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# SOLVING FUZZY FRACTIONAL WAVE EQUATION BY THE VARIATIONAL ITERATION METHOD IN FLUID MECHANICS

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ABSTRACT. In this paper, we are extending fractional partial differential equations to fuzzy fractional partial differential equation under Riemann-Liouville and Caputo fractional derivatives, namely Variational iteration methods, and this method have applied to the fuzzy fractional wave equation with initial conditions as in fuzzy. It is explained by one and two-dimensional wave equations with suitable fuzzy initial conditions.

## 1. INTRODUCTION

In now a day, there have been many implementations in obtaining exact solutions in the subject of a fuzzy fractional partial differential equation. Amiri defines the fuzzy generalized Pantograph Equation under Hukuhara differentiability [1]. The concept of the fuzzy derivative was initially defined by Chang and Zadeh [2]. The concept of the fuzzy matrix is first introduced by Thomason 1977 [3]. VIM for obtaining the analytical solution of nonlinear fuzzy initial value problem (NLFIVP) relating to the fuzzy Duffing‘s equation without changing of the first-order system [4]. Applications of VIM, and finding the exact solution of fractional order by using FIVP is compared [5]. Jafari had explained Huan VIM for fractional Riccati differential equation with following NL equations,

$$^{c}D^{\alpha}_{0+}u(t) = A(t) + B(t)u + C(t)u^2, t > 0, m-1 < \alpha \leq m, \tag{1.1}$$

with fuzzy initial condition,

$$u^{j}(0) = \tilde{C}_j, j = 1, \cdots, m-1, m \epsilon \mathbf{N}$$

Where A(t), B(t) and C(t) are given functions, $\tilde{C}_j, j = 1, 2, \cdots, m-1, m \in \mathbf{N}$, are arbitrary fuzzy numbers and $\alpha$ is an order of the fractional derivative [6]. N-th order fuzzy differential equation for VIM is done by Abbasbandy at.al [7]. Using Laplace transforms method

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with fuzzy fractional differential equations Kumar had found the exact solutions [8]. Fuzzy fractional differential equations under Riemann Liouville H-differentiability by fuzzy Laplace transforms are done in [9]. A comparison of the ADM, VIM, and NIM in one dimension equations is done by Ghadle and Khan [10]. Time-fractional diffusion equation, time-fractional telegraphic equation, and the time fraction wave equation in three dimensions with three systematic schemes, namely the ADM, VIM and the NIM is done in [11].

In this paper, we are taking a very important first and second-order linear partial differential wave equation as they occur in classical physics such as mechanical wave-like water wave, sound wave, and seismic wave or a light wave. It arises in fields like Tsunami prediction, traffic-induced vibrations, soil dynamics, acoustics, electromagnetic and fluid dynamics. In 1992 and 1993 Matsuno import the NL evolution equation for one-dimensional gravity wave equation in fluid of arbitrary depth by complex function theory and the set of equations and further he studied non-uniform bottom and for interval wave, and he extended it to two-dimensional third-order NL surface wave by taking the Fourier transform method, including the effects of pressure forcing on the free surface and surface tension [12].

VIM is a nearly accurate-analytical method was proposed by He [13, 14, 15] and has been performed by many physics and engineering problems [16, 17]. VIM was used in [18] to obtain the approximate solution for the first-order LFIVP. Also besides, it was proved that VIM is more valuable and much quicker than ADM. In [19] VIM was used to find the accurate-analytical result for FDE's including NL first-order FIVP. The VIM is been proven to be an efficient method as compared to other methods for solving the fuzzy fractional wave equation.

It is assumed that the existence of vague parameters in wave equations with variable coefficients. Since Zadeh explain fuzzy sets theory [20] is a strong tool for representing blurred and dealing out vague in mathematical models, hence, the idea in solving wave equations with fuzzy parameters using the similar approach as Buckley and Feuring [21] using VIM [22].

In comparison with the paper [23], Chadli investigated the problems with fuzzy parameters, FIV and fuzzy forcing functions recommend by a new theorem for discovering the exact fuzzy solutions, witch extended to the Buckley-Feuring for the proposed models [24]. The application of VIM is easy and computation of the following approximations is straightforward. Strategy based on VIM [25] is introduced by two types of solutions, the Buckley-Feuring solution, and the Seikkala solution.

In the second section, we have explained VIM by using examples like linear inhomogeneous time-fractional wave equation in one and two dimensions with suitable FIC and analysis of the VIM are used from [26].

