Honam Mathematical J. **41** (2019), No. 4, pp. 781–798
https://doi.org/10.5831/HMJ.2019.41.4.781

A STUDY OF POLY-BERNOULLI POLYNOMIALS ASSOCIATED WITH HERMITE POLYNOMIALS WITH q-PARAMETER

Waseem A. Khan* and Divesh Srivastava

Abstract. This paper is designed to introduce a Hermite-based-poly-Bernoulli numbers and polynomials with q-parameter. By making use of their generating functions, we derive several summation formulae, identities and some properties that is connected with the Stirling numbers of the second kind. Furthermore, we derive symmetric identities for Hermite-based-poly-Bernoulli polynomials with q-parameter by using generating functions.

1. Introduction

The poly-Bernoulli polynomials have been the area of interest for many researchers in recent decade. A wide-ranging applications from number theory and combinatorics to other fields of applied mathematics belongs to the poly-Bernoulli polynomials (see [1]-[22]). Chad Brewbaker in [3, 4] showed the number of (0, 1)-matrices with n-rows and k columns uniquely reconstructible from their row and column sums are the poly- Bernoulli numbers of negative index. Another application of poly-Bernoulli numbers is in Zeta function theory. Stephane Launois in [17, 18] proved the cardinality of sub-poset of the reverse Bruhat ordering is equal to the poly-Bernoulli numbers. Also Jonas Sjöstrand [22] found a relation between poly-Bernoulli numbers and the number of elements in a Bruhat interval. Also he showed the Poincare polynomial (for value q = 1) of some particularly interesting intervals in the finite Weyl

Received March 6, 2019. Revised April 12, 2019. Accepted May 23, 2019.
2010 Mathematics Subject Classification. 11B68, 11B73, 11B75, 33C10, 33C45.
Key words and phrases. Hermite polynomials, Hermite-based-poly-Bernoulli polynomials with q-parameter, Stirling numbers of the second kind, polylogarithm function.
*Corresponding author

group can be written in terms of poly-Bernoulli numbers. Moreover Peter Cameron in [6] showed that the number of acyclic orientations of a complete bipartite graph is a poly-Bernoulli number.

Kaneko [13] introduced and studied poly-Bernoulli numbers which generalize the classical Bernoulli numbers. poly-Bernoulli numbers $B_n^{(k)}$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, appear in the following power series:

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \tag{1.1}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, |z| < 1, \tag{1.2}$$

and

$$\mathrm{Li}_1(z) = -\ln(1-z),\ \mathrm{Li}_0(z) = \frac{z}{1-z},\ \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \ldots$$

When $k \geq 1$, the left hand side of (1.1) can be written in the form of iterated integrals

$$e^t \frac{1}{e^t-1} \int_0^t \frac{1}{e^t-1} \cdots \int_0^t \frac{1}{e^t-1} \int_0^t \frac{t}{e^t-1} dt dt \cdots dt = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}.$$

Here it is obvious that

$$B_n^{(1)} = B_n.$$

Recently, Jolany et al. [10, 11] generalized the concept of poly-Bernoulli polynomials as follows:

Let $a, b, c > 0$ and $a \neq b$. The generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$, the generalized poly-Bernoulli polynomials $B_n^{(k)}(x;a,b)$ and the generalized polynomials $B_n^{(k)}(x;a,b,c)$ are appeared in the following series respectively.

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)}(a,b) \frac{t^n}{n!}, |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{1.3}$$

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x,a,b) \frac{t^n}{n!}, |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{1.4}$$

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t-a^{-t}} c^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x,a,b,c) \frac{t^n}{n!}, |t| < \frac{2\pi}{|\ln a + \ln b|}. \tag{1.5}$$

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [18-20] are defined as

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \tag{1.6}$$

It is easily concluded from definition (1.6) that

$$H_n(2x,-1) = H_n(x) \tag{1.7}$$

and

$$H_n(x,-\frac{1}{2}) = He_n(x), \tag{1.8}$$

where $H_n(x)$ and $He_n(x)$ being ordinary Hermite polynomials.
Also

$$H_n(x,0) = x^n.$$

The generating function for Hermite polynomial H_n(x,y) are given by (see [19-21]):

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.9}$$

Recently, Pathan and Khan [19] introduced the generalized Hermite poly-Bernoulli polynomials ${}_HB_n^{(k)}(x,y;a,b,e)$ are defined by

$$\frac{\mathrm{Li}_k(1-(ab)^{-t})}{b^t - a^{-t}} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(k)}(x,y;a,b,e)\frac{t^n}{n!}, \tag{1.10}$$

$$\left(|\, t \,| < 2\pi/(|\ln a + \ln b\,|), x, y \in \mathbb{R}\right).$$

The classical Stirling numbers of the second kind $S_2(n,m)$ are defined by the following relations

$$x^n = \sum_{m=0}^{n} S_2(n,m)(x)_m, \tag{1.11}$$

where $(x)n = x(x-1)(x-2)...(x-(n-1))$ is falling factorial.
The Stirling numbers of the second kind is defined by (see [12, 14]):

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.12}$$

By the definitions of the polylogarithm function $\mathrm{Li_k(x)}$ and the Stirling numbers of the second kind, we get the following result.

