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THE HARDY TYPE INEQUALITY ON METRIC MEASURE SPACES

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Abstract. In this paper, we prove that if a metric measure space satisfies the volume doubling condition and the Hardy type inequality with the same exponent n ($n \geq 3$), then it has exactly the n-dimensional volume growth. Besides, three interesting applications of this fact have also been given. The first one is that we prove that complete noncompact *smooth* metric measure space with non-negative weighted Ricci curvature on which the Hardy type inequality holds with the best constant are isometric to the Euclidean space with the same dimension. The second one is that we show that if a complete n-dimensional Finsler manifold of nonnegative n-Ricci curvature satisfies the Hardy type inequality with the best constant, then its flag curvature is identically zero. The last one is an interesting rigidity result, that is, we prove that if a complete n-dimensional Berwald space of non-negative n-Ricci curvature satisfies the Hardy type inequality with the best constant, then it is isometric to the Minkowski space of dimension n.

1. Introduction

Denote by $C_0^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support in an n-dimensional ($n \geq 3$) Euclidean space $\mathbb{R}^n$. Let p, q, γ be constants satisfying

$$1 \leq q < n, \quad q \leq p \leq \frac{nq}{n-q}, \quad \gamma = -1 + n\left(\frac{1}{q} - \frac{1}{p}\right), \tag{1.1}$$

and let $|x|$ be the Euclidean length of $x \in \mathbb{R}^n$. We know that there exists a positive constant C such that

$$\left(\int_{\mathbb{R}^n} |x|^{\gamma p} |u|^p dx\right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}^n} |\nabla u|^q dx\right)^{\frac{1}{q}}, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

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which is usually called *Hardy inequality* [24]. Denote by $K(n,q,\gamma)$ the *best constant* of the above Hardy inequality, which is given by

$$K(n,q,\gamma)^{-1} = \inf_{u\in C^0_\infty(\mathbb{R}^n)-\{0\}} \frac{\left(\int_{\mathbb{R}^n} |\nabla u|^q dx\right)^{\frac{1}{q}}}{\left(\int_{\mathbb{R}^n} |x|^{\gamma p}|u|^p dx\right)^{\frac{1}{p}}}.$$

When $\gamma p + q > 0$ and $q > 1$, the equality in (1.2) can be achieved with

$$u(x) = \left(\lambda + |x|^{\frac{q+\gamma p}{q-1}}\right)^{\frac{q-n}{q+\gamma p}}, \qquad \lambda > 0,$$

and $C = K(n,q,\gamma)$ (see, e.g., [11, 23, 24]). In this situation, the function u is called the extremal function (or minimizer) of the Hardy inequality (1.2). Observe that when $q = 2$, the Hardy inequality (1.2) becomes the following Caffarelli-Kohn-Nirenberg inequality (cf. [9])

$$\left(\int_{\mathbb{R}^n} |x|^{\gamma p}|u|^p dx\right)^{\frac{1}{p}} \le C\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\frac{1}{2}}, \qquad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{1.3}$$

In 2011, Adriano-Xia [1] showed that complete non-compact Riemannian manifolds with asymptotically non-negative Ricci curvature satisfying some Hardy type inequalities are not far from the Euclidean space of the same dimension. This rigidity result is interesting and it reveals the deep connection between the validity of given functional inequalities and the geometric structure of manifolds with specified curvature assumption. This kind of topic is attractive and some interesting results have been obtained. For complete non-compact Riemannian manifolds with non-negative Ricci curvature (or asymptotically non-negative Ricci curvature) satisfying some Sobolev or Hardy type inequalities, similar rigidity results can also be expected (see, e.g., [1,3–5,10,12,13,22,34,36]). Recently, Mao [26,27,29] has proven that complete non-compact *smooth* metric measure spaces with non-negative *weighted* Ricci curvature on which a functional inequality of some specified type (for instance, the Caffarelli-Kohn-Nirenberg type inequality, the Gagliardo-Nirenberg type inequality, and so on) is satisfied are close to the Euclidean space with the same dimension. Even more, this kind of rigidity results can be improved to metric measure spaces (see, e.g., [14, 19, 20]).

Let (X, d, μ) be a metric measure space and μ be the Borel measure on X such that $0 < \mu(U) < \infty$ for any nonempty bounded open set $U \subset X$. Let $\mathrm{Lip}_0(X)$ be the space of Lipschitz functions with compact support on X, and set

$$|Du|(x) := \limsup_{y\to x} \frac{|u(y) - u(x)|}{d(x,y)},$$

which is the local Lipschitz constant of u at $x \in X$.

In 2013, Kristály-Ohta [20] studied metric measure spaces satisfying the volume doubling condition mentioned therein and the Caffarelli-Kohn-Nirenberg inequality with the same exponent $n \ge 3$, and then they proved that those spaces have exactly the n-dimensional volume growth. Inspired by Kristály-Ohta's work and interesting results mentioned above, we study the Hardy type inequality on complete metric measure spaces in this paper and can get the following.

Theorem 1.1. *Let p, q, γ be constants satisfying (1.1) and $q>1$, $n\geq 3$, $x_0\in X$, $C\geq K(n,q,\gamma)$ with $K(n,q,\gamma)$ the best constant mentioned before, and $C_0\geq 1$. Assume that for any $u\in \mathrm{Lip}_0(X)$, the Hardy type inequality*

$$\left(\int_X d(x_0,x)^{\gamma p}|u(x)|^p d\mu(x)\right)^{\frac{1}{p}} \leq C\left(\int_X |Du|^q(x)d\mu(x)\right)^{\frac{1}{q}} \tag{1.4}$$

and the volume conditions

$$\frac{\mu(B(x,R))}{\mu(B(x,r))}\leq C_0\cdot\left(\frac{R}{r}\right)^n \quad \text{for all } x\in X, \text{ and } 0<r<R, \tag{1.5}$$

$$\liminf_{r\to 0}\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}=1 \tag{1.6}$$

hold on a proper metric measure space (X,d,μ), where $B(x,r):=\{y\in X: d(x,y)<r\}$, $\mathbb{B}_n(r):=\{x\in\mathbb{R}^n: |x|<r\}$, and μ_E is the n-dimensional Lebesgue measure. Then, for any $x\in X$ and $\rho>0$, we have

$$\mu(B(x,\rho))\geq C_0^{-1}\left(C^{-1}K(n,q,\gamma)\right)^{\frac{pq}{p-q}}\mu_E(\mathbb{B}_n(\rho)). \tag{1.7}$$

In particular, (X,d,μ) has the n-dimensional volume growth

$$C_0^{-1}\left(C^{-1}K(n,q,\gamma)\right)^{\frac{pq}{p-q}} w_n\rho^n\leq \mu(B(x_0,\rho))\leq C_0 w_n\rho^n$$

for all $\rho>0$, where w_n is the volume of the unit ball in $\mathbb{R}^n$.