## 2. Numerical Result

In this section, we present the illustrative examples, by using definitions of fuzzy and Triangular fuzzy numbers and also using the key lemma, essential for the build the solutions, the properties of Modified Fuzzy Riemann-Liouville derivative are explained and is solved with VIM [26] and using (1.1).

**Example 1.** *Consider the following one dimensional linear inhomogeneous fuzzy fractional wave equation.*

$$^{c}D_{0+}^{\alpha}u(t)+u_{x}=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x)+t\cos(x),t>0,x\in R,0<\alpha\leq 1, \tag{2.1}$$

*Subject to the fuzzy initial condition,*

$$u(0)=\tilde{0}[r-1,1+r],$$

*The exact solution of (2.1) is,*

$$\begin{aligned}u(x,t)=&t\sin(x)-[\sum_{i=1}^{\infty}\frac{1}{x^{i}}\frac{t^{\alpha}i}{\Gamma(\alpha_{i}+1)}],i=1,2\cdots\\=&t\sin(x)-E_{\alpha}\frac{1}{x}t^{\alpha}\end{aligned}$$

*Where $E_{\alpha}$ is Mittage-Leffler function.*

*Now, the correction function for (2.1) form as below,*

$$\begin{aligned}\underline{u}_{n+1}(t,x,r)=&\underline{u}_{n}(t,x,r)+\frac{1}{\Gamma(\alpha+1)}\int_{0}^{t}[\lambda(\xi)\frac{d^{\alpha}}{d\xi^{\alpha}}\underline{u}_{n}(\xi,x,r)+\frac{d}{dx}\underline{u}_{n}(\xi,x,r)\\&-\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x)-t\cos(x)]d\xi^{\alpha},\end{aligned} \tag{2.2}$$

$$\begin{aligned}\overline{u}_{n+1}(t,x,r)=&\overline{u}_{n}(t,x,r)+\frac{1}{\Gamma(\alpha+1)}\int_{0}^{t}[\lambda(\xi)\frac{d^{\alpha}}{d\xi^{\alpha}}\overline{u}_{n}(\xi,x,r)+\frac{d}{dx}\overline{u}_{n}(\xi,x,r)\\&-\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x)-t\cos(x)]d\xi^{\alpha},\end{aligned} \tag{2.3}$$

*Where*

$$\frac{d^{\alpha}}{d\xi^{\alpha}}[u_{n}(\xi,x_{1},x_{2},r)]^{r}={}^{c}D_{0+}^{\alpha}[u_{n}(\xi,x_{1},x_{2},r)]^{r}$$

*This yield the stationary conditions*

$$\lambda'(\xi)=0,1+\underline{\lambda}(\xi)=0,\overline{\lambda}'(\xi)=0,1+\overline{\lambda}(\xi)=0$$

*which gives,*

$$\underline{\lambda}(\xi)=\overline{\lambda}(\xi)=-1$$

*Using the Lagrangian multiplier in (2.2) and (2.3) the iteration formula is shown below,*

$$\begin{aligned}\underline{u}_{n+1}(t,x,r) =&\underline{u}_n(t,x,r) - \frac{1}{\Gamma(\alpha+1)}\int_0^t [\frac{d^\alpha}{d\xi^\alpha}\underline{u}_n(\xi,x,r) + \frac{d}{dx}\underline{u}_n(\xi,x,r)\\ &- \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) - t\cos(x)]d\xi^\alpha,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\overline{u}_{n+1}(t,x,r) =&\overline{u}_n(t,x,r) + \frac{1}{\Gamma(\alpha+1)}\int_0^t [\frac{d^\alpha}{d\xi^\alpha}\overline{u}_n(\xi,x,r) + \frac{d}{dx}\overline{u}_n(\xi,x,r)\\ &- \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) - t\cos(x)]d\xi^\alpha\end{aligned} \tag{2.5}$$

*Beginning with*

$$\underline{u}_0(t,x_1,x_2,r) = (r-1)\frac{t^\alpha}{\Gamma(\alpha+1)}, \overline{u}_0(t,x_1,x_2,r) = (1-r)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

*By the iteration formula (2.4) and (2.5), we get the other components as,*

$$\begin{aligned}\underline{u}_1(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x),\\ \overline{u}_1(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x),\\ \underline{u}_2(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x),\\ \overline{u}_2(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x),\\ \underline{u}_3(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x),\\ \overline{u}_3(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x),\\ \underline{u}_4(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\sin(x),\\ \overline{u}_4(t,x_1,x_2,r) =&t\sin(x) + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\sin(x),\\ &\vdots\end{aligned} \tag{2.6}$$

*And so on. The nth approximate fuzzy solution of the method approximates the exact series solution. So, we approximate fuzzy solutions.*