$$\mathrm{Li_k(1-e^{-t})} = \sum_{\mathrm{m=1}}^{\infty} \frac{\mathrm{(1-e^{-t})^m}}{\mathrm{m^k}} = \sum_{\mathrm{n=1}}^{\infty}\sum_{\mathrm{m=1}}^{\mathrm{n}} \frac{\mathrm{(-1)^{n+m}}}{\mathrm{m^k}} \mathrm{m!S_2(n,m)}\frac{\mathrm{t^n}}{\mathrm{n!}}. \tag{1.13}$$

Note that, $\tilde{S}_k(m) = \sum_{i=1}^{m} i^k$ is a power sum polynomials (see [9], [12]).
The exponential generating function of the power sum polynomials are expressed by

$$\frac{e^{(m+1)t}-1}{e^t-1} = \sum_{m=0}^{\infty} \tilde{S}_k(m)\frac{t^m}{m!}. \tag{1.14}$$

This paper is designed in five sections. In section two, we introduce the generating functions of Hermite-based-poly-Bernoulli polynomials with q-parameter and its properties. Section 3 deals with different summation formulae involving Hermite-based-poly-Bernoulli polynomials with q-parameter. In section 4, a relationship that are connected with the Stirling numbers of the second kind and Hermite poly-Bernoulli polynomials with q-parameter is obtained and the last section 5 is comprises with several symmetric identities for there polynomials.

2. Hermite-based-poly-Bernoulli polynomials with q-parameter

In this section, we introduce Hermite-based-poly-Bernoulli polynomials with q-parameter by using generating functions. Also, we give some identities of these polynomials and find a relation that is connected with Hermite polynomials and classical Bernoulli polynomials.

Definition 2.1. *For $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}$, the Hermite-based-poly-Bernoulli polynomials ${}_HB_{n,q}^{(k)}(x,y)$ with q-parameter are defined by means of the following generating function:*

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{qt}-1}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}, \tag{2.1}$$

where

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty}\frac{t^n}{n^k},$$

is the k^{th} polylogarithm function.

Making $x = y = 0$, $B_{n,q}^{(k)}(x,y) = B_{n,q}^{(k)}$ are called the generalized poly-Bernoulli numbers with q-parameter.

When the condition allow $q = 1$, it is trivial that the Hermite-based-poly-Bernoulli polynomials with q-parameter reduced to Hermite-based-poly-Bernoulli polynomials (see [16]).

Remark 2.2. *On replacing x by $2x$ and $y = -1$ in (2.1), we obtain*

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{qt}-1}e^{2xt-t^2} = \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x)\frac{t^n}{n!}. \tag{2.2}$$

Remark 2.3. *For $y = 0$ in (2.1), the result reduces to*

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{qt}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(k)}(x)\frac{t^n}{n!}, \tag{2.3}$$

where $B_{n,q}^{(k)}(x)$ are called the q-poly-Bernoulli polynomials.

Remark 2.4. *On setting $k = q = 1$, (2.1) reduces to*

$$\frac{t}{e^t-1}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}, \tag{2.4}$$

where ${}_HB_n(x,y)$ is called the Hermite-Bernoulli polynomials which is defined by Dattoli et al. [8].

Theorem 2.5. *The Hermite-based-poly-Bernoulli polynomials with q-parameter holds the following relation:*

$$ {}_HB_{n,q}^{(k)}(x,y) = \sum_{m=0}^{n}\binom{n}{m}B_{n-m,q}^{(k)}H_m(x,y). \tag{2.5}$$

Proof. By using equation (1.9) and (2.1), we obtain (2.5). □

Theorem 2.6. *The Hermite-based-poly-Bernoulli polynomials with q-parameter holds the following relation:*

$$ {}_HB_{n,q}^{(k)}(mx,y) = \sum_{l=0}^{n}\binom{n}{l}(m-1)^{n-l}\,{}_HB_{l,q}^{(k)}(x,y)x^{n-l}. \tag{2.6}$$

Proof. By using the definition (2.1), we replace x by mx as

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(mx,y)\frac{t^n}{n!} &= \frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1}e^{mxt+yt^2}.\\ &= \frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1}e^{(m-1)xt+yt^2}e^{xt}.\end{aligned}$$

$$\sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(mx,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{l=0}^{n}\binom{n}{l}(m-1)^{n-l}\,{}_HB_{l,q}^{(k)}(x,y)x^{n-l}\right]\frac{t^n}{n!}.$$

Finally, on equating the coefficients of t^n in above equation, we get the explicit result (2.6). □

Theorem 2.7. *The Hermite-based-poly-Bernoulli polynomials with q-parameter holds the following relation:*

$$ {}_HB_{n,q}^{(k)}(x,y) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\sum_{p=0}^{\left[\frac{n}{2}\right]}\frac{n!}{p!(n-2p)!}\binom{m+1}{r} \times \frac{(-1)^{r+1}(x-r+ql-qm)^{n-2p}y^p}{(m+1)^k}. \tag{2.7}$$

