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ENHANCED SEMI-ANALYTIC METHOD FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT. In this paper, we propose a new semi-analytic approach based on the generalized Taylor series for solving nonlinear differential equations of fractional order. Assuming the solution is expanded as the generalized Taylor series, the coefficients of the series can be computed by solving the corresponding recursive relation of the coefficients which is generated by the given problem. This method is called the generalized differential transform method(GDTM). In several literatures the standard GDTM was applied in each sub-domain to obtain an accurate approximation. As noticed in [19], however, a direct application of the GDTM in each sub-domain loses a term of memory which causes an inaccurate approximation. In this work, we derive a new recursive relation of the coefficients that reflects an effect of memory. Several illustrative examples are demonstrated to show the effectiveness of the proposed method. It is shown that the proposed method is robust and accurate for solving nonlinear differential equations of fractional order.

1. INTRODUCTION

The beginning of fractional calculus is considered to be the by Leibniz's letter to L'Hospital in 1695, where the notation for differentiation of non-integer order $1/2$ is discussed. Since then, for the three centuries, this topic has been investigated by the pure mathematicians. Because of the non-local property of the fractional derivative, many researchers found out that a mathematical model with fractional calculus is better than the integer order to describe many phenomena which involve the effect of memory. Recently, the fractional calculus has been playing more important roles in many science and engineering fields[1, 2, 3, 4]. A number of numerical methods for solving differential equations of the fractional have been

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introduced. Authors in [5] presented the predictor-corrector approach based on the Adams-Bashforth-Moulton type numerical method that has been successful to obtain the stable approximations for solving many fractional differential equations. In [6], authors proposed a high order method by employing a block-by-block approach which is common method for solving the integral equations. Some of semi-analytic methods such as the Adomian decomposition method(ADM)[7, 8], homotopy analysis method(HAM) [9, 10], homotopy perturbation method(HPM)[11, 12], variational iteration method(VIM)[13, 14, 15] and generalized differential transform method(GDTM)[16, 17, 18, 19, 20, 21] have been introduced to provide analytic or numeric approximations.

In this paper, we focus on the generalized differential transform method which is based on the generalized Taylor series. From the given fractional differential equation, the GDTM provides a simple recurrence relation of the generalized Taylor series' coefficients of the solution. However, the approximate solution by the GDTM has a limitation of accuracy due to the local convergence property of the generalized Talyor series. To overcome this limitation the standard GDTM has been applied in each sub-domain [22, 23, 24]. As presented in [19], a direct application of the GDTM in each sub-domain with a fixed initial condtion does not impose an effect of memory which is the main property of differential operator of fractional order. Thus it causes an increasing error as the fractional order is decreasing. In this work, we provide the new recurrence relations that contains the effect of memory. The paper is organized as follows. Section 2 introduces some preliminary results for the fractional calculus that we shall use. In Section 3, we present the basic ideas and some properties of GDTM. In Section 4, the new recurrence relation of complex nonlinear functions are introduced. In Section 5, numerical results of several examples are demonstrated by using new recurrence relations and are compared with the ones obtained by another numerical method. Finally, we give a conclusion in Section 6.

2. Fractional calculus

In this section we introduce some basic definitions and properties of the fractional integration and differentiation. There are two main definitions of fractional calculus: Riemann-Liouville's and Caputo's definition. We adopt the Caputo's definition in this work.

Definition 2.1. *A real valued function $f(t)$, $t > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there etists a real number p, $p < \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$ with of a function $f(t) \in C_\mu$, $t > 0, \mu \geq -1$ is defined by*

$$J_a^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t (t-s)^{\alpha-1} f(s)ds, & a \geq 0, x > a, \\ f(t), \quad \alpha = 0. \end{cases}$$

The operator J_a^α satisfies the following properties: For $f(t) \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$,

1. $J_a^\alpha J_a^\beta f(t) = J^\beta J_a^\alpha f(t) = J_a^{\alpha+\beta} f(t)$,
2. $J_a^\alpha (t-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}$.

Definition 2.3. *The fractional derivative in the Caputo sense of* $f(t)$ [1], $f(t) \in C_{-1}^m, m \in \mathbb{N}, t > 0$ *is defined by*

$$D_a^\alpha f(t) = \begin{cases} J_a^{m-\alpha}\left(\dfrac{d^m}{dt^m} f(t)\right), & m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m. \end{cases}$$

For $\alpha > 0, \gamma > -1$ and constant C, Caputo fractional derivative has some basic properites as follows:

1. $D_a^\alpha C = 0$,
2. $D_a^\alpha (t-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}(t-a)^{\gamma-\alpha}$.