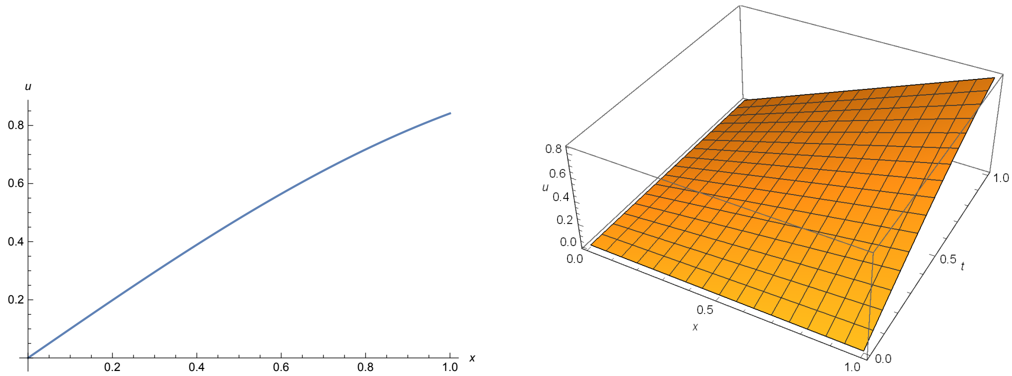



FIGURE 1. **(a)**Exact solution **(b)** Approximate fuzzy solution.

TABLE 1. For approximate fuzzy solution at $\alpha = 0.5$ and $\underline{u}(t,x,r) \cong \underline{u}_2(t,x,r)$.

| | | | | $\underline{u}(t,x,r)$ | | | |
|---|---|---|---|---|---|---|---|
| $\alpha \rightarrow$ | *0.0* | *0.1* | *0.2* | *0.3* | *0.4* | *0.5* | *u exact* |
| $t\downarrow$ | | | | | | | |
| *0.0* | *0.0* | *0.0* | *0.0* | *0.0* | *0.0* | *0.0* | *0.0* |
| *0.1* | *0.1* | *0.2* | *0.157266* | *0.132049* | *0.109454* | *0.105* | *0.1* |
| *0.2* | *0.4* | *0.331563* | *0.284579* | *0.253263* | *0.232919* | *0.22* | *0.2* |
| *0.3* | *0.6* | *0.514014* | *0.449207* | *0.401899* | *0.368299* | *0.345* | *0.3* |
| *0.4* | *0.8* | *0.702251* | *0.623205* | *0.561463* | *0.514632* | *0.48* | *0.4* |
| *0.5* | *1.0* | *0.895057* | *0.805054* | *0.730744* | *0.671295* | *0.625* | *0.5* |

TABLE 2. For approximate solution at $\alpha = 0.5$ and $\overline{u}(t,x,r) \cong \overline{u}_2(t,x,r)$.

| | | | | $\overline{u}(t,x,r)$ | | | |
|---|---|---|---|---|---|---|---|
| $\alpha \rightarrow$ | *0.0* | *0.1* | *0.2* | *0.3* | *0.4* | *0.5* | *u exact* |
| $t\downarrow$ | | | | | | | |
| *0.0* | *0.0* | *0.0* | *0.0* | *0.0* | *0.0* | *0.0* | *0.0* |
| *0.1* | *0.1* | *0.2* | *0.157266* | *0.132049* | *0.11757* | *0.109454* | *0.105* |
| *0.2* | *0.2* | *0.4* | *0.331563* | *0.284579* | *0.253263* | *0.232919* | *0.22* |
| *0.3* | *0.3* | *0.6* | *0.514014* | *0.449207* | *0.401899* | *0.368299* | *0.345* |
| *0.4* | *0.4* | *0.8* | *0.702251* | *0.623205* | *0.561463* | *0.514632* | *0.48* |
| *0.5* | *0.5* | *1.0* | *0.895057* | *0.805054* | *0.730744* | *0.671295* | *0.625* |

**Example 2.** *Consider the following one dimensional linear inhomogeneous fuzzy fractional wave equation.*

$$ {}^{c}D^{\alpha}_{0+}u(t) + u_x = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) + t\cos(x), t > 0, x \in R, 0 < \alpha \leq 1, \tag{2.7}$$

*Subject to the fuzzy initial condition,*

$$u(0) = \tilde{1} = [0.5 + 0.5r, 1.5 - 0.5r], (r - cut, r \in (0, 1], 0 \leq r \leq 1)$$