Proof. From generating relation (2.1) and equation (1.2), we have

$$\frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1}e^{xt+yt^2} = \left((-1)\sum_{l=0}^{\infty}e^{lqt}\right)\left(\sum_{m=0}^{\infty}\frac{(1-e^{-t})^{m+1}}{(m+1)^k}\right)e^{xt+yt^2}$$

$$= (-1)\sum_{l=0}^{\infty}\sum_{m=0}^{l}e^{(l-m)qt}\frac{(1-e^{-t})^{m+1}}{(m+1)^k}e^{xt+yt^2}$$

$$= \sum_{l=0}^{\infty}\sum_{m=0}^{l}\left((-1)\frac{e^{(l-m)qt}}{(m+1)^k}\right)\left(\sum_{r=0}^{m+1}\binom{m+1}{r}(-1)^r e^{(x-r)t}\right)e^{yt^2}$$

$$= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{r=0}^{m+1}\sum_{p=0}^{\left[\frac{n}{2}\right]}\binom{m+1}{r}(-1)^{r+1}\frac{(x-r+ql-qm)^{n-2p}}{(m+1)^k}\frac{y^pt^n}{p!(n-2p)!}. \tag{2.8}$$

On equating the coefficients of same powers of t^n from (2.8), we have our result. □

3. Summation formulae for Hermite poly-Bernoulli polynomials with q-parameter

In this section, we define summation formulae involving Hermite poly-Bernouli polynomials ${}_HB_{n,q}^{(k)}(x,y)$ with parameter q and their basic properties.

Theorem 3.1. *The following summation formulae for Hermite poly-Bernoulli polynomials with q-parameter holds true:*

$$ {}_HB^{(k)}_{l+p,q}(z,y) = \sum_{m,n=0}^{l,p} \left(\begin{array}{c} l \\ m \end{array}\right)\left(\begin{array}{c} p \\ n \end{array}\right)(z-x)^{m+n}\,{}_HB^{(k)}_{l+p-m-n,q}(x,y). \tag{3.1} $$

Proof. We replace t by $t+u$ and rewrite the generating function (2.1) as

$$ \frac{\mathrm{Li}_k(1-(e)^{-(t+u)})}{e^{q(t+u)}-1}e^{y(t+u)^2} = e^{-x(t+u)}\sum_{l,p=0}^{\infty} {}_HB^{(k)}_{l+p,q}(x,y)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.2} $$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$ e^{(z-x)(t+u)}\sum_{m,l=0}^{\infty} {}_HB^{(k)}_{l+p,q}(x,y)\frac{t^l}{l!}\frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_HB^{(k)}_{l+p,q}(z,y)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.3} $$

On expanding exponential function, (3.3) gives

$$ \sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{l,p=0}^{\infty} {}_HB^{(k)}_{l+p,q}(x,y)\frac{t^l}{l!}\frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_HB^{(k)}_{l+p,q}(z,y)\frac{t^l}{l!}\frac{u^p}{p!}, \tag{3.4} $$

which on using formula [23, p.52(2)]

$$ \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, $$

in the left hand side becomes

$$ \sum_{m,n=0}^{\infty}\frac{(z-x)^{m+n}t^m u^n}{m!n!}\sum_{l,p=0}^{\infty} {}_HB^{(k)}_{l+p,q}(x,y)\frac{t^l}{l!}\frac{u^p}{p!} = \sum_{l,p=0}^{\infty} {}_HB^{(k)}_{l+p,q}(z,y)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.5} $$

Now replacing l by $l-m$, p by $p-n$ in the left hand side of (3.5), we get

$$ \sum_{l,p=0}^{\infty}\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}\frac{(z-x)^{m+n}}{m!n!}\,{}_HB^{(k)}_{l+p-m-n,q}(x,y)\frac{t^l}{l!}\frac{u^p}{p!} $$
$$ = \sum_{l,p=0}^{\infty} {}_HB^{(k)}_{l+p,q}(z,y)\frac{t^l}{l!}\frac{u^p}{p!}. \tag{3.6} $$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result. □

Remark 3.2. *By taking $l = 0$ in Equation (3.1), we immediately deduce the following result.*

Corollary 3.3. *The following summation formula holds true:*

$$ {}_HB^{(k)}_{p,q}(z,y) = \sum_{n=0}^{p} \left(\begin{array}{c} p \\ n \end{array} \right) (z-x)^n \, {}_HB^{(k)}_{p-n,q}(x,y). \tag{3.7}$$

Remark 3.4. *On replacing z by $z+x$ and setting $y=0$ in Theorem 3.1, we have*

$$ {}_HB^{(k)}_{l+p,q}(z+x) = \sum_{m,n=0}^{l,p} \left(\begin{array}{c} l \\ m \end{array} \right) \left(\begin{array}{c} p \\ n \end{array} \right) (z)^{m+n} B^{(k)}_{l+p-m-n,q}(x). \tag{3.8}$$

Whereas by setting $z=0$ in Theorem 3.1, we get another result involving poly-Bernoulli polynomials of one and two variables.

$$ B^{(k)}_{l+p,q}(y) = \sum_{m,n=0}^{l,p} \left(\begin{array}{c} l \\ m \end{array} \right) \left(\begin{array}{c} p \\ n \end{array} \right) (-x)^{m+n} \, {}_HB^{(k)}_{l+p-m-n,q}(x,y). \tag{3.9}$$