3. GENERALIZED DIFFERENTIAL TRANSFORM METHOD

In this section we describe the basic concept, definitions and some properties of the generalized differential transform method(GDTM) For the fractional differential operator D_a^α, $m-1 < \alpha \le m$, in the sense of Caputo, let us define $(D_a^\alpha)^n$ by

$$(D_a^\alpha)^n = D_a^\alpha \cdot D_a^\alpha \cdots D_a^\alpha, \text{(n-times)}.$$

Theorem 3.1. *(Generalized Taylor's Formula) Suppose that* $(D_a^\alpha)^k f(t) \in C(a,b]$ *for* $k = 0, 1, ..., n+1$, *where* $0 < \alpha \le 1$, *then we have* [17]

$$f(t) = \sum_{i=0}^{n} \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_a^\alpha)^i f)(a) + \frac{((D_a^\alpha)^{n+1} f)(\eta)}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}$$

with $0 \le \eta \le t$, $\forall t \in (a,b]$

For an analytic function $f(t)$ let us define the generalized differential transform(GDT) of the kth derivative as follows:

$$F(k) = \frac{1}{\Gamma(\alpha k+1)}[(D_a^\alpha)^k f(t)]_{t=0},$$

where $0 < \alpha \le 1, k = 0, 1, 2, \cdots$., and the generalized differential inverse transform of $F(k)$ is defined as follows:

$$f(t) = \sum_{k=0}^{\infty} F(k)(t-a)^{\alpha k}.$$

In case of $\alpha = 1$, then the GDT reduces to the classical differential transform. Several fundamental properties of the GDT are listed in [22, 24]

3.1. **Generalized differential transforms for nonlinear functions.** In this section we describe how to construct the generalized differential transforms $F(k)$ for several nonlinear functions [20].

Let us consider $f(y(t)) = e^{ay(t)}$, where a is a constant. Taking the fractional derivative D^α to $f(y(t))$ gives

$$D^\alpha f(y(t)) = J^{1-\alpha}\left(\frac{d}{dt}f(y(t))\right) = J^{1-\alpha}\left(f'(y(t))y'(t)\right) = J^{1-\alpha}\left(af(y(t))y'(t)\right).$$

Theorem 3.2. *Suppose that $f(y(t)) = \sum_{k=0}^\infty F(k)t^{\alpha k}$ and $y(t) = \sum_{k=0}^\infty Y(k)t^{\alpha k}$. Then we have*

$$F(k) = \begin{cases} e^{aY(0)}, \quad k = 0, \\ \dfrac{a}{k}\displaystyle\sum_{r=0}^{k-1}(k-r)F(r)Y(k-r), \quad k \geq 1. \end{cases}$$

Now we consider the case of logarithmic nonlinearity. Let $f(y(t)) = \ln(a + by(t)$, where$a$ and b are constants.

Theorem 3.3. *Suppose that $f(y(t)) = \sum_{k=0}^\infty F(k)t^{\alpha k}$ and $y(t) = \sum_{k=0}^\infty Y(k)t^{\alpha k}$. Then we have*

$$F(k) = \begin{cases} \ln(a + bY(0)), \quad k = 0, \\ \dfrac{1}{k}\displaystyle\sum_{r=0}^{k-1}(k-r)G(r)Y(k-r), \quad k \geq 1, \end{cases}$$

where

$$G(k) = \begin{cases} \dfrac{b}{a + bY(0)}, \quad k = 0, \\ -\dfrac{b}{a + bY(0)}\displaystyle\sum_{r=0}^{k-1}G(r)Y(k-r), \quad k \geq 1. \end{cases}$$

Let us consider $f(y(t)) = \sin(ay(t))$ and $g(f(t)) = \cos(ay(t))$, where a is a constant.

Theorem 3.4. *Suppose that $f(y(t)) = \sum_{k=0}^\infty F(k)t^{\alpha k}$, $g(y(t)) = \sum_{k=0}^\infty G(k)t^{\alpha k}$ and $y(t) = \sum_{k=0}^\infty Y(k)t^{\alpha k}$. Then we have*

$$F(k) = \begin{cases} \sin(aY(0)), \quad k = 0, \\ \dfrac{a}{k}\displaystyle\sum_{r=0}^{k-1}(k-r)G(r)Y(k-r), \quad k \geq 1, \end{cases}$$

$$G(k) = \begin{cases} \cos(aY(0)), \quad k = 0, \\ -\dfrac{a}{k}\displaystyle\sum_{r=0}^{k-1}(k-r)F(r)Y(k-r), \quad k \geq 1. \end{cases}$$

Theorem 3.5. *For the hyperbolic functions $f(y(t)) = \sinh(ay(t))$ and $g(ay(t)) = \cosh(ay(t))$, where a is a consider, suppose that $f(y(t)) = \sum_{k=0}^{\infty} F(k)t^{\alpha k}$, $g(y(t)) = \sum_{k=0}^{\infty} G(k)t^{\alpha k}$ and $y(t) = \sum_{k=0}^{\infty} Y(k)t^{\alpha k}$. Then we have*

$$F(k) = \begin{cases} \sinh(aY(0)), & k = 0, \\ \dfrac{a}{k}\displaystyle\sum_{r=0}^{k-1}(k-r)G(r)Y(k-r), & k \geq 1, \end{cases}$$

$$G(k) = \begin{cases} \cosh(aY(0)), & k = 0, \\ \dfrac{a}{k}\displaystyle\sum_{r=0}^{k-1}(k-r)F(r)Y(k-r), & k \geq 1. \end{cases}$$