*The exact solution of (2.7) is,*

$$\begin{aligned} u(x,t) =& t\sin(x) - [\sum_{i=1}^{\infty} \frac{1}{x^i} \frac{t^{\alpha i}}{\Gamma(\alpha_i + 1)}], i = 1, 2 \cdots \\ =& t\sin(x) - E_\alpha \frac{1}{x} t^\alpha \end{aligned}$$

*Where $E_\alpha$ is Mittage-Leffler function.*

*Now, the correction function for (2.7) form as below,*

$$\begin{aligned} \underline{u}_{n+1}(t,x,r) =& \underline{u}_n(t,x,r) + \frac{1}{\Gamma(\alpha+1)} \int_0^t [\lambda(\xi) \frac{d^\alpha}{d\xi^\alpha} \underline{u}_n(\xi,x,r) + \frac{d}{dx} \underline{u}_n(\xi,x,r) \\ &- \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) - t\cos(x)] d\xi^\alpha, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \overline{u}_{n+1}(t,x,r) =& \overline{u}_n(t,x,r) + \frac{1}{\Gamma(\alpha+1)} \int_0^t [\lambda(\xi) \frac{d^\alpha}{d\xi^\alpha} \overline{u}_n(\xi,x,r) + \frac{d}{dx} \overline{u}_n(\xi,x,r) \\ &- \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) - t\cos(x)] d\xi^\alpha, \end{aligned} \tag{2.9}$$

*Where*

$$\frac{d^\alpha}{d\xi^\alpha}[u_n(\xi, x_1, x_2, r)]^r =^c D_{0+}^\alpha [u_n(\xi, x_1, x_2, r)]^r$$

*This yield the stationary condition*

$$\begin{aligned} \lambda'(\xi) = 0, 1 + \underline{\lambda}(\xi) = 0, \overline{\lambda}'(\xi) = 0, 1 + \overline{\lambda}(\xi) = 0 \\ \underline{\lambda}(\xi) = \overline{\lambda}(\xi) = -1 \end{aligned}$$

*Using the Lagrangian multiplier in (2.8) and (2.9), the iteration formula is shown below,*

$$\begin{aligned} \underline{u}_{n+1}(t,x,r) =& \underline{u}_n(t,x,r) - \frac{1}{\Gamma(\alpha+1)} \int_0^t [\frac{d^\alpha}{d\xi^\alpha} \underline{u}_n(\xi,x,r) + \frac{d}{dx} \underline{u}_n(\xi,x,r) \\ &- \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) - t\cos(x)] d\xi^\alpha, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \overline{u}_{n+1}(t,x,r) =& \overline{u}_n(t,x,r) + \frac{1}{\Gamma(\alpha+1)} \int_0^t [\frac{d^\alpha}{d\xi^\alpha} \overline{u}_n(\xi,x,r) + \frac{d}{dx} \overline{u}_n(\xi,x,r) \\ &- \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) - t\cos(x)] d\xi^\alpha, \end{aligned} \tag{2.11}$$

*Beginning with*

$$\underline{u}_0(t,x_1,x_2,r)=(0.5+0.5r),\overline{u}_0(t,x_1,x_2,r)=(1.5-0.5r)$$

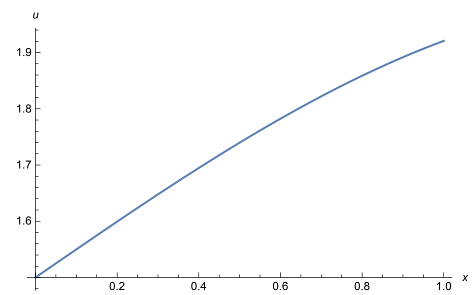
*By the iteration formula (2.10) and (2.11), we get the other components as,*

$$\begin{aligned}
\underline{u}_1(t,x_1,x_2,r) =&(0.5+0.5r)+t\sin(x)+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x),\\
\overline{u}_1(t,x_1,x_2,r) =&(1.5-0.5r)+t\sin(x)+\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x),\\
\underline{u}_2(t,x_1,x_2,r) =&(0.5+0.5r)+t\sin(x)+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x),\\
\overline{u}_2(t,x_1,x_2,r) =&(1.5-0.5r)+t\sin(x)+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x),\\
\underline{u}_3(t,x_1,x_2,r) =&(0.5+0.5r)+t\sin(x)+\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x),\\
\overline{u}_3(t,x_1,x_2,r) =&(1.5-0.5r)+t\sin(x)+\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x),\\
\underline{u}_4(t,x_1,x_2,r) =&(0.5+0.5r)+t\sin(x)+\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\sin(x),\\
\overline{u}_4(t,x_1,x_2,r) =&(1.5-0.5r)+t\sin(x)+\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\sin(x),
\end{aligned}$$

*And so on. The nth approximate fuzzy solution of the method approximates the exact series solution. So, we approximate fuzzy solutions.*

$$[u(t,x)]^r=[\underline{u}_{10}(t,x,r),\overline{u}_{10}(t,x,r)]$$

*Using VIM is plotted for* $\alpha=0.5$.