Remark 3.5. *For $y=0$ in Theorem 3.1, the following summation formula holds true:*

$$ B^{(k)}_{l+p,q}(z) = \sum_{m,n=0}^{l,p} \left(\begin{array}{c} l \\ m \end{array} \right) \left(\begin{array}{c} p \\ n \end{array} \right) (z-x)^{n+m} B^{(k)}_{l+p-m-n,q}(x). \tag{3.10}$$

Theorem 3.6. *The following summation formula for Hermite-based-poly-Bernoulli polynomials with q-parameter holds true:*

$$ {}_HB^{(k)}_{n,q}(x+u,y) = \sum_{l=0}^{n} \binom{n}{l} \, {}_HB^{(k)}_{l,q}(x,y) u^{n-l}. \tag{3.11}$$

Proof. Making x as $x+u$ in (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} {}_HB^{(k)}_{l,q}(x+u,y)\frac{t^n}{n!} &= \frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1} e^{(x+u)t+yt^2} \\ &= \frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1} e^{xt+yt^2} e^{ut} \\ &= \sum_{l=o}^{\infty} {}_HB^{(k)}_{l,q}(x,y)\frac{t^l}{l!} \sum_{m=0}^{\infty} \frac{(ut)^m}{m!} \\ &= \sum_{m=o}^{\infty} \sum_{l=0}^{m} \binom{n}{l} {}_HB^{(k)}_{l,q}(x,y) u^{m-l} \frac{t^m}{m!} \end{aligned}$$

$$\sum_{n=0}^{\infty} {}_HB_{l,q}^{(k)}(x+u,y)\frac{t^n}{n!} = \sum_{n=o}^{\infty}\sum_{l=0}^{n}\binom{n}{l}{}_HB_{l,q}^{(k)}(x,y)u^{n-l}\frac{t^n}{n!}.$$

On comparing the coefficients of t^n, we get the explicit result (3.11). □

Theorem 3.7. *The following summation formula for Hermite-based-poly-Bernoulli polynomials with q-parameter holds true:*

$$ {}_HB_{n,q}^{(k)}(x+u,y+v) = \sum_{l=0}^{n}\binom{n}{l}{}_HB_{l,q}^{(k)}(x,y)H_{n-l}(u,v). \qquad (3.12)$$

Proof. On changing x and y both by $x+u$ and $y+v$, respectively, in (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x+u,y+v)\frac{t^n}{n!} &= \frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{t}})}{e^{qt}-1}e^{(x+u)t+(y+v)t^2} \\ &= \frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{t}})}{e^{qt}-1}e^{xt+yt^2}e^{ut+vt^2} \\ &= \sum_{n=o}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}\sum_{m=0}^{\infty}H_m(u,v)\frac{t^m}{m!} \\ &= \sum_{n=o}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_HB_{n-m,q}^{(k)}(x,y)H_m(u,v)\frac{t^n}{n!}\end{aligned}$$

$$\sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x+u,y+v)\frac{t^n}{n!} = \sum_{n=o}^{\infty}\sum_{m=0}^{n}\binom{n}{l}{}_HB_{n-m,q}^{(k)}(x,y)H_m(u,v)\frac{t^n}{n!}.$$

On comparing the coefficients of t^n, we get the explicit result (3.12). □

Remark 3.8. *On making $q=1$ in (3.11) and (3.12) respectively, we obtained a known result of Khan et al. [16].*

Theorem 3.9. *The following summation formula for Hermite-based-poly-Bernoulli polynomials with q-parameter holds true:*

$$ {}_HB_{n,q}^{(k)}(x,y) = \sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]} y^j x^{n-m-2j}B_{m,q}^{(k)}\frac{n!}{m!j!(n-2j-m)!}. \qquad (3.13)$$

Proof. We start with

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{qt}-1}e^{xt+yt^2} = \left(\sum_{m=0}^{\infty}B_{m,q}^{(k)}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j\frac{t^{2j}}{j!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} x^{n-m} B_{m,q}^{(k)} \right) \frac{t^n}{n!} \left(\sum_{j=0}^{\infty} y^j \frac{t^{2j}}{j!} \right).$$

$$\sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x,y) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n-2j} \sum_{j=0}^{[\frac{n}{2}]} x^{n-m-2j} y^j \binom{n-2j}{m} B_{m,q}^{(k)} \right) \frac{t^n}{(n-2j)!j!}.$$

Equating the coefficients of $\frac{t^n}{n!}$, we get the result (3.13). □

Theorem 3.10. *The following summation formula for Hermite-based-poly-Bernoulli polynomials with q-parameter holds true:*

$$ {}_H B_{n,q}^{(k)}(x+1,y) = \sum_{j=0}^{[\frac{n}{2}]} \sum_{m=0}^{n-2j} \binom{n-2j}{m} y^j B_{m,q}^{(k)}(x). \tag{3.14}$$