Remark 1.2. (1) When $p=q=2$, then $\gamma=-1$, and correspondingly, the Hardy type inequality (1.4) degenerates into

$$\left(\int_X \frac{|u(x)|^2}{d(x_0,x)^2}d\mu(x)\right)^{\frac{1}{2}}\leq C\left(\int_X |Du|^2(x)d\mu(x)\right)^{\frac{1}{2}}$$

for $u\in\mathrm{Lip}_0(X)$, which is a special case of the Caffarelli-Kohn-Nirenberg type inequality $(\mathbf{CKN})_C^{x_0}$ in [20, Theorem 1.1]. However, based on this fact, one cannot get the conclusion that when $p=q=2$, [20, Theorem 1.1] covers our Theorem 1.1. In fact, even when $p=q=2$, Theorem 1.1 is different from [20, Theorem 1.1], since in the assumption of [20, Theorem 1.1] if $p=2$, then $a=1$, but which is not permitted.

(2) Similar to [14, Remark 1.3(2)], here the validity of the Hardy type inequality implies that (X,d) is non-compact. In fact, if (X,d) is bounded, then one can choose $q=p=2$, then $\gamma=-1$, which lets (1.4) become the Caffarelli-Kohn-Nirenberg type inequality mentioned above, and in this setting, $u+\ell$ with $\ell\to\infty$ clearly violates the validity of (1.4).

(3) As pointed out in [20, Remark 1.2(b)] and also mentioned in [14, Remark 1.3(4)], if (X, d, μ) satisfies the *volume doubling condition*

$$\mu(B(x, 2r)) \leq \Lambda \mu(B(x, r)) \quad \text{for some } \Lambda \geq 1 \text{ and all } x \in X,\ r > 0,$$

then it is easy to get that the volume condition (1.5) is satisfied with, e.g., $n \geq \log_2 \Lambda$ and $C_0 = 1$. Therefore, (1.5) can be comprehended as the volume doubling condition with the explicit exponent n. Besides, one can regard the volume condition (1.6) as a generalization of the classical Bishop-Gromov volume comparison for complete manifolds with non-negative Ricci curvature.

(4) As shown in [14, Remark 1.3(4)], by the volume conditions (1.5) and (1.6), one can get $\mu(B(x_0, R)) \leq C_0 w_n R^n$ for any $R > 0$. This implies that the last assertion of Theorem 1.1 (i.e., the n-dimensional volume growth) can be obtained directly provided (1.7) is proved.

(5) The assertion of having n-dimensional volume growth implies that, for instance, the cylinder $\mathbb{S}^{n-1} \times \mathbb{R}$ does not satisfy (1.4) for any $x \in X$ and C.

(6) Here we would like to repeat the second part of [14, Remark 1.3(5)], which is helpful for readers to know deeply about the volume conditions (1.5) and (1.6). The volume doubling condition (1.5) implies that the Hausdorff dimension $\dim_H X$ of (X, d) is at most n. Besides, as in (3), by the volume conditions (1.5) and (1.6), we have

$$\frac{\mu(B(x_0, R))}{\mu_E(\mathbb{B}_n(R))} \leq C_0 \frac{\mu(B(x_0, r))}{\mu_E(\mathbb{B}_n(r))}$$

for $x_0 \in X$ and $0 < r < R$, which implies that

$$\limsup_{R \to 0} \frac{\mu(B(x_0, R))}{\mu_E(\mathbb{B}_n(R))} \leq \liminf_{r \to 0} C_0 \frac{\mu(B(x_0, r))}{\mu_E(\mathbb{B}_n(r))} = C_0.$$

Therefore, we know that the *Ahlfors n-regularity* at x_0 in the sense that $\eta^{-1} r^n \leq \mu(B(x_0, r)) \leq \eta r^n$ for some $\eta \geq 1$ and small $r > 0$, which means that $\dim_H X = n$. The volume doubling condition and the Ahlfors regularity are important in analysis on metric measure spaces. For this fact, see, e.g., [17] for the details. Note that the choice of the constant 1 chosen at the right hand side of (1.6) is only for simplicity. In fact, by (1.5), we know that $\eta_{x_0} := \liminf_{r \to 0} \frac{\mu(B(x_0, r))}{\mu_E(\mathbb{B}_n(r))}$ is positive. So, one can normalize μ so as to satisfy (1.6) once η_{x_0} is bounded.

As the first application of Theorem 1.1, we can use it to get a rigidity result for *smooth* metric measure spaces with nonnegative weighted Ricci curvature and satisfying the sharp Hardy type inequality. However, before stating this result, we need to briefly introduce the notions of smooth metric measure spaces and weighted Ricci curvature.

A smooth metric measure space, which is also known as the weighted measure space, is actually a Riemannian manifold equipped with some measure (which is conformal to the usual Riemannian measure). More precisely, for a given complete n-dimensional Riemannian manifold (M, g) with the metric

g, the triple $(M, g, e^{-f} dv_g)$ is called a smooth metric measure space, with f a *smooth real-valued* function on M and dv_g the Riemannian volume element related to g (sometimes, we also call dv_g the volume density). For a geodesic ball $B(x, r)$, we can define its *weighted (or f-)volume* $\mathrm{Vol}_f[B(x, r)]$ as follows

$$\mathrm{Vol}_f[B(x,r)] = \int_{B(x,r)} e^{-f} dv_g. \tag{1.8}$$

On a smooth metric measure space $(M, g, e^{-f} dv_g)$, the so called ∞-*Bakry-Émery Ricci tensor* Ric_f is defined by

$$\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess} f,$$

which is also called the *weighted Ricci curvature*. Bakry and Émery [6, 7] introduced firstly and investigated extensively the generalized Ricci tensor above and its relationship with diffusion processes.