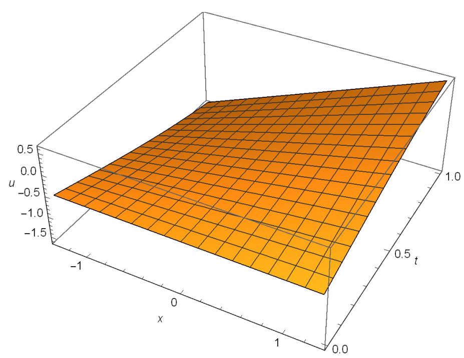




FIGURE 2. **(a)**Exact solution **(b)** Approximate fuzzy solution.

TABLE 3. For approximate solution at $\alpha = 0.5$ and $\underline{u}(t,x,r) \cong \underline{u}_{10}(t,x,r)$.

| | | | $\underline{u}(t,x,r)$ | | | | |
|---|---|---|---|---|---|---|---|
| $\alpha \rightarrow$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | u exact |
| $t\downarrow$ | | | | | | | |
| 0.0 | 0.500000 | 0.550000 | 0.600000 | 0.650000 | 0.700000 | 0.750000 | 0.50000 |
| 0.1 | 0.600000 | 0.650000 | 0.700000 | 0.750000 | 0.800000 | 0.850000 | 0.55000 |
| 0.2 | 0.700000 | 0.750000 | 0.800000 | 0.850000 | 0.900000 | 0.950000 | 0.60000 |
| 0.3 | 0.800001 | 0.850001 | 0.900001 | 0.950001 | 1.000001 | 1.050001 | 0.65000 |
| 0.4 | 0.900006 | 0.950006 | 1.000006 | 1.050006 | 1.100006 | 1.150006 | 0.70000 |
| 0.5 | 1.000022 | 1.050022 | 1.100022 | 1.150022 | 1.200022 | 1.250022 | 0.75000 |

TABLE 4. For approximate solution at $\alpha = 0.5$ and $\overline{u}(t,x,r) \cong \overline{u}_{10}(t,x,r)$.

| | | | $\overline{u}(t,x,r)$ | | | | |
|---|---|---|---|---|---|---|---|
| $\alpha \rightarrow$ | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | u exact |
| $t\downarrow$ | | | | | | | |
| 0.0 | 1.500000 | 1.450000 | 1.400000 | 1.350000 | 1.30000 | 1.250000 | 0.50000 |
| 0.1 | 1.600000 | 1.550000 | 1.500000 | 1.450000 | 1.40000 | 1.350000 | 0.55000 |
| 0.2 | 1.700000 | 1.650000 | 1.600000 | 1.550000 | 1.500000 | 1.450000 | 0.60000 |
| 0.3 | 1.800001 | 1.750001 | 1.700001 | 1.650001 | 1.600001 | 1.550001 | 0.65000 |
| 0.4 | 1.900006 | 1.850006 | 1.800006 | 1.750006 | 1.700006 | 1.650006 | 0.70000 |
| 0.5 | 2.000022 | 1.950022 | 1.900022 | 1.850022 | 1.800022 | 1.750022 | 0.75000 |

**Example 3.** *Consider the following two dimensional linear inhomogeneous fuzzy fractional wave equation.*

$$\frac{d^\alpha}{dt^\alpha}u(t,x_1,x_2) = \frac{d^2}{dx_1^2}u(t,x_1,x_2) + \frac{d^2}{dx_2^2}u(t,x_1,x_2) - 2u(t,x_1,x_2),$$
$$1 < \alpha \leq 2, t > 0, x_1; x_2 \in R^2 \tag{2.12}$$

*Subject to the fuzzy initial condition,*

$$u(0) = [r-1, r+1]$$

*Now the correction functional for (2.12) form as below,*

$$\begin{aligned}\underline{u}_{n+1}(t,x_1,x_2,r) =& \underline{u}_n(t,x_1,x_2,r) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(\xi)[\frac{d^\alpha}{d\xi^\alpha}\underline{u}_n(\xi,x_1,x_2,r) \\ &- \frac{d^2}{dx_1^2}\underline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\underline{u}_n(\xi,x_1,x_2,r) \\ &+ 2\underline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha, \end{aligned} \tag{2.13}$$

$$\begin{aligned}\overline{u}_{n+1}(t,x_1,x_2,r) =&\overline{u}_n(t,x_1,x_2,r) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(\xi)[\frac{d^\alpha}{d\xi^\alpha}\overline{u}_n(\xi,x_1,x_2,r)\\ &- \frac{d^2}{dx_1^2}\overline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\overline{u}_n(\xi,x_1,x_2,r)\\ &+ 2\overline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha,\end{aligned} \tag{2.14}$$