Proof. By the definition of Hermite-based-poly-Bernoulli polynomials, we have

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^{qt}-1} e^{(x+1)t+yt^2} = \sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x+1,y) \frac{t^n}{n!} \tag{3.15}$$

$$= \left(\sum_{m=0}^{\infty} B_{m,q}^{(k)}(x) \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} y^j \frac{t^{2j}}{j!} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} B_{m,q}^{(k)}(x) \frac{t^n}{n!} \left(\sum_{j=0}^{\infty} y^j \frac{t^{2j}}{j!} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} y^j B_{m,q}^{(k)}(x) \frac{t^{n+2j}}{n!j!}.$$

Replacing n by $n-2j$, we have

$$\sum_{n=0}^{\infty} {}_H B_{n,q}^{(k)}(x+1,y) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[\frac{n}{2}]} \sum_{m=0}^{n-2j} \binom{n-2j}{m} y^j B_{m,q}^{(k)}(x) \right) \frac{t^n}{n!}. \tag{3.16}$$

Combining (3.15) and (3.16) and equating the coefficients of $\frac{t^n}{n!}$ leads to formula (3.14). □

Theorem 3.11. *The following summation formula for Hermite-based-poly-Bernoulli polynomials with q-parameter holds true:*

$$ {}_HB_{n,q}^{(k)}(x+1,y) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) {}_HB_{n-m,q}^{(k)}(x,y). \tag{3.17} $$

Proof. By the definition of Hermite-based-poly-Bernoulli polynomials, we have

$$ \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x+1,y)\frac{t^n}{n!} + \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!} = \frac{\mathrm{Li}_k(1-e^{-t})}{e^{qt}-1} e^{xt+yt^2}(e^t+1) $$

$$ = \left(\sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{t^m}{m!}\right) + \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HB_{n-m,q}^{(k)}(x,y)\frac{t^n}{(n-m)!m!} + \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}. $$

Finally, equating the coefficients of $\frac{t^n}{n!}$, we get (3.17).

Theorem 3.7. The following summation formula for Hermite-based-poly-Bernoulli polynomials with q-parameter holds true:

$$ {}_HB_{n-m,q}^{(k)}(-x,y) = (-1)^n {}_HB_{n,q}^{(k)}(x,y). \tag{3.18} $$

Proof. We replace t by $-t$ in (2.1) and then subtract the result from (2.1) itself finding

$$ e^{yt^2}\left[\frac{\mathrm{Li}_k(1-e^{-t})}{e^{qt}-1}(e^{xt}-e^{-xt})\right] = \sum_{n=0}^{\infty}[1-(-1)^n]{}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}, $$

which is equivalent to

$$ \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(-x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[1-(-1)^n]{}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!} $$

$$ \sum_{n=0}^{\infty} {}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!} - {}_HB_{n-m,q}^{(k)}(-x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[1-(-1)^n]{}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}. $$

and thus by equating the coefficients of $\frac{t^n}{n!}$, we get (3.18). $\square$

4. Relation with Stirling numbers of the second kind

In this section, by using the generating function of the Stirling numbers of the second kind, we derive some interesting relations that is associated with the generalized Hermite-based-poly-Bernoulli polynomials with q-parameter.

Theorem 4.1. *The following relation holds true:*

$$ {}_HB_{n,q}^{(k)}(x,y)=\sum_{r=0}^{n}\binom{n}{r}\sum_{l=1}^{r+1}\frac{(-1)^{l+r+1}l!S_2(r+1,l)}{l^k(r+1)}B_{n-r,q}(x,y). \tag{4.1}$$

Proof. By Definition 2.1, we have

$$\sum_{n=0}^{\infty}{}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}=\frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1}e^{xt+yt^2}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n+1}\frac{(-1)^{l+n+1}l!S_2(n+1,l)}{l^k(n+1)}\frac{t^n}{n!}\sum_{n=0}^{\infty}{}_HB_{n,q}(x,y)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{l=0}^{r+1}\binom{n}{r}\frac{(-1)^{l+n+1}l!S_2(r+1,l)}{l^k(r+1)}{}_HB_{n-r,q}(x,y)\frac{t^n}{n!}. \tag{4.2}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, the proof of Theorem 4.1 is now complete. □

Theorem 4.2. *The following relation holds true:*

$$ {}_HB_{n,q}^{(k)}(x,y)=\sum_{l=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}(x)_lS_2(m,l)B_{n-m,q}^{(k)}y^p\frac{n!}{p!(n-2p)!}. \tag{4.3}$$

Proof. Using Equations (1.13) and (2.1), we obtain

$$\sum_{n=0}^{\infty}{}_HB_{n,q}^{(k)}(x,y)\frac{t^n}{n!}=\frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1}[(e^t-1)+1]^xe^{yt^2}$$

$$=\frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{e^{qt}-1}\sum_{l=0}^{\infty}(x)_l\frac{(e^t-1)^l}{l!}\sum_{p=0}^{\infty}\frac{y^pt^{2p}}{p!}$$

$$=\sum_{n=0}^{\infty}B_{n,q}^{(k)}\frac{t^n}{n!}\sum_{l=0}^{\infty}(x)_l\sum_{m=l}^{\infty}S_2(m,l)\frac{t^m}{m!}\sum_{p=0}^{\infty}\frac{y^pt^{2p}}{p!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{\infty}\binom{n}{m}B_{n-m,q}^{(k)}(x)_l S_2(m,l)\frac{t^n}{n!}\sum_{p=0}^{\infty}\frac{y^p t^{2p}}{p!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{\infty}\sum_{p=0}^{\left[\frac{n}{2}\right]}\binom{n-2p}{m}B_{n-2p-m,q}^{(k)}(x)_l S_2(m,l)\frac{n!}{p!(n-2p)!}\frac{y^p t^n}{n!}.$$