By applying Theorems 1.1 and 3.1, we can obtain the following.

Theorem 1.3. *Let p, q, γ, $K(n, q, \gamma)$ be as in Theorem 1.1, and let*

$$\big(M, g, e^{-f} dv_g\big)$$

be an n-dimensional ($n \geq 3$) complete noncompact smooth metric measure space with non-negative ∞-Bakry-Émery Ricci curvature. For a point $x_0 \in M$ at which $f(x_0)$ is away from $-\infty$, assume that the radial derivative $\partial_t f$ satisfies $\partial_t f \geq 0$ along all minimal geodesic segments starting from x_0, with $t := d(x_o, \cdot)$ the distance to x_0 (on M). Furthermore, for any $u \in C_0^\infty(M)$, if the following Hardy type inequality

$$\left(\int_M t^{\gamma p} |u|^p e^{-f} d\nu_g\right)^{\frac{1}{p}} \leq K(n, p, \gamma) \left(\int_M |\nabla u|^q e^{-f} d\nu_g\right)^{\frac{1}{q}} \tag{1.9}$$

holds, then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Moreover, in this case, we have $f \equiv f(x_0)$ is a constant function, and $e^{-f} dv_g = e^{-f(x_0)} dv_{g_{\mathbb{R}^n}}$, where $g_{\mathbb{R}^n}$ and $dv_{g_{\mathbb{R}^n}}$ are the usual Euclidean metric and the Euclidean volume density related to $g_{\mathbb{R}^n}$, respectively.

Remark 1.4. When $\gamma = 0$ and the weighted function f is chosen to be $f \equiv 0$, Theorem 1.3 degenerates into the main theorem in [21].

By Theorem 4.2 (equivalently, see also Shen [32] or Ohta [30]), we know that for Finsler manifolds with non-negative n-Ricci curvature (for this notion, see Definition 4.1 for the precise statement), the volume doubling condition (1.5) holds with $C_0 = 1$. For complete Finsler manifolds with non-negative n-Ricci curvature, when the Hardy type inequality (1.4) is satisfied with the best constant (i.e., $C = K(n, q, \gamma)$), by applying Theorems 1.3 and 4.2, we can prove the following result.

Theorem 1.5. *Let p, q, γ, $K(n, q, \gamma)$ be as in Theorem 1.1, and let (X, F) be an n-dimensional ($n \geq 3$) complete Finsler manifold. Fix a positive smooth* *measure μ on X and assume that the n-Ricci curvature* Ric_n *of* (X, F, μ) *is non-negative. If the Hardy type inequality* (1.4) *is satisfied with the best constant* (*i.e.,* $C = K(n, q, \gamma)$) *and* $\lim_{r\to 0} \frac{\mu(B(x_0,r))}{w_n r^n} = 1$*, then under the volume doubling condition* (1.5)*, we have the flag curvature of* (X, F) *is identically zero.*

Remark 1.6. Finsler manifolds are special metric measure spaces with prescribed Finsler structures. See Subsection 4.2 for a brief introduction to Finsler manifolds.

As the last application of Theorem 1.1, we can use it to get a rigidity result for Berwald spaces, with nonnegative Ricci curvature and satisfying the sharp Hardy type inequality, as follows.

Theorem 1.7. *Let p, q, γ, $K(n, q, \gamma)$ be as in Theorem 1.1, and let (X, F) be an n-dimensional ($n \geq 3$) complete Berwald space with non-negative Ricci curvature. If for some $x_0 \in X$ and the n-dimensional Hausdorff measure of (X, F), the Hardy type inequality* (1.4) *is satisfied with the best constant* (*i.e.,* $C = K(n, q, \gamma)$)*, then (X, F) is isometric to a Minkowski space.*

The paper is organized as follows. The proof of Theorem 1.1 will be given in Section 2. In Section 3, we first recall some fundamental but necessary knowledge in Riemannian geometry and an important volume comparison result in the setting of smooth metric measure spaces, and then use them to prove Theorem 1.3. In the last section, some basic notions and results from Finsler geometry will be introduced first, and then based on them Theorems 1.5 and 1.7 will be proved.

2. Proof of Theorem 1.1

Proof. Consider two auxiliary functions $F, G : (0, +\infty) \to \mathbb{R}$ defined by

$$F(\lambda) := \frac{q + p\gamma}{p(n-q) - q - p\gamma} \int_X \frac{d(x_0, x)^{p\gamma}}{\left(\lambda + d(x_0, x)^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)-q-p\gamma}{q+p\gamma}}} d\mu(x)$$

and

$$G(\lambda) := \frac{q + p\gamma}{p(n-q) - q - p\gamma} \int_{\mathbb{R}^n} \frac{|x|^{p\gamma}}{\left(\lambda + |x|^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)-q-p\gamma}{q+p\gamma}}} d\mu_E(x)$$

respectively, which are well defined and of class C^1.