3.2. **Enhanced multistage generalized differential transform method.** Let us consider the following initial value problem of fractional order:

$$D^{\alpha}y(t) = f(t, y(t)), \quad t > 0, \quad y(0) = y_0. \tag{3.1}$$

Applying the GDTM with $y(t) = \sum_{k=0}^{\infty} Y(k)t^{\alpha k}$ and $f(t) = \sum_{k=0}^{\infty} F(k)t^{\alpha k}$ we have the recursive relation as follows

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}Y(k+1) = F(k), \quad k = 0, 1, \cdots. \tag{3.2}$$

All GDTs $Y(k)$ are given by solving the above recursive relation (3.2) with $Y(0) = y_0$. GDTM is based on the generalized Taylor series in Theorem 3.1. Thus it is clear that an accurate approximation can be obtained near $t = 0$ if the series is expanded at $t = 0$. In order to obtain a reliable approximation in a large domain $\Omega = (0, T)$, the GDTM can be applied in each subdomain $\Omega_i = (t_i, t_{i+1})$ where $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T, t_i = i \times h$ and $h = T/N$. Suppose that GDTM is applied to the problem (3.1) on Ω_i with an initial condition $y(t_i)$, that is,

$$D^{\alpha}y(t) = f(t, y(t)), \quad t \in \Omega_i. \tag{3.3}$$

Then we have the analytic solution $y(t)$ by taking the Riemann-Louville integral operator to (3.3)

$$y(t) = y(t_i) + \frac{1}{\Gamma(\alpha)}\int_{t_i}^{t}(t-\tau)^{\alpha-1}f(\tau, y(\tau))d\tau. \tag{3.4}$$

In the similar way, we have the analytic solution $y(t)$ from (3.1)

$$y(t) = y(t_0) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau, y(\tau))d\tau.$$

Then it can be rewritten as

$$
\begin{aligned}
y(t) &= y(t_i) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t} (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \\
&+ \sum_{j=0}^{i-1} \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} \left\{ (t-\tau)^{\alpha-1} - (t_i-\tau)^{\alpha-1} \right\} f(\tau, y(\tau)) d\tau.
\end{aligned} \tag{3.5}
$$

Comparing (3.4) with (3.5) it is clear that the analytic solution for the subdomain problem (3.3) loses some information which was called as *memory*. Authors in [19] proposed the new method in evaluating an initial condition $y(t_{i+1})$ by using the piecewise linear interpolation of $f(t)$

$$
y(t_{i+1}) \approx y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{k=0}^{i+1} a_{k,i+1} f(t_k, s_{k,n_k}(t_k)), \tag{3.6}
$$

where $s_{k,n_k}(t)$ is the partial sum of $y(t)$ in (3.4), $s_{k,n_k}(t) = \sum_{i=0}^{n_k} Y(i)(t-t_k)^{\alpha i}$, and

$$
a_{k,i+1} = \begin{cases} i^{\alpha+1} - (i-\alpha)(i+1)^\alpha, & \text{if } k = 0, \\ (i-k+2)^{\alpha+1} + (i-k)^{\alpha+1} - 2(i-k+1)^{\alpha+1}, & \text{if } 1 \le k \le i, \\ 1, & \text{if } k = i+1. \end{cases}
$$

As seen in (3.6) all values $y(t_{i+1})$ are approximated by using the partial sum of GDTM in each subdomain Ω_i and numerical calculation of integration. In this work it is motivated to obtain a new recursive relation of the GDTs which contain the memory terms in (3.5). Let us first introduce the following calculation which is useful to derive the new recursive relation.

Lemma 3.6. *For $j = 0, 1, \cdots, i-1$, we have*

$$
\int_{t_j}^{t_{j+1}} (t_i-\tau)^{\alpha-1} (\tau-t_j)^{\alpha k} d\tau = h^{\alpha(k+1)} (i-j)^{\alpha(k+1)} B(\frac{1}{i-j}; \alpha k+1, \alpha),
$$

where $B(x; a, b)$ is the incomplete beta function

$$
B(x; , a, b) = \int_0^x t^{a-1}(1-t)^{b-1} dt.
$$

Proof. Using a simple change of variable we have

$$
\begin{aligned}
\int_{t_j}^{t_{j+1}} (t_i-\tau)^{\alpha-1}(\tau-t_j)^{\alpha k} d\tau &= h^{\alpha(k+1)} \int_0^1 (i-j-s)^{\alpha-1} s^{\alpha k} ds \\
&= h^{\alpha(k+1)} (i-j)^{\alpha-1} \int_0^1 (1 - \frac{s}{i-j})^{\alpha-1} s^{\alpha k} ds
\end{aligned}
$$

Then the proof is completed by the change of variable with $\dfrac{s}{i-j} = t$ and the definition incomplete beta function $B(x; a, b)$. □