Comparison the coefficients of equal powers of t leads to our result. □

Theorem 4.3. *The following relation holds true:*

$$_HB_{n,q}^{(k)}(x+1,y) - {}_HB_{n,q}^{(k)}(x,y) = \sum_{l=0}^{n}\binom{l}{n}{}_HB_{l-n,q}^{(k)}(x,y)S_2(n,1). \quad (4.4)$$

Proof. In order to proof the above result we start with the Left-hand side of the above equation as

$$\sum_{n=0}^{\infty}\left[{}_HB_{n,q}^{(k)}(x+1,y) - {}_HB_{n,q}^{(k)}(x,y)\right]\frac{t^n}{n!} = \frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{t}})}{e^{qt}-1}e^{xt+yt^2}(e^t-1).$$

By using the result (2.1) and (1.12), we have

$$\sum_{n=0}^{\infty}\left[{}_HB_{n,q}^{(k)}(x+1,y) - {}_HB_{n,q}^{(k)}(x,y)\right]\frac{t^n}{n!} = \sum_{l=0}^{\infty}{}_HB_{l,q}^{(k)}(x,y)\frac{t^l}{l!}\sum_{n=1}^{\infty}S_2(n,1)\frac{t^n}{n!}$$

$$= \sum_{l=0}^{\infty}\sum_{n=0}^{l}\binom{l}{n}{}_HB_{l-n,q}^{(k)}(x,y)S_2(n,1)\frac{t^n}{n!}. \quad (4.5)$$

Comparing the coefficient on both sides, we obtain the desired result. □

5. General symmetric identities

In this section, we consider several symmetric properties of the Hermite-based-poly-Bernoulli polynomials with q-parameter. Such type of identities introduced by Khan et al. (see [16]) and Pathan and Khan (see [19], [20], [21]).

Theorem 5.1. *Let m_1, m_2 be non-negative integers and $k \in \mathbb{Z}$, the following symmetric relation holds true:*

$$\sum_{r=0}^{n}\binom{n}{r}m_1^{n-r}m_2^{r}B_{n-r,q}^{(k)}(m_2x,m_2y)B_{r,q}^{(k)}(m_1x,m_1y)$$

$$=\sum_{r=0}^{n}\binom{n}{r}m_2^{n-r}m_1^{r}B_{n-r,q}^{(k)}(m_1x,m_1y)B_{r,q}^{(k)}(m_2x,m_2y). \tag{5.1}$$

Proof. Consider

$$G(t)=\frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{m_1t}})\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{m_2t}})}{(e^{qm_1t-1})(e^{qm_1t-1})}e^{2m_1m_2xt+2m_1^2m_2^2yt^2}$$

$$G(t)=\frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{m_1t}})}{(e^{qm_1t-1})}e^{m_1m_2xt+m_1^2m_2^2yt^2}\frac{Li_k(1-e^{-m_2t})}{(e^{qm_2t-1})}e^{m_1m_2xt+m_1^2m_2^2yt^2}$$

$$G(t)=\sum_{n=0}^{\infty}{}_HB_{n,q}^{(k)}(m_2x,m_2y)\frac{(m_1t)^n}{n!}\sum_{r=0}^{\infty}{}_HB_{n,q}^{(k)}(m_1x,m_1y)\frac{(m_2t)^r}{r!}$$

$$G(t)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}m_1^{n-r}m_2^{r}{}_HB_{r,q}^{(k)}(m_1x,m_1y){}_HB_{n-r,q}^{(k)}(m_2x,m_2y)\frac{t^n}{n!}. \tag{5.2}$$

Similarly, we can say in another way

$$G(t)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}m_2^{n-r}m_1^{r}{}_HB_{r,q}^{(k)}(m_2x,m_2y){}_HB_{n-r,q}^{(k)}(m_1x,m_1y)\frac{t^n}{n!}. \tag{5.3}$$

Comparing the coefficient of Equation (5.2) and (5.3), it is clear to Theorem 5.1. □

Using the Equation (1.15), we have the following symmetric identity of the Hermite-based-poly-Bernoulli polynomials.