*Where*

$$\frac{d^\alpha}{d\xi^\alpha}[u_n(\xi,x_1,x_2,r)]^r = ^c D_0^\alpha[u_n(\xi,x_1,x_2,r)]^r$$

*This yield the stationary condition*

$$\lambda'(\xi) = 0, 1 + \underline{\lambda}(\xi) = 0, \overline{\lambda}'(\xi) = 0, 1 + \overline{\lambda}(\xi) = 0$$

$$\underline{\lambda}(\xi) = \overline{\lambda}(\xi) = -1$$

*Using the Lagrangian multiplier in (2.13) and (2.14), the iteration formula is shown below,*

$$\begin{aligned}\underline{u}_{n+1}(t,x_1,x_2,r) =&\underline{u}_n(t,x_1,x_2,r) - \frac{1}{\Gamma(\alpha+1)}\int_0^t [\frac{d^\alpha}{d\xi^\alpha}\underline{u}_n(\xi,x_1,x_2,r)\\ &- \frac{d^2}{dx_1^2}\underline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\underline{u}_n(\xi,x_1,x_2,r)\\ &+ 2\underline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha,\end{aligned} \tag{2.15}$$

$$\begin{aligned}\overline{u}_{n+1}(t,x_1,x_2,r) =&\overline{u}_n(t,x_1,x_2,r) - \frac{1}{\Gamma(\alpha+1)}\int_0^t [\frac{d^\alpha}{d\xi^\alpha}\overline{u}_n(\xi,x_1,x_2,r)\\ &- \frac{d^2}{dx_1^2}\overline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\overline{u}_n(\xi,x_1,x_2,r)\\ &+ 2\overline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha,\end{aligned} \tag{2.16}$$

*Beginning with*

$$\underline{u}_0(t,x_1,x_2,r) = (r-1)\frac{t^\alpha}{\Gamma(\alpha+1)}, \overline{u}_0(t,x_1,x_2,r) = (1-r)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

*By the iteration formula (2.15) and (2.16), we get the other components as,*

$$\begin{aligned}\underline{u}_1(t,x_1,x_2,r) =& -2(r-1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \overline{u}_1(t,x_1,x_2,r) = -2(1-r)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\ \underline{u}_2(t,x_1,x_2,r) =&4(r-1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \overline{u}_2(t,x_1,x_2,r) = 4(1-r)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\ \underline{u}_3(t,x_1,x_2,r) =& -8(r-1)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \overline{u}_3(t,x_1,x_2,r) = -8(1-r)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},\end{aligned}$$

$$\underline{u}_4(t,x_1,x_2,r) = 16(r-1)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}, \overline{u}_4(t,x_1,x_2,r) = 16(1-r)\frac{t^{5\alpha}}{\Gamma(5\alpha+1)},$$
$$\vdots$$

*And so on. The nth approximate fuzzy solution of the method approximates the exact series solution. So, we approximate fuzzy solutions.*

**Example 4.** *Consider the following two dimensional linear inhomogeneous fuzzy fractional wave equation.*

$$\frac{d^\alpha}{d\xi^\alpha}u(t,x_1,x_2) = \frac{d^2}{dx_1^2}u(t,x_1,x_2) + \frac{d^2}{dx_2^2}u(t,x_1,x_2) - 2u(t,x_1,x_2),$$
$$1 < \alpha \leq 2, t > 0, x_1; x_2 \in R^2 \tag{2.17}$$

*Subject to the fuzzy initial condition,*

$$u(0) = \tilde{1} = [0.5 + 0.5r, 1.5 - 0.5r], (r - cut, r \in (0,1], 0 \leq 1)$$

*Now the correction function for (2.17) form as below,*

$$\begin{aligned}\underline{u}_{n+1}(t,x_1,x_2,r) =& \underline{u}_n(t,x_1,x_2,r) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(\xi)[\frac{d^\alpha}{d\xi^\alpha}\underline{u}_n(\xi,x_1,x_2,r) \\ &- \frac{d^2}{dx_1^2}\underline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\underline{u}_n(\xi,x_1,x_2,r) \\ &+ 2\underline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\overline{u}_{n+1}(t,x_1,x_2,r) =& \overline{u}_n(t,x_1,x_2,r) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(\xi)[\frac{d^\alpha}{d\xi^\alpha}\overline{u}_n(\xi,x_1,x_2,r) \\ &- \frac{d^2}{dx_1^2}\overline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\overline{u}_n(\xi,x_1,x_2,r) \\ &+ 2\overline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha,\end{aligned} \tag{2.19}$$

