$|f''(c)| \geq 2|f(c)|^2\left(1-\frac{1}{4}\ln H_2(1)\right).$$

□

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