Theorem 5.2. *Let m_1, m_2 be non-negative integers and $k \in \mathbb{Z}$, the following symmetric relation holds true:*

$$\sum_{r=0}^{n}\binom{n}{r}\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{m_2t}})\mathrm{q^{n-r}m_1^rm_2^{n-r}}{}_{\mathrm{H}}\mathrm{B_{r,q}^{(k)}(m_2x,m_2y)\tilde{S}_{n-r}(m_1-1)}$$

$$=\sum_{r=0}^{n}\binom{n}{r}\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{m_1t}})\mathrm{q^{n-r}m_2^rm_1^{n-r}}{}_{\mathrm{H}}\mathrm{B_{r,q}^{(k)}(m_1x,m_1y)\tilde{S}_{n-r}(m_2-1)}. \tag{5.4}$$

Proof. Consider

$$H(t)=\frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{m_1t}})\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{m_2t}})(\mathrm{e^{qm_1m_2t}}-1)(\mathrm{e^{qm_1m_2xt+qm_1^2m_2yt^2}})}{(e^{qm_1t}-1)(e^{qm_2t}-1)}$$

$$H(t) = \mathrm{Li_k}(1-\mathrm{e^{-m_2t}})\sum_{\mathrm{r=0}}^{\infty} {}_{\mathrm{H}}\mathrm{B^{(k)}_{r,q}}(\mathrm{m_2x, m_2y})\frac{(\mathrm{m_1t})^{\mathrm{r}}}{\mathrm{r!}}\sum_{\mathrm{n=0}}^{\infty}\tilde{\mathrm{S}}_{\mathrm{r}}(\mathrm{m_1}-1)\frac{(\mathrm{qm_2t})^{\mathrm{n}}}{\mathrm{n!}}$$

$$H(t) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\mathrm{Li_k}(1-\mathrm{e^{-m_2t}})\mathrm{q^{n-r}m_1^rm_2^{n-r}}$$

$${}_{H}B^{(k)}_{r,q}(m_2x, m_2y)\tilde{S}_{n-r}(m_1-1)\frac{t^n}{n!}. \quad (5.5)$$

Similarly we can write

$$H(t) = \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\mathrm{Li_k}(1-\mathrm{e^{-m_1t}})\mathrm{q^{n-r}m_2^rm_1^{n-r}}$$

$${}_{H}B^{(k)}_{r,q}(m_1x, m_1y)\tilde{S}_{n-r}(m_2-1)\frac{t^n}{n!}. \quad (5.6)$$

On comparing the coefficients of equal powers of t, we have our result. □

Theorem 5.3. *Let m_1, m_2 be non-negative integers and $k \in \mathbb{Z}$, then the following symmetric relation holds true:*

$$B^{(k)}_{n,q}\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}B^{(k)}_{n,q}\frac{(m_2t)^n}{n!}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}q^{n-1}m_1^{r-1}m_2^{n-r}$$
$$\times\tilde{S}_{n-r}(m_2-1)q^{-1}m_2^{-1}{}_{H}B_{n,q}(m_1x, m_2y)$$
$$= B^{(k)}_{n,q}\frac{(m_2t)^n}{n!}\sum_{n=0}^{\infty}B^{(k)}_{n,q}\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}q^{n-1}m_2^{r-1}m_1^{n-r}$$
$$\times\tilde{S}_{n-r}(m_1-1)q^{-1}m_1^{-1}{}_{H}B_{n,q}(m_2x, m_1y). \quad (5.7)$$

Proof. Let $h(t)$

$$F(t) = \frac{\mathrm{Li_k}(1-\mathrm{e^{-m_1t}})\mathrm{Li_k}(1-\mathrm{e^{-m_2t}})(\mathrm{e^{qm_1m_2t}}-1)(\mathrm{e^{qm_1m_2xt+q^2m_1^2m_2^2yt}})}{(e^{qm_1t}-1)^2(e^{qm_2t}-1)^2}$$
$$= \sum_{n=0}^{\infty}B^{(k)}_{n,q}\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}B^{(k)}_{n,q}\frac{(m_2t)^n}{n!}\sum_{r=0}^{\infty}\tilde{S}_{n-r}(m_2-1)\frac{(qm_1t)^r}{r!}q^{-1}m_2^{-1}$$
$$\times\sum_{n=0}^{\infty}{}_{H}B_{n,q}(m_1x, m_1y)\frac{(qm_2t)^n}{n!}$$
$$= \sum_{n=0}^{\infty}B^{(k)}_{n,q}\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}B^{(k)}_{n,q}\frac{(m_2t)^n}{n!}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}q^{n-1}m_1^{r-1}m_2^{n-r}$$
$$\times\tilde{S}_{n-r}(m_2-1)q^{-1}m_2^{-1}{}_{H}B_{n,q}(m_1x, m_1y)\frac{t^n}{n!}. \quad (5.8)$$

Similarly we can say

$$F(t)=\sum_{n=0}^{\infty}B_{n,q}^{(k)}\frac{(m_2t)^n}{n!}\sum_{n=0}^{\infty}B_{n,q}^{(k)}\frac{(m_1t)^n}{n!}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}q^{n-1}m_2^{r-1}m_1^{n-r}$$
$$\times\tilde{S}_{n-r}(m_1-1)q^{-1}m_1^{-1}{}_HB_{n,q}(m_2x,m_2y)\frac{t^n}{n!}. \tag{5.9}$$

Comparing the coefficient of $\frac{t^n}{n!}$ form the above Equations (5.8) and (5.9), we find the symmetric identity (5.7). □

6. Concluding Remark

The poly Bernoulli polynomials associated with Hermite polynomials with q-parameter plays a major role in number theory and combinatorics to other field of applied mathematics. We can relate generalized polynomials by defining a Gould-Hopper based polynomials with q-parameter as

$$\frac{\mathrm{Li_k}(1-\mathrm{e}^{-\mathrm{t}})}{e^{qt}-1}e^{xt+yt^s}=\sum_{n=0}^{\infty}{}_HB_{n,q}^{(k,s)}(x,y)\frac{t^n}{n!}. \tag{6.1}$$

The equation (2.1) may be derived from (6.1), if we set $s=2$ in (6.1).