Theorem 3.7. *Suppose that $y(t)$ is the solution of (3.1). Let $y_i(t) = \sum_{k=0}^{\infty} Y_i(k)(t-t_i)^{\alpha k}$ and $f_j(t, y(t)) = \sum_{k=0}^{\infty} F_j(k)(t-t_j)^{\alpha k}, j = 0, 1, \cdots, i$. Then we have the following recursive relation with $Y_i(k)$ and $F_i(k)$*

$$Y_i(k+1) = \begin{cases} \dfrac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} F_i(k), \quad i = 0, \\ \dfrac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}(F_i(k) - F_{i-1}(k)) + \dfrac{2^{\alpha(k+1)} B(\frac{1}{2}; \alpha k+1, \alpha)}{\Gamma(\alpha)} F_{i-1}(k), \quad i = 1, \\ \dfrac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}(F_i(k) - F_{i-1}(k)) + \dfrac{2^{\alpha(k+1)} B(\frac{1}{2}; \alpha k+1, \alpha)}{\Gamma(\alpha)} F_{i-1}(k) \\ + \displaystyle\sum_{j=0}^{i-2} \frac{F_j(k)}{\Gamma(\alpha)} \Big[(i+1-j)^{\alpha(k+1)} B(\frac{1}{i+1-j}; \alpha k+1, \alpha) \\ -(i-j)^{\alpha(k+1)} B(\frac{1}{i-j}; \alpha k+1, \alpha) \Big], \quad i \geq 2, \end{cases}$$

where $B(x; a, b)$ is the incomplete beta function.

Proof. For $i \geq 1$, the value of $y(t_{i+1})$ can be rewritten by

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau \\ &+ \sum_{j=0}^{i-1} \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} \left\{ (t_{i+1} - \tau)^{\alpha-1} - (t_i - \tau)^{\alpha-1} \right\} f(\tau, y(\tau)) d\tau. \end{aligned} \tag{3.7}$$

Substituting the generalized Taylor series $y(t) = \sum_{k=0}^{\infty} Y_i(k)(t-t_i)^{\alpha k}$, $f(t, y(t)) = \sum_{k=0}^{\infty} F_j(k)(t-t_j)^{\alpha k}$ into (3.7) we have

$$\begin{aligned} \sum_{k=0}^{\infty} Y_i(k) h^{\alpha k} &= y(t_i) + \sum_{k=0}^{\infty} F_i(k) \left\{ \frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - \tau)^{\alpha-1} (\tau - t_i)^{\alpha k} d\tau \right\} \\ &+ \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{i-1} \frac{F_j(k)}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} \left\{ (t_{i+1} - \tau)^{\alpha-1} - (t_i - \tau)^{\alpha-1} \right\} (\tau - t_j)^{\alpha k} d\tau \right\}. \end{aligned} \tag{3.8}$$

Now we evaluate the integrals of the right hand side of (3.8). The property of the Riemann-Liouville integral operator J^α gives the exact formulation of the first integral in (3.8):

$$\frac{1}{\Gamma(\alpha)} \int_{t_i}^{t_{i+1}} (t_{i+1} - \tau)^{\alpha-1} (\tau - t_i)^{\alpha k} d\tau = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} h^{\alpha(k+1)}. \tag{3.9}$$

Applying Lemma 3.6, the second integral of the right hand side of (3.8) can be written by

$$\begin{aligned}&\int_{t_j}^{t_{j+1}} \left\{(t_{i+1}-\tau)^{\alpha-1}-(t_i-\tau)^{\alpha-1}\right\}(\tau-t_j)^{\alpha k}d\tau\\ &=\Big[(i+1-j)^{\alpha(k+1)}B(\frac{1}{i+1-j};\alpha k+1,\alpha)\\ &\quad -(i-j)^{\alpha(k+1)}B(\frac{1}{i-j};\alpha k+1,\alpha)\Big]h^{\alpha(k+1)}.\end{aligned} \tag{3.10}$$

For $j=i-1$, the incomplete beta function can be evaluated by

$$B(\frac{1}{i-j};\alpha k+1,\alpha)=\frac{\Gamma(\alpha)\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}. \tag{3.11}$$

Substituting $(3.9), (3.10)$ and (3.11) into (3.8) combined with $y(t_i)=Y_i(0)$ and collecting the coefficients of $h^{\alpha(k+1)}$ the proof is completed. □

3.3. **Approximation of generalized differential transforms.** In the previous sections we discussed how to construct the generalized differential transform for several nonlinear forms and the recursive relations of the generalized differential transforms in multistage approach. In the MsGDTM, it is the most important to construct the corresponding recursive relations. For the given following fractional differential equation

$$D_{t_i}^{\alpha}y_i(t)=f_i(t,y_i(t)),\quad t\in\Omega_i=(t_i,t_{i+1})\quad i\geq 0,$$

it is easy to construct the recursive relation

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}Y_i(k+1)=F_i(k),\quad k=0,1,2,\cdots,$$

where $y_i(t)=\sum_{k=0}^{\infty}Y_i(k)(t-t_i)^{\alpha k}$ and $f_i(t)=\sum_{k=0}^{\infty}F_i(k)(t-t_i)^{\alpha k}$. The above recursive relation can be easily solved if $F_i(k)$ are known. Suppose that $f_i(t,y_i(t))=f_{i,1}(y_i(t))+f_{i,2}(t)$. For the nonlinear function $f_{i,1}(y_i(t))$, we discussed how to construct the corresponding generalized differential transform. In fractional calculus, however, it is usually difficult to find the fractional derivative for any function. Since the GDTs of $f_{i,2}(t)$ are obtained by the generalized Taylor series