*Where*

$$\frac{d^\alpha}{d\xi^\alpha}[u_n(\xi,x_1,x_2,r)]^r = ^c D_0^\alpha[u_n(\xi,x_1,x_2,r)]^r$$

*This yield the stationary condition*

$$\lambda'(\xi) = 0, 1 + \underline{\lambda}(\xi) = 0, \overline{\lambda}'(\xi) = 0, 1 + \overline{\lambda}(\xi) = 0$$
$$\underline{\lambda}(\xi) = \overline{\lambda}(\xi) = -1$$

*Using the Lagrangian multiplier in (2.18) and (2.19), the iteration formula is shown below,*

$$\begin{aligned}\underline{u}_{n+1}(t,x_1,x_2,r) =&\underline{u}_n(t,x_1,x_2,r) - \frac{1}{\Gamma(\alpha+1)}\int_0^t [\frac{d^\alpha}{d\xi^\alpha}\underline{u}_n(\xi,x_1,x_2,r)\\ &- \frac{d^2}{dx_1^2}\underline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\underline{u}_n(\xi,x_1,x_2,r)\\ &+ 2\underline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\overline{u}_{n+1}(t,x_1,x_2,r) =&\overline{u}_n(t,x_1,x_2,r) - \frac{1}{\Gamma(\alpha+1)}\int_0^t [\frac{d^\alpha}{d\xi^\alpha}\overline{u}_n(\xi,x_1,x_2,r)\\ &- \frac{d^2}{dx_1^2}\overline{u}_n(\xi,x_1,x_2,r) - \frac{d^2}{dx_2^2}\overline{u}_n(\xi,x_1,x_2,r)\\ &+ 2\overline{u}_n(\xi,x_1,x_2,r)]d\xi^\alpha,\end{aligned} \tag{2.21}$$

*Beginning with*

$$\underline{u}_0(t,x_1,x_2,r) = (r-1)\frac{t^\alpha}{\Gamma(\alpha+1)}, \overline{u}_0(t,x_1,x_2,r) = (1-r)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

*By the iteration formula (2.20) and (2.21), we get the other components as,*

$$\begin{aligned}\underline{u}_1(t,x_1,x_2,r) =&(0.5+0.5r)(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}),\\ \overline{u}_1(t,x_1,x_2,r) =&(1.5-0.5r)(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}),\\ \underline{u}_2(t,x_1,x_2,r) =&(0.5+0.5r)(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}),\\ \overline{u}_2(t,x_1,x_2,r) =&(1.5-0.5r)(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}),\\ \underline{u}_3(t,x_1,x_2,r) =&(0.5+0.5r)(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{8t^{3\alpha}}{\Gamma(3\alpha+1)}),\\ \overline{u}_3(t,x_1,x_2,r) =&(1.5-0.5r)(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{8t^{3\alpha}}{\Gamma(3\alpha+1)}),\\ \underline{u}_4(t,x_1,x_2,r) =&(0.5+0.5r)(1-\frac{2t^\alpha}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{8t^{3\alpha}}{\Gamma(3\alpha+1)}\\ &+\frac{16t^{4\alpha}}{\Gamma(4\alpha+1)}),\end{aligned}$$

$$\overline{u}_4(t,x_1,x_2,r) =(1.5-0.5r)(1-\frac{2t^{\alpha}}{\Gamma(\alpha+1)}+\frac{4t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{8t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{16t^{4\alpha}}{\Gamma(4\alpha+1)}),$$

$$\vdots$$

*And so on. The nth approximate fuzzy solution of the method approximates the exact series solution. So, we approximate fuzzy solutions.*