By the layer cake representation of functions, we have

$$F(\lambda) = \frac{q + p\gamma}{p(n-q) - q - p\gamma} \int_0^{+\infty} \mu\left\{ x \in X; \frac{d(x_0, x)^{p\gamma}}{\left(\lambda + d(x_0, x)^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)-q-p\gamma}{q+p\gamma}}} > s \right\} ds.$$

Taking into account that $\operatorname{diam}(X) = \infty$ and making the variable change

$$s = \frac{\rho^{p\gamma}}{\left(\lambda + \rho^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)-q-p\gamma}{q+p\gamma}}},$$

it follows that

$$F(\lambda) = \frac{q+p\gamma}{p(n-q)-q-p\gamma} \int_0^{+\infty} \mu(B(x_0,\rho)) f(\lambda,\rho) d\rho, \tag{2.1}$$

where

$$f(\lambda,\rho) = \frac{\rho^{p\gamma-1}\left[-p\gamma\lambda + \frac{q(n-1)}{q-1}\rho^{\frac{q+p\gamma}{q-1}}\right]}{\left(\lambda + \rho^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)}{q+p\gamma}}}.$$

Similar to the previous argument, we can obtain

$$G(\lambda) = \frac{q+p\gamma}{p(n-q)-q-p\gamma} \int_0^{+\infty} \mu_E(\mathbb{B}_n(\rho)) f(\lambda,\rho) d\rho. \tag{2.2}$$

On the other hand, since the volume conditions (1.5) and (1.6) hold on (X, d, μ), we have

$$\mu(B(x_0,\rho)) \leq C_0 \mu_E(\mathbb{B}_n(\rho)). \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), it is easy to get that

$$0 \leq F(\lambda) \leq C_0 G(\lambda). \tag{2.4}$$

For each $\lambda > 0$, consider the sequence of functions $u_{\lambda,k} : X \to \mathbb{R}$, $k \in \mathbb{N}$, defined as follows

$$u_{\lambda,k}(x) := \max\{0, \min\{0, k - d(x_0,x)\} + 1\} \left(\lambda + \max\{d(x_0,x), k^{-1}\}^{\frac{q+\gamma p}{q-1}}\right)^{\frac{q-n}{q+\gamma p}}.$$

Since (X, d) is proper, $\operatorname{supp}(u_{\lambda,k}) = \{x \in X : d(x_0,x) \leq k+1\}$ is compact. Therefore, we have $u_{\lambda,k} \in \operatorname{Lip}_0(X)$ for every $\lambda > 0$ and $k \in \mathbb{N}$. Set

$$u_\lambda(x) := \lim_{k\to\infty} u_{\lambda,k}(x) = \left(\lambda + d(x_0,x)^{\frac{q+\gamma p}{q-1}}\right)^{\frac{q-n}{q+\gamma p}}.$$

Similar to the explanation in Step 2 of the proof of [20, Theorem 1.1], one can show that $u_\lambda(x)$ verifies the Hardy type inequality (1.4). In fact, since the functions $u_{\lambda,k}(x)$ satisfy the Hardy type inequality (1.4), a simple approximation based on (2.4) shows that u_λ also satisfies (1.4). Applying a chain rule for the local Lipschitz constant and the fact that $x \mapsto d(x_0,x)$ is 1-Lipschitz (thus $|Dd(x_0,\cdot)|(x) \leq 1$ for all x), we can get

$$\left(\int_X \frac{d(x_0,x)^{p\gamma}}{\left(\lambda + d(x_0,x)^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)}{q+p\gamma}}} d\mu(x)\right)^{\frac{q}{p}}$$

$$
\begin{aligned}
&\leq \left(\frac{C(n-q)}{q-1}\right)^q \int_X \frac{d(x_0,x)^{\frac{(1+p\gamma)q}{q-1}}}{\left(\lambda + d(x_0,x)^{\frac{q+p\gamma}{q-1}}\right)^{\frac{q(n+p\gamma)}{q+p\gamma}}} d\mu(x) \\
&= \left(\frac{C(n-q)}{q-1}\right)^q \int_X \frac{d(x_0,x)^{p\gamma+\frac{q+p\gamma}{q-1}}}{\left(\lambda + d(x_0,x)^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)}{q+p\gamma}}} d\mu(x) \\
&= \left(\frac{C(n-q)}{q-1}\right)^q \Bigg[\int_X \frac{d(x_0,x)^{p\gamma}}{\left(\lambda + d(x_0,x)^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)-q-p\gamma}{q+p\gamma}}} d\mu(x) \\
&\qquad\qquad -\lambda \int_X \frac{d(x_0,x)^{p\gamma}}{\left(\lambda + d(x_0,x)^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)}{q+p\gamma}}} d\mu(x)\Bigg].
\end{aligned}
$$

By the definition of $F(\lambda)$, we know that the above inequality can be rewritten as follows

$$
l\left(-F'(\lambda)\right)^{\frac{q}{p}} \leq \frac{p(n-q)-q-p\gamma}{q+p\gamma} F(\lambda) + \lambda F'(\lambda), \tag{2.5}
$$

where

$$
l = \left(\frac{q-1}{C(n-q)}\right)^q.
$$

Since $v_\lambda(x) := \left(\lambda + |x|^{\frac{q+\gamma p}{q-1}}\right)^{\frac{q-n}{q+\gamma p}}$, $\lambda > 0$, is a minimizer of the Hardy inequality in $\mathbb{R}^n$, one can obtain that for every $\lambda > 0$, the following equality

$$
\left(\int_{\mathbb{R}^n} |x|^{\gamma p} |v_\lambda(x)|^p d\mu_E(x)\right)^{\frac{1}{p}} = K(n,q,\gamma) \left(\int_{\mathbb{R}^n} |\nabla v_\lambda(x)|^q d\mu_E(x)\right)^{\frac{1}{q}} \tag{2.6}
$$

holds. By the definition of $G(\lambda)$ and a similar argument as above, (2.6) can be rewritten as follows

$$
\widetilde{l}\left(-G'(\lambda)\right)^{\frac{q}{p}} = \frac{p(n-q)-q-p\gamma}{q+p\gamma} G(\lambda) + \lambda G'(\lambda), \tag{2.7}
$$

where

$$
\widetilde{l} = \left(\frac{q-1}{K(n,q,\gamma)\cdot(n-q)}\right)^q.
$$

Taking $G(\lambda) = G(1)\lambda^{-\frac{n-q}{q+p\gamma}}$ into (2.7), we have

$$
\widetilde{l}\left(\frac{n-q}{q+p\gamma}\right)^{\frac{q}{p}} = \frac{p(n-q)-n-p\gamma}{q+p\gamma} G(1)^{1-\frac{q}{p}}. \tag{2.8}
$$