$$F_{i,2}(k)=\frac{1}{\Gamma(\alpha k+1)}[(D^{\alpha})^k f_{i,2}(t)]_{t=t_i},$$

it is not easy to find the GDTs for the $f_{i,2}(t)$. To overcome this difficulty we propose the following method to find approximations of $F_{i,2}(k)$. Suppose that $f_{i,2}(t)$ can be written by the generalized Taylor series on $\Omega_i=(t_i,t_{i+1})$

$$f_{i,2}(t)=\sum_{k=0}^{\infty}F_{i,2}(k)(t-t_i)^{\alpha k}.$$

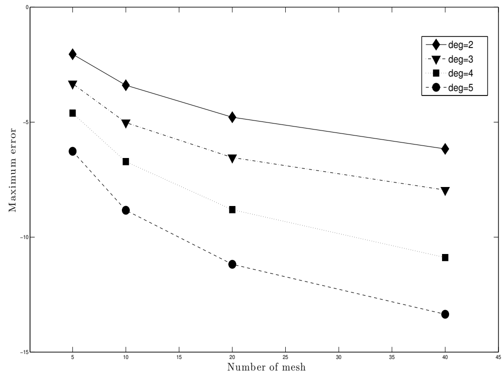


FIGURE 1. Comparisons of maximum norm error between the exact and the approximations by Taylor collocation approach; $f_{i,2}(t) = \sin(2t)$ with the degree $n = 2, 3, 4$ and 5 in (3.12).

In order to approximate $f_{i,2}(t)$ with the $n-$term partial sum we consider

$$f_{i,2}(t) \approx \sum_{k=0}^{n} F_{i,2}(k)(t-t_i)^{\alpha k} \equiv s_{i,n}(t). \tag{3.12}$$

Since $f_{i,2}(t_i) = F_{i,2}(0)$, there are n unknowns $F_{i,2}(k)$, $k = 1, \cdots, n$. To determine the $F_{i,2}(k)$ we consider the Taylor collocation approach. For the appropriate collocation points t_{i_j}, $j = 1, \cdots, n$, we set $f_{i,2}(t_{i_j}) = \sum_{k=0}^{n} F_{i,2}(k)(t_{i_j} - t_i)^{\alpha k}$. Then we have the system $AF = B$, where $a_{p,q} = (t_{i_p} - t_i)^{q\alpha}, b_p = f_{i,2}(t_{i_p})$ and $F^T = (F_{i,2}(1), \cdots, F_{i,2}(n))$. In Fig. 1, the maximum norm errors between the exact $f_{i,2}(t) = \sin(2t)$ and the approximations by the Taylor collocation approach with a number of mesh$(5, 10, 20$ and $40)$ at a degree $\alpha = 0.5$. Here, the collocation points are selected uniformly in each sub interval $\bar{\Omega}_i$. It is shown that the more accurate approximations are obtained when the degree is increasing and the size of mesh is decreasing.

Remark There are many numerical methods to determine the GDTs $F_{i,2}(k)$ such as the least square. However, we employ the uniform collation method in each sub interval because of a simple computational work.

4. NUMERICAL ILLUSTRATIONS

In this section we demonstrate numerical results of several nonlinear fractional differential equations by using the proposed recurrence relations. To show the effectiveness of the proposed method we also present the numerical results obtained by several methods such as the fractional Adams-Bashforth-Moulton method(FABM)[4], the standard multistage generalized differential transform method(SGDTM)[22, 23], the standard multistage generalized differential transform method with memory(SGDTM-M)[19], the enhanced multistage generalized differential transform method (EGDTM) which is given by Theorem 3.7 and the enhanced multistage generalized differential transform method with memory(EGDTM-M). The method with memory means that the initial condition $y(t_{i+1})$ is approximated by (3.6) and the partial sum $s_{k,n_k}(t)$ is obtained by the (standard or enhanced) GDTM. For all numerical examples, the degree of partial sum in all GDTMs is set by $n = 5$. That is, $y_i(t) \approx \sum_{k=0}^{5} Y_i(k)(t - t_i)^{\alpha k}$.

Example 1 Consider the following nonlinear fractional differential equation[10]:

$$D^\alpha y(t) + \lambda_1 y(t) + \lambda_2 y^2(t) = \sin(4t) - \sin(t), \quad t > 0, \tag{4.1}$$

where $0 < \alpha \leq 1$ and $\lambda_i, i = 1, 2$, subject to the initial condition $y(0) = 5.0$.

Applying the properties of GDTM to (4.1), we have the follow recurrence relation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} Y(k+1) + \lambda_1 Y(k) + \lambda_2 \sum_{r=0}^{k} Y(r)Y(k-r) = F(k), \quad k \geq 0,$$

where $F(k)$ is the GDT of $f(t) = \sin(4t) - \sin(t)$. Here the constants λ_i are set by $\lambda_1 = 0.25, \lambda_2 = -0.05$. The time step is chosen by $h = 10^{-5}$ in FABM and $h = 10^{-1}$ in all GDTMs, respectively. In each subdomian, $F(k)$ is approximated by using the collocation method in the previous section.