This process can easily be extended to established multi-variable Hermite poly-Bernoulli polynomials with q-parameter, multi-variable Hermite poly-Euler polynomials with q-parameter and multi-variable Hermite poly-Hermite polynomials with q-parameter.

7. Acknowledgement

The present work acknowledged by Integral university, with acknowledgement no "IU/R&D/2019-MCN534"

References

[1] T. Arakawa and M. Kaneko, *On poly-Bernoulli numbers*, Comment. Math. Univ. Sanct. Pauli, **48(2)** (1999), 159-167.
[2] A. Bayad and Y. Hamahata, *Polylogarithms and poly-Bernoulli polynomials*, Kyushu. J. Math. **65** (2011), 15-24.
[3] C. Brewbaker, *Lonesum (0, 1)-matrices and poly-Bernoulli numbers of negative index*, Masters thesis, Iowa State University, 2005.

[4] C. Brewbaker, *A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues*, Integers **8** (2008), A02.

[5] M. Cenkci, T. Komatsu, *poly-Bernoulli numbers and polynomials with a q-parameter*, J. Number Theory **152** (2015), 38-54.

[6] P. Cameron, *poly-Bernoulli numbers and acyclic orientations*, CIRCA/Algebra seminar, St Andrews 29 January 2014.

[7] U. Duran, M. Acikgoz, S. Araci, *Hermite based poly-Bernoulli polynomials with a q-parameter*, Adv. Stud. Contemp. Math. **28(2)** (2018), 285-296.

[8] G. Dattoli, S. Lorenzutta and C. Cesarano, *Finite sums and generalized forms of Bernoulli polynomials*, Rendiconti di Mathematica **19** (1999), 385-391.

[9] K. W. Hwang, B. R. Nam and N. S. Jung, *A note on q-analogue of poly-Bernoulli numbers and polynomials*, J. Appl. Math. & Informatics **35** (2017), 611-621.

[10] H. Jolany, M. R. Darafsheh and R. E. Alikelaye, *Generalizations of poly-Bernoulli numbers and polynomials*, Int. J. Math. Comb. **2** (2010), 7-14.

[11] H. Jolany and R. B. Corcino, *Explicit formula for generalization of poly-Bernoulli numbers and polynomials with a,b,c parameters*, J. Classical Anal. **6(2)** (2015), 119-135.

[12] N. S. Jung, C. S. Ryoo, *Symmetric identities for degenerate q-poly-Bernoulli numbers and polynomials*, J. Appl. Math. & Informatics **36** (2018), 29-38.

[13] M. Kaneko, *poly-Bernoulli numbers*, Journal de thorie des nombres de Bordeaux **9** (1997), 221-228.

[14] B. Kurt, *Some identities for the generalized poly-Genocchi polynomials with the parameters a, b and c*, Journal of Math. Anal. **8** (2017), 156-163.

[15] B. Kurt, *Identities and relation on the poly-Genocchi polynomials*, Journal of Inequalities and Special Functions **9(1)** (2018), 1-8.

[16] W. A. Khan, I. A. Khan and S. Zia, *Some properties of Hermite poly-Bernoulli numbers and polynomials*, Appl. Math. Inf. Sci. **7** (2016), 1-8.

[17] S. Launois, *Combinatorics of H -primes in quantum matrices*, J. Algebra **309(1)** (2007), 139-167.

[18] M. A. Pathan and W. A. Khan, *Some new classes of generalized Hermite-based Apostol-Euler and Apostol-Genocchi polynomials*, Fasciculi Mathematici **55(1)** (2015), 153-170.

[19] M. A. Pathan and W. A. Khan, *A new class of generalized polynomials associated with Hermite and poly-Bernoulli polynomials*, Miskolc Math. J. (Accepted) (2018).

[20] M. A. Pathan and W. A. Khan, *A new class of generalized polynomials associated with Hermite and Bernoulli polynomials*, Le Matematiche **LXX** (2015), 53-70.

[21] M. A. Pathan and W. A. Khan, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials*, Mediterr. J. Math. **12** (2015), 679-695.

[22] J. Sjöstrand, *Bruhat intervals as rooks on skew Ferrers boards*, J. Combin. Theory Ser. A **114** (2007), 1182-1198.

[23] H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, Ellis Horwood Limited. Co. New York (1984).

Waseem A. Khan
Department of Mathematics, Faculty of Science,
Integral University,
Lucknow-226026, India.
E-mail: waseem08_khan@rediffmail.com

Divesh Srivastava
Department of Mathematics, Faculty of Science,
Integral University,
Lucknow-226026, India.
E-mail: divesh2712@gmail.com