Consider the constant A given by

$$l\left(\frac{n-q}{q+p\gamma}\right)^{\frac{q}{p}}=\frac{p(n-q)-n-p\gamma}{q+p\gamma}A^{1-\frac{q}{p}}. \tag{2.9}$$

It is not hard to check the function

$$H_0(\lambda)=A\lambda^{-\frac{n-q}{q+p\gamma}},\qquad \lambda\in(0,+\infty),$$

satisfies the differential equation

$$l\left(-H_0'(\lambda)\right)^{\frac{q}{p}}=\frac{p(n-q)-q-p\gamma}{q+p\gamma}H_0(\lambda)+\lambda H_0'(\lambda). \tag{2.10}$$

By (2.8) and (2.9), we get

$$A=\left(\frac{K(n,q,\gamma)}{C}\right)^{\frac{pq}{p-q}}G(1),$$

which implies

$$H_0(\lambda)=\left(\frac{K(n,q,\gamma)}{C}\right)^{\frac{pq}{p-q}}\lambda^{-\frac{n-q}{q+p\gamma}}G(1)=\left(\frac{K(n,q,\gamma)}{C}\right)^{\frac{pq}{p-q}}G(\lambda). \tag{2.11}$$

Now, we would like to show that when $C>\Phi$ for every $\lambda>0$, we have

$$F(\lambda)\geq H_0(\lambda). \tag{2.12}$$

First, we *claim* that if $F(\lambda_0)<H_0(\lambda_0)$ for some $\lambda_0>0$, then $F(\lambda)<H_0(\lambda),\forall\lambda\in(0,\lambda_0]$. We prove this claim by contradiction. Suppose the claim is not true. Then there exists some $\widetilde{\lambda}\in(0,\lambda_0)$ such that $F(\widetilde{\lambda})\geq H_0(\widetilde{\lambda})$. Set $\lambda_1:=\sup\{\lambda<\lambda_0;F(\lambda)=H_0(\lambda)\}$, then for any $\lambda\in[\lambda_1,\lambda_0]$, $0<F(\lambda)\leq H_0(\lambda)$, and furthermore, together with (2.5), we have

$$l\left(-F'(\lambda)\right)^{\frac{q}{p}}\leq\frac{p(n-q)-q-p\gamma}{q+p\gamma}H_0(\lambda)+\lambda F'(\lambda). \tag{2.13}$$

For every $\lambda>0$, we define the function $z_\lambda:(0,\infty)\to\mathbb{R}$ by $z_\lambda(s)=ls^{\frac{q}{p}}+\lambda s$, which is increasing. Hence, when $\lambda\in[\lambda_1,\lambda_0]$, one can infer from (2.13) and (2.10) that

$$\begin{aligned}z_\lambda(-F'(\lambda))&=l(-F'(\lambda))^{\frac{q}{p}}-\lambda F'(\lambda)\\&\leq\frac{p(n-q)-q-p\gamma}{q+p\gamma}H_0(\lambda)=z_\lambda(-H_0'(\lambda)),\end{aligned}$$

which implies $F'(\lambda)\geq H_0'(\lambda),\forall\lambda\in[\lambda_1,\lambda_0]$. Thus, the function $F-H_0$ is increasing on $[\lambda_1,\lambda_0]$, which implies that

$$0=(F-H_0)(\lambda_1)\leq(F-H_0)(\lambda_0)<0.$$

This is a contradiction. Thus the above *claim* is true.

Due to (1.6), for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(B(x_0,\rho)) \geq (1-\varepsilon)\mu_E(\mathbb{B}(\rho))$ for all $0 \leq \rho \leq \delta$. Together with (2.1) and making a change of the variable $\rho = \lambda^{\frac{q-1}{q+p\gamma}} t$, we can get

$$F(\lambda) \geq \frac{(q+p\gamma)(1-\varepsilon)}{p(n-q)-q-p\gamma} \int_0^\delta \mu_E(\mathbb{B}_n(\rho)) f(\lambda,\rho) d\rho$$
$$= \frac{(q+p\gamma)(1-\varepsilon)}{p(n-q)-q-p\gamma} \lambda^{-\frac{n-q}{q+p\gamma}} \int_0^{\delta/\lambda^{\frac{q-1}{q+p\gamma}}} \mu_E(\mathbb{B}_n(t)) f(1,t) dt.$$

On the other hand, from (2.2) we can easily get

$$G(\lambda) = \frac{q+p\gamma}{p(n-q)-q-p\gamma} \lambda^{-\frac{n-q}{q+p\gamma}} \int_0^\infty \mu_E(\mathbb{B}_n(t)) f(1,t) dt.$$

Hence, from the above two expressions, one can obtain

$$\liminf_{\lambda\to 0} \frac{F(\lambda)}{G(\lambda)} \geq 1-\varepsilon.$$

Letting $\epsilon \to 0$ results in

$$\liminf_{\lambda\to 0} \frac{F(\lambda)}{G(\lambda)} \geq 1.$$

When $C > K(n,q,\gamma)$, we infer from the above inequality and (2.11) that

$$\liminf_{\lambda\to 0} \frac{F(\lambda)}{H_0(\lambda)} = \left(\frac{C}{K(n,q,\gamma)}\right)^{\frac{pq}{p-q}} \liminf_{\lambda\to 0} \frac{F(\lambda)}{G(\lambda)} \geq \left(\frac{C}{K(n,q,\gamma)}\right)^{\frac{pq}{p-q}} > 1.$$

Together with the previous *claim*, we know that $F(\lambda) \geq H_0(\lambda), \forall \lambda > 0$. Thus, for any $\lambda > 0$, we can get from (2.1), (2.2), (2.11) that

$$\int_0^{+\infty} \left\{\mu(B(x_0,\rho)) - b\mu_E(\mathbb{B}_n(\rho))\right\} f(\lambda,\rho) d\rho \geq 0, \tag{2.14}$$

where $b = \left(C^{-1}K(n,q,\gamma)\right)^{\frac{pq}{p-q}}$. By the volume condition (1.5), for a fixed $\rho > 0$ we have

$$C_0 \frac{\mu(B(x_0,\rho))}{\mu_E(\mathbb{B}_n(\rho))} \geq \frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}$$

for any $r > \rho \geq 0$. We can assume

$$b_0 := \limsup_{r\to\infty} \frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))}.$$