The comparisons of numerical results are shown in Figure 2. For the large fractional orders $\alpha = 0.9, 0.7$, the results obtained by most GDTMs exept SGDTM are nearly the same and close to the result by FABM. As α is getting smaller, however, numerical approximations by SGDTM and SGDTM-M blow up in a short time. The EGDTM-M gives almost the same with the result by FABM.

Example 2 Consider the following nonlinear fractional differential equation[18]:

$$D^\alpha y(t) = e^{-y(t)} + y(t), \quad t > 0, \tag{4.2}$$

where $0 < \alpha \leq 1$, subject to the initial condition $y(0) = 0.5$

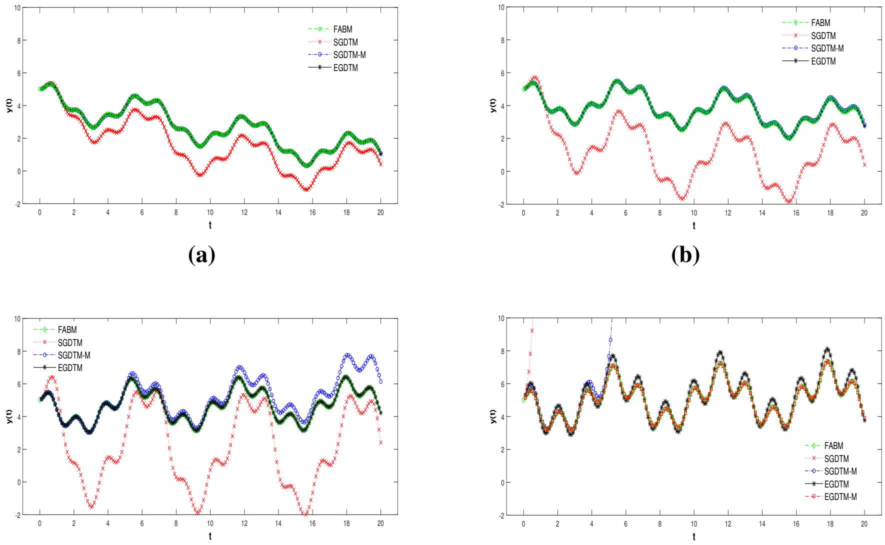
Applying the properties of GDTM in Theorem 3.1 and 3.2 to (4.2), we have the following recurrence relation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} Y(k+1) = F(k) + Y(k), \quad k \geq 0,$$

where $F(k)$ is the GDT of $e^{y(t)}$ and is given by as follows

$$F(k) = \frac{1}{k} \sum_{r=0}^{k-1} (k-r) F(r) Y(k-r), \quad k \geq 1. \tag{4.3}$$

From the initial condition $y(0) = 0$, we have $Y(0) = 0$ and $F(0) = e^{Y(0)}$.



(a) (b) (c) (d)

FIGURE 2. Comparisons of approximate solutions with $\alpha = (a)0.9, (b)0.7, (c)0.5$ and (d)0.3 from top to bottom

TABLE 1. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.9$ in Example 2

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	0.649371	0.649370	0.649334	0.649370	0.649334
0.3	0.927463	0.979716	0.930360	0.927889	0.927654
0.5	1.226325	1.369327	1.235908	1.227390	1.226824
0.7	1.563724	1.843697	1.584128	1.565723	1.564655
0.9	1.955064	2.435985	1.991838	1.958409	1.956606

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	0.787364	0.894235	0.787994	0.787406	0.787363
0.4	1.073037	1.373282	1.074893	1.073139	1.073038
0.6	1.389262	1.986059	1.393022	1.389450	1.389268
0.8	1.751642	2.800329	1.758230	1.751953	1.751653
1.0	2.176198	3.909788	2.186879	2.176683	2.176217

TABLE 2. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.7$ in Example 2

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	0.759755	0.759726	0.758609	0.759726	0.758609
0.3	1.120497	1.395246	1.146719	1.123690	1.120122
0.5	1.482904	2.293112	1.565537	1.490042	1.483219
0.7	1.885367	3.646766	2.058448	1.898037	1.886619
0.9	2.351559	5.755351	2.62769	2.372091	2.354152

t	y_h	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	0.944264	1.765056	0.951892	0.944619	0.944238
0.4	1.298381	4.381588	1.318474	1.299112	1.298363
0.6	1.677574	10.595925	1.716414	1.678822	1.677564
0.8	2.109090	25.612480	2.175534	2.111068	2.109091
1.0	2.615695	61.910497	2.721971	2.618702	2.615712

TABLE 3. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.5$ in Example 2

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	0.956630	0.955848	0.950455	0.955848	0.950455
0.3	1.428010	2.352869	1.599497	1.441996	1.426029
0.5	1.884920	5.229080	2.452074	1.913593	1.884701
0.7	2.392817	11.541742	3.666449	2.441988	2.395139
0.9	2.986370	25.473154	5.436456	3.064643	2.992272