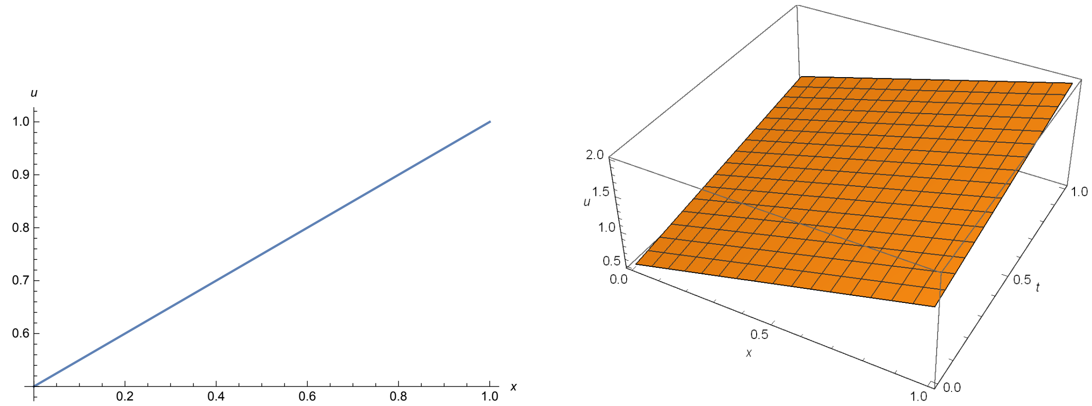



FIGURE 3. **(a)**Exact solution **(b)** Approximate fuzzy solution.

TABLE 5. For approximate solution at $\alpha$=0.5 and $\underline{u}(t,x_1,x_2,r) \cong \underline{u}_2(t,x_1,x_2,r)$.

| | | | | $\underline{u}(t,x_1,x_2,r)$ | | | |
|---|---|---|---|---|---|---|---|
| $\alpha \rightarrow$ | *0.0* | *0.1* | *0.2* | *0.3* | *0.4* | *0.5* | *u exact* |
| $t\downarrow$ 0.0 | *0.500000* | *0.550000* | *0.600000* | *0.650000* | *0.700000* | *0.750000* | *0.50000* |
| *0.1* | *0.343175* | *0.377493* | *0.411810* | *0.446128* | *0.480445* | *0.514763* | *0.55000* |
| *0.2* | *0.395373* | *0.434911* | *0.474448* | *0.513986* | *0.553523* | *0.59306* | *0.60000* |
| *0.3* | *0.481961* | *0.530157* | *0.578354* | *0.62655* | *0.674746* | *0.722942* | *0.65000* |
| *0.4* | *0.586350* | *0.644985* | *0.703620* | *0.762255* | *0.820890* | *0.879526* | *0.70000* |
| *0.5* | *0.702115* | *0.772327* | *0.842539* | *0.912750* | *0.982962* | *1.053170* | *0.75000* |

TABLE 6. For approximate solution at $\alpha = 0.5$ and $\overline{u}(t, x_1, x2, r) \cong \overline{u}_2(t, x, r)$.

| | | | | $\overline{u}(t, x_1, x_2, r)$ | | | |
|---|---|---|---|---|---|---|---|
| $\alpha \rightarrow$ | *0.0* | *0.1* | *0.2* | *0.3* | *0.4* | *0.5* | *u exact* |
| $t\downarrow$ | | | | | | | |
| *0.0* | *1.50000* | *1.450000* | *1.40000* | *1.350000* | *1.300000* | *1.250000* | *0.50000* |
| *0.1* | *1.02953* | *0.995208* | *0.96089* | *0.926573* | *0.892255* | *0.857938* | *0.55000* |
| *0.2* | *1.18612* | *1.146580* | *1.10705* | *1.067510* | *1.027970* | *0.988434* | *0.60000* |
| *0.3* | *1.44588* | *1.397690* | *1.34949* | *1.301300* | *1.253100* | *1.204900* | *0.65000* |
| *0.4* | *1.75905* | *1.700420* | *1.64178* | *1.583150* | *1.524510* | *1.465880* | *0.70000* |
| *0.5* | *2.10635* | *2.036130* | *1.96592* | *1.895710* | *1.825500* | *1.755290* | *0.75000* |

## 3. INTERPRETATION OF GRAPH

In example (1) taking equation (2.6) i.e $\underline{u}_2(t, x_1, x_2, r)$ and $\overline{u}_2(t, x_1, x_2, r)$ which is approximately equal to $u(t, x_1, x_2, r)$ and varying $\alpha$ and t for $\underline{u}_2(t, x_1, x_2, r)$ and $\overline{u}_2(t, x_1, x_2, r)$ which is given in table (1) (2) and comparison of exact and approximation solution is interpreted by Figure (1). Similarly, in example (2) we get approximation in 10th iteration and in example (4) we get approximation in 2nd iteration.

## 4. CONCLUSION

The article consists of, VIM is explored to calculate the fractional order fuzzy wave equations. The result we obtained is accomplished by the VIM is in infinite series form, which can be articulated by an inherent form with proper fuzzy IC, the final result is shown by graphically, how approximation the method is.

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