In order to prove (1.7) in the case that $C > K(n,q,\gamma)$, it suffices to show that $b_0 \geq b$. We will prove this by contradiction. By the definition of b_0, we know that for some $\rho_0 > 0$, there exists $\varepsilon_0 > 0$ such that

$$\frac{\mu(B(x_0,\rho))}{\mu_E(\mathbb{B}_n(\rho))} \leq b - \varepsilon_0, \qquad \forall \rho \geq \rho_0. \tag{2.15}$$

Substituting (2.15) into (2.14), and together with (2.2) and (2.3), for every $\lambda > 0$, we have

$$
\begin{aligned}
0 \le & \int_0^{+\infty} \{\mu(B(x_0,\rho)) - b\mu_E(\mathbb{B}_n(\rho))\} f(\lambda,\rho)d\rho \\
\le & \int_0^{\rho_0} \mu(B(x_0,\rho))f(\lambda,\rho)d\rho + (b-\varepsilon_0)\int_{\rho_0}^{+\infty} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho \\
& - b\int_0^{+\infty} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho \\
\le & \, C_0\int_0^{\rho_0} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho - b\int_0^{\rho_0} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho \\
& - \varepsilon_0\int_{\rho_0}^{+\infty} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho \\
= & \, (C_0 - b + \varepsilon_0)\int_0^{\rho_0} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho - \varepsilon_0\int_0^{+\infty} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho \\
= & \, (C_0 - b + \varepsilon_0)\int_0^{\rho_0} \mu_E(\mathbb{B}_n(\rho))f(\lambda,\rho)d\rho - \varepsilon_0\frac{q+p\gamma}{p(n-q)-q-p\gamma}\lambda^{-\frac{n-q}{q+p\gamma}}G(1).
\end{aligned}
$$

Since $f(\lambda,\rho) = \frac{\rho^{p\gamma-1}\left[-p\gamma\lambda + \frac{q(n-1)}{q-1}\rho^{\frac{q+p\gamma}{q-1}}\right]}{\left(\lambda+\rho^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)}{q+p\gamma}}}$, one has

$$
\begin{aligned}
& \int_0^{\rho_0} \rho^n f(\lambda,\rho)d\rho \\
= & \int_0^{\rho_0} \frac{\rho^{n+p\gamma-1}\left[-p\gamma\lambda + \frac{q(n-1)}{q-1}\rho^{\frac{q+p\gamma}{q-1}}\right]}{\left(\lambda+\rho^{\frac{q+p\gamma}{q-1}}\right)^{\frac{p(n-q)}{q+p\gamma}}}d\rho \\
\le & \frac{-p\gamma\rho_0^{n+p\gamma}}{n+p\gamma}\lambda^{\frac{-p(n-q)}{q+p\gamma}+1} + \left(\frac{q(n-1)}{q-1}\frac{\rho_0^{n+p\gamma+\frac{q+p\gamma}{q-1}}}{n+p\gamma+\frac{q+p\gamma}{q-1}}\right)\lambda^{\frac{-p(n-q)}{q+p\gamma}}.
\end{aligned}
$$

Combining the above two inequalities, we can get an inequality of the following type

$$
M_1\lambda^{-\frac{n-q}{q+p\gamma}} \le M_2\lambda^{\frac{-p(n-q)}{q+p\gamma}+1} + M_3\lambda^{\frac{-p(n-q)}{q+p\gamma}}, \qquad \forall\lambda > 0,
$$

where M_1, M_2, $M_3 > 0$ are constants independent of λ. Observe that $\frac{-p(n-q)}{q+p\gamma} + 1 + \frac{n-q}{q+p\gamma} = \frac{(1-p)(n-q)}{q+p\gamma} + 1 < 0$. Letting $\lambda \to +\infty$ in the above inequality, one can obtain a contradiction immediately. This means that (1.7) holds in the case that $C > K(n,q,\gamma)$.

When $C = K(n,q,\gamma)$, we can also get (1.7). In fact, in this case, for any fixed $\delta > 0$, we have

$$\left(\int_X d(x_0,x)^{\gamma p}|u(x)|^p d\mu(x)\right)^{\frac{1}{p}} \leq (\delta + K(n,q,\gamma)) \left(\int_X |Du|^q(x) d\mu(x)\right)^{\frac{1}{q}}.$$

Therefore, for any $x \in X$, by the previous argument, we have

$$\mu(B(x,\rho)) \geq C_0^{-1}\left(\frac{K(n,q,\gamma)}{K(n,q,\gamma)+\delta}\right)^{\frac{pq}{p-q}} \mu_E(\mathbb{B}_n(\rho)), \qquad \forall \rho > 0.$$

Letting $\delta \to 0$, we can obtain

$$\mu(B(x,\rho)) \geq C_0^{-1}\mu_E(\mathbb{B}_n(\rho)), \qquad \forall \rho > 0,$$

which implies (1.7) holds in the case that $C = K(n,q,\gamma)$.

This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.3

In this section, we would like to give the proof of Theorem 1.3. However, before that, we need to introduce some notions. For more details, we refer readers to [15, 25–27, 29].