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	1.202300	7.647129	1.266979	1.204033	1.202172
0.4	1.652715	78.712858	1.815157	1.655913	1.652670
0.6	2.130147		2.444142	2.135379	2.130197
0.8	2.676929		3.220666	2.685048	2.677112
1.0	3.325064		4.206744	3.337233	3.325436

For several fractional orders $\alpha = 0.9, 0.7$ and 0.5, the comparisons of numerical results are listed in Table 1, 2 and 3. The time step is set by $h = 10^{-7}$ in FABM. In all GDTMs, two time steps are chosen by $h = 10^{-1}$ and 10^{-2}. All numerical results are shown at two different sets of points; $\{0.1, \cdots, 0.9\}, \{0.2, \cdots, 1.0\}$. With a fine time step $h = 10^{-7}$, we assume that the numerical results by FABM are almost the exact solutions. As α is getting smaller, the numerical results by the SGDTM and SGDTM-M are more inaccurate. However, for all α, the two EGDTMs give very accurate numerical results. Especially, the EGDTM-M gives the most accurate numerical results. A close observation of the numercal results by EGDTM and EGDTM-M compared with the ones by FABM reveals that the errors are reduced in one more decimal place for all α. It is worth noting that EGDTM-M uses the time step $h = 10^{-2}$ and their numerical results agree with the ones by FABM in three decimal places.

Example 3 Consider the following nonlinear fractional differential equation:

$$D^\alpha y(t) = y(t) - y(t) \ln y(t), \quad t > 0, \tag{4.4}$$

where $0 < \alpha \leq 1$, subject to the initial condition $y(0) = 1$.

Applying the properties of GDTM in Theorem 3.1 and 3.3 to (4.4), we have the following recurrence relation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} Y(k+1) = Y(k) - \sum_{r=0}^{k} Y(r) F(k-r), \quad k \geq 0,$$

where $F(k)$ is the GDT of $\ln y(t)$ and is given as follows

$$F(k) = \frac{1}{k} \sum_{r=0}^{k-1} (k-r) G(r) Y(k-r), \quad k \geq 1.$$

and

$$G(s) = -\frac{1}{Y(0)} \sum_{r=0}^{s-1} G(r) Y(s-r), \quad s \geq 1.$$

From the initial condition $y(0) = 0$, we have $Y(0) = 1, F(0) = \ln(Y(0)) = 0$ and $G(0) = 1/Y(0) = 1$, respectively. For $\alpha = 0.9, 0.7$ and 0.5, the comparisons of numerical results are listed in Table 4, 5 and 6. For all α, the numerical results by SGDTM have the large errors. For small α, they are blowing up in a short time. But all other numerical methods gives very close approximations. However, it is also observed that EGDTM-M gives the best approximations for all α.

Example 4 Consider the following nonlinear fractional differential equation:

$$D^\alpha y(t) = y(t) e^{-y(t)} + \lambda_1 \cos(y(t)) + \lambda_2, \quad t > 0, \tag{4.5}$$

where $0 < \alpha \leq 1$, subject to the initial condition $y(0) = 0.5$. $\lambda_1 = 2, \lambda_2 = -1$. Applying the properties of GDTM in Theorem 3.2 and 3.4 to (4.5), we have the following recurrence

TABLE 4. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.9$ in Example 3

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	1.130479	1.130478	1.130335	1.130479	1.130334
0.3	1.344291	1.383388	1.343574	1.344253	1.344030
0.5	1.529469	1.615847	1.528442	1.529354	1.529183
0.7	1.690853	1.820994	1.689695	1.690642	1.690529
0.9	1.830688	1.996474	1.829532	1.830381	1.830479

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	1.241643	1.324116	1.241630	1.241634	1.241641
0.4	1.440025	1.620346	1.440005	1.440000	1.440022
0.6	1.612986	1.873568	1.612962	1.612946	1.612983
0.8	1.763334	2.079830	1.763309	1.763279	1.763331
1.0	1.893185	2.242099	1.893162	1.893120	1.893184

TABLE 5. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.7$ in Example 3

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	1.217061	1.217033	1.216744	1.217033	1.216744
0.3	1.451052	1.615579	1.4462721	1.450510	1.450835
0.5	1.617347	1.938792	1.611429	1.616336	1.617246
0.7	1.746556	2.181436	1.740452	1.745224	1.746541
0.9	1.850559	2.354905	1.844718	1.849043	1.850601

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	1.346483	1.791094	1.346256	1.346405	1.346476
0.4	1.539956	2.283125	1.539659	1.539817	1.539953
0.6	1.685666	2.526836	1.685355	1.685490	1.685664
0.8	1.801214	2.636395	1.800917	1.801019	1.801213
1.0	1.895317	2.683673	1.895044	1.895113	1.895316

TABLE 6. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.5$ in Example 3

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	1.343828	1.343272	1.344184	1.343272	1.344183
0.3	1.560016	1.905574	1.540004	1.58480	1.560406
0.5	1.686500	2.265874	1.665304	1.684451	1.686848
0.7	1.776320	2.473694	1.755858	1.774125	1.776608
0.9	1.845368	2.587958	1.826130	1.843191	1.845602

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	1.470799	2.437412	1.468388	1.470504	1.47073
0.4	1.629524	2.685293	1.626965	1.629190	1.629511
0.6	1.734685	2.714556	1.732225	1.734353	1.734667
0.8	1.812875	2.717863	1.810572	1.812557	1.812854
1.0	1.874539	2.718234	1.872401	1.874239	1.874516

relation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}Y(k+1) = \sum_{r=0}^{n} Y(r)F_1(k-r) + \lambda_1 F_2(k) + \lambda_2\delta(k), \quad k \geq 0,$$

where $F(k)$ is the GDT of $\ln y(t)$ and is given as follows

$$F_1(k) = -\frac{1}{k}\sum_{r=0}^{k-1}(k-r)F_1(r)Y(k-r), \quad k \geq 1.$$

and

$$F_2(k) = -\frac{1}{k}\sum_{r=0}^{k-1}(k-r)G_2(r)Y(k-r), \quad k \geq 1,$$

$$G_2(k) = \frac{1}{k}\sum_{r=0}^{k-1}(k-r)F_2(r)Y(k-r), \quad k \geq 1.$$

From the initial condition $y(0) = 0.$, we have $Y(0) = 0.5, F_1(0) = e^{-Y(0)}, F_2(0) = -\cos(Y(0))$ and $G_2(0) = \sin(Y(0))$, respectively. For $\alpha = 0.9, 0.7$ and 0.5, the comparisons of numerical results are listed in Table 7, 8 and 9. In this example, the SGDTM gives inaccurate numerical apporoximations for all α. Even though the numerical results by SGDTM are not blowing up for $\alpha = 0.5$ which are shown in the previous example, it converges to the wrong appromxations

TABLE 7. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.9$ in Example 4

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	0.631075	0.623627	0.631612	0.631074	0.631112
0.3	0.812286	0.830642	0.810220	0.811832	0.812445
0.5	0.935241	0.979059	0.929292	0.934554	0.935603
0.7	1.019272	1.078052	1.010588	1.018537	1.019733
0.9	1.077082	1.141454	1.066942	1.076401	1.077548

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	0.730662	0.799228	0.729789	0.730607	0.730665
0.4	0.879623	0.995300	0.877827	0.879536	0.879627
0.6	0.981216	1.109492	0.978885	0.981122	0.981221
0.8	1.050835	1.172457	1.048312	1.050746	1.050840
1.0	1.098973	1.206306	1.096481	1.098896	1.098978

TABLE 8. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.7$ in Example 4

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	0.705789	0.685727	0.711093	0.705873	0.708152
0.3	0.872581	0.956617	0.856594	0.870899	0.874362
0.5	0.961191	1.105143	0.934942	0.959380	0.962495
0.7	1.016447	1.178759	0.985826	1.014845	1.017375
0.9	1.053915	1.213785	1.021873	1.052572	1.054578

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	0.804740	1.072371	0.797543	0.804443	0.804775
0.4	0.922656	1.211311	0.912294	0.922362	0.922678
0.6	0.991719	1.238148	0.980452	0.991470	0.991733
0.8	1.036841	1.243155	1.025631	1.036638	1.036851
1.0	1.068395	1.244083	1.057633	1.068230	1.068402

at $t = 1.0$. The EGDTM-M with $h = 10^{-2}$ has the best approximations which agree to the four decimal places.

TABLE 9. Comparison of the numerical results by the MsGDTM(top:($h = 10^{-1}$), bottom: ($h = 10^{-2}$)) and FABM($h = 10^{-7}$) for $\alpha = 0.5$ in Example 4

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.1	0.794293	0.756083	0.818586	0.797983	0.805632
0.3	0.916474	1.057309	0.882245	0.914413	0.919256
0.5	0.972229	1.174639	0.928288	0.970404	0.973746
0.7	1.006537	1.218496	0.959699	1.005039	1.007513
0.9	1.030463	1.234764	0.982983	1.029199	1.031123

t	FABM	SGDTM	SGDTM-M	EGDTM	EGDTM-M
0.2	0.870733	1.227964	0.846505	0.870145	0.870814
0.4	0.948300	1.244019	0.921004	0.947881	0.948322
0.6	0.991116	1.244289	0.963630	0.990801	0.991119
0.8	1.019446	1.244294	0.992535	1.019197	1.019441
1.0	1.040013	1.244294	1.013902	1.039808	1.040003

5. CONCLUSION

In this work we proposed a new scheme to obtain an accrurate numerical approximation for the nonlinear differential equations of the fractional order. The proposed method is based on the generalized Taylor series which is called the generalized differential transform method(GDTM). The conventional GDTM was applied in each sub-domain to obtain the accurate approximations in whole domain. However, it has been shown that this approach does not contain the effect of memory which is the main characteristic in the diffential equations of the fractional order. We derive a new recursive relation in GDTM, which contains the effect of memory. From the several illustrative examples, it is shown that the proposed method has the promising numerical results with low computational cost.

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