3.1. Some basic notions

Denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$. Given an n-dimensional ($n \geq 2$) complete Riemannian manifold (M,g) with the metric g, for a point $x \in M$, let S_x^{n-1} be the unit sphere with center x in the tangent space T_xM, and let $Cut(x)$ be the cut-locus of x, which is a closed set of zero n-Hausdorff measure. Clearly,

$$\mathbb{D}_x = \left\{t\xi \,|\, 0 \leq t < d_\xi,\ \xi \in S_x^{n-1}\right\}$$

is a star-shaped open set of T_xM, and through which the exponential map $\exp_x : \mathbb{D}_x \to M\backslash Cut(x)$ gives a diffeomorphism from $\mathbb{D}_x$ to the open set $M\backslash Cut(x)$, where d_ξ is defined by

$$d_\xi = d_\xi(x) := \sup\{t > 0 \,|\, \gamma_\xi(s) := \exp_x(s\xi) \text{ is the unique minimal geodesic joining } x \text{ and } \gamma_\xi(t)\}.$$

We can introduce two important maps used to construct the geodesic spherical coordinate chart at a prescribed point on a Riemannian manifold. For a fixed vector $\xi \in T_xM$, $|\xi| = 1$, let $\xi^\perp$ be the orthogonal complement of $\{\mathbb{R}\xi\}$ in T_xM, and let $\tau_t : T_xM \to T_{\exp_x(t\xi)}M$ be the parallel translation along $\gamma_\xi(t)$. The path of linear transformations $\mathbb{A}(t,\xi) : \xi^\perp \to \xi^\perp$ is defined by

$$\mathbb{A}(t,\xi)\eta = (\tau_t)^{-1}Y_\eta(t),$$

where $Y_\eta(t) = d(\exp_x)_{(t\xi)}(t\eta)$ is the Jacobi field along $\gamma_\xi(t)$ satisfying $Y_\eta(0) = 0$, and $(\nabla_t Y_\eta)(0) = \eta$. Moreover, for $\eta \in \xi^\perp$ set

$$\mathcal{R}(t)\eta = (\tau_t)^{-1}R(\gamma_\xi'(t), \tau_t\eta)\gamma_\xi'(t),$$

where the curvature tensor $R(X,Y)Z$ is defined by $R(X,Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X,Y]}Z$. Then $\mathcal{R}(t)$ is a self-adjoint operator on $\xi^{\perp}$, whose trace is the radial Ricci tensor $\mathrm{Ric}_{\gamma_\xi(t)}\left(\gamma_\xi'(t), \gamma_\xi'(t)\right)$. Clearly, the map $\mathbb{A}(t,\xi)$ satisfies the Jacobi equation $\mathbb{A}'' + \mathcal{R}\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0,\xi) = 0$, $\mathbb{A}'(0,\xi) = I$. By Gauss's lemma, the Riemannian metric of $M\backslash Cut(x)$ in the geodesic spherical coordinate chart can be expressed by

$$ds^2(\exp_x(t\xi)) = dt^2 + |\mathbb{A}(t,\xi)d\xi|^2, \qquad \forall t\xi \in \mathbb{D}_x. \tag{3.1}$$

We consider the metric components $g_{ij}(t,\xi)$, $i,j \geq 1$, in a coordinate system $\{t, \xi_a\}$ formed by fixing an orthonormal basis $\{\eta_a, a \geq 2\}$ of $\xi^{\perp} = T_\xi S_x^{n-1}$, and then extending it to a local frame $\{\xi_a, a \geq 2\}$ of S_x^{n-1}. Define a function $J > 0$ on $\mathbb{D}_x\backslash\{x\}$ by

$$J^{n-1} = \sqrt{|g|} := \sqrt{\det[g_{ij}]}. \tag{3.2}$$

Since $\tau_t : S_x^{n-1} \to S_{\gamma_\xi(t)}^{n-1}$ is an isometry, we have

$$\langle d(\exp_x)_{t\xi}(t\eta_a), d(\exp_x)_{t\xi}(t\eta_b)\rangle_g = \langle \mathbb{A}(t,\xi)(\eta_a), \mathbb{A}(t,\xi)(\eta_b)\rangle_g,$$

and then $\sqrt{|g|} = \det \mathbb{A}(t,\xi)$. So, by applying (3.1) and (3.2), the volume $\mathrm{vol}(B(x,r))$ of a geodesic ball $B(x,r)$, with radius r and center x, on M is given by

$$\begin{aligned} \mathrm{Vol}(B(x,r)) &= \int_{S_x^{n-1}} \int_0^{\min\{r,d_\xi\}} \sqrt{|g|}\,dt\,d\sigma \\ &= \int_{S_x^{n-1}} \left(\int_0^{\min\{r,d_\xi\}} \det(\mathbb{A}(t,\xi))dt \right) d\sigma, \end{aligned} \tag{3.3}$$

where $d\sigma$ denotes the $(n-1)$-dimensional volume element on $\mathbb{S}^{n-1} \equiv S_x^{n-1} \subseteq T_xM$. As in Section 1, let $r(z) = d(x,z)$ be the intrinsic distance to the point $x \in M$. Since for any $\xi \in S_x^{n-1}$ and $t_0 > 0$, we have $\nabla r(\gamma_\xi(t_0)) = \gamma_\xi'(t_0)$ when the point $\gamma_\xi(t_0) = \exp_x(t_0\xi)$ is away from the cut locus of x (cf. [16]), then, by the definition of a non-zero tangent vector "*radial*" to a prescribed point on a manifold given in the first page of [18], we know that for $z \in M\backslash(Cut(x) \cup x)$ the unit vector field

$$v_z := \nabla r(z)$$

is the radial unit tangent vector at z. We also need the following fact about $r(z)$ (cf. Prop. 39 on p. 266 of [31]),

$$\partial_r \Delta r + \frac{(\Delta r)^2}{n-1} \leq \partial_r \Delta r + |\mathrm{Hess} r|^2 = -\mathrm{Ric}(\partial_r, \partial_r), \quad \text{with } \Delta r = \partial_r \ln(\sqrt{|g|}),$$

with $\partial_r = \nabla r$ as a differentiable vector (cf. Prop. 7 on p. 47 of [31] for the differentiation of ∂_r), where Δ is the Laplace operator on M and $\mathrm{Hess} r$ is the

Hessian of $r(z)$. Then, together with (3.2), we have

$$J'' + \frac{1}{(n-1)}\mathrm{Ric}\left(\gamma'_\xi(t), \gamma'_\xi(t)\right) J \leq 0, \quad (3.4)$$
$$J(t,\xi) = t + O(t^2), \quad J'(t,\xi) = 1 + O(t).$$

3.2. A volume comparison theorem in smooth metric measure spaces

We also need the following volume comparison theorem proven by Wei and Wylie (cf. [35, Theorem 1.2]) which is the key point to prove Theorem 1.3.

Theorem 3.1 ([35]). *Let $(M, g, e^{-f}dv_g)$ be an n-dimensional $(n \geq 2)$ complete smooth metric measure space with $\mathrm{Ric}_f \geq (n-1)H$. Fix $x_0 \in M$. If $\partial_t f \geq -a$ along all minimal geodesic segments from x_0, then for $R \geq r > 0$ (assume $R \leq \pi/2\sqrt{H}$ if $H > 0$),*

$$\frac{\mathrm{Vol}_f[B(x_0,R)]}{\mathrm{Vol}_f[B(x_0,r)]} \leq e^{aR}\frac{\mathrm{Vol}^n_H(R)}{\mathrm{Vol}^n_H(r)},$$

where $\mathrm{Vol}^n_H(\cdot)$ is the volume of the geodesic ball with the prescribed radius in the space n-form with constant sectional curvature H, and, as before, $\mathrm{vol}_f(\cdot)$ denotes the weighted (or f-)volume of the given geodesic ball on M. Moreover, equality in the above inequality holds if and only if the radial sectional curvatures are equal to H and $\partial_t f \equiv -a$. In particular, if $\partial_t f \geq 0$ and $\mathrm{Ric} \geq 0$, then M has f-volume growth of degree at most n.

3.3. Proof of Theorem 1.3

Proof. For the complete and *non-compact* smooth metric measure n-space $(M, g, e^{-f}dv_g)$, if $\partial_t f \geq 0$ (along all minimal geodesic segments from x_0) and $\mathrm{Ric}_f \geq 0$, then by Theorem 3.1 we have

$$\frac{\mathrm{Vol}_f[B(x_0,R)]}{\mathrm{Vol}_f[B(x_0,r)]} \leq e^{0\cdot R}\cdot\frac{V_0(R)}{V_0(r)} = \left(\frac{R}{r}\right)^n,$$

where, as before, $V_0(\cdot)$ denotes the volume of the ball with the prescribed radius in $\mathbb{R}^n$. Clearly, here the volume doubling condition (1.5) is satisfied with $C_0 = 1$.

For $(M, g, e^{-f}dv_g)$, in order to apply Theorem 1.1 to prove Theorem 1.3, we need to normalize the original measure $e^{-f}dv_g$ such that the volume condition (1.6) can be satisfied. In fact, we need to choose the positive measure μ to be $\mu = e^{f(x_0)-f}dv_g$. Then by applying (1.8), (3.2), (3.3) and (3.4), we can get

$$\begin{aligned}\lim_{r\to 0}\frac{\mu(B(x_0,r))}{\mu_E(\mathbb{B}_n(r))} &= \lim_{r\to 0}\frac{e^{f(x_0)}\cdot\mathrm{Vol}_f[B(x_0,r)]}{V_0(r)}\\ &= \lim_{r\to 0}\frac{\int_{\mathbb{S}^{n-1}}\left(\int_0^{\min\{r,d_\xi\}} J^{n-1}(t,\xi)\cdot e^{-f}dt\right)d\sigma}{e^{-f(x_0)}\int_{\mathbb{S}^{n-1}}\int_0^r t^{n-1}dtd\sigma}\end{aligned}$$

$$= \frac{J'(0,\xi)\cdot e^{-f(x_0)}}{e^{-f(x_0)}} = 1$$

by applying L'Hôpital's rule n-times, which implies (1.6) is satisfied. Therefore, if in addition the Hardy type inequality (1.9) is satisfied, then by applying Theorem 1.1, we can get

$$\mu(B(x_0,\rho)) = w_n\rho^n = \mu_E(\mathbb{B}_n(\rho))$$

for all $\rho > 0$. Furthermore, together with Theorem 3.1, we know that the *radial* sectional curvatures of M are equal to 0 and $f \equiv f(x_0)$. Applying generalized Bishop-type volume comparisons (cf. [15, Theorem 3.3, Corollary 3.5 and Theorem 4.2]) for complete manifolds with radial curvature bounded, it follows that $B(x_0,\rho)$ is isometric to $\mathbb{B}_n(\rho)$ for all $\rho > 0$, which implies that (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. This completes the proof of Theorem 1.3. □

Remark 3.2. The Bishop-type volume comparisons [15, Theorem 3.3, Corollary 3.5 and Theorem 4.2] have been improved to more general versions by Mao [28].

4. Proofs of Theorem 1.5 and Theorem 1.7

In this section, we firstly recall some notions in Finsler geometry. For more details, we refer readers to [8].

4.1. Basic notions and useful facts from Finsler geometry

Let X be a connected n-dimensional C^∞ manifold and $TX = \bigcup_{x\in X} T_xX$ be its tangent bundle. The pair (X, F) is a *Finsler manifold* if the continuous function $F : TX \to [0, +\infty)$ satisfies the following conditions:

(i) $F \in C^\infty(TX \setminus \{0\})$;

(ii) $F(x, tv) = tF(x, v)$ for all $t \geq 0$ and $(x, v) \in TX$;

(iii) The $n\times n$ matrix $g_{ij}(x,v) := \frac{1}{2}\frac{\partial^2(F^2)}{\partial v^i\partial v^j}(x,v)$, where $v = v^i\frac{\partial}{\partial x^i}$, is positive definite for all $(x, v) \in TX \setminus \{0\}$.

We note that: if $F(x, tv) = |t|F(x, v)$ for all $t \in \mathbb{R}$ and $(x, v) \in TM$, then (M, F) is a *reversible Finsler manifold*; if $g_{ij}(x) = g_{ij}(x, v)$ is independent of v, then (X, F) degenerates into a Riemannian manifold. A *Minkowski space* consists of a finite dimensional vector space V and a Minkowski norm which induces a Finsler metric on V by translation (i.e., $F(x, v)$ is independent of x). A Finsler manifold (X, F) is called a *locally Minkowski space* if any point in X admits a local coordinate system $\{x^i\}$ on its neighborhood such that $F(x, v)$ depends only on v and not on x. *Berwald spaces* are just a bit more general than Riemannian and locally Minkowskian. A Berwald space is that all its tangent spaces are linearly isometric to a common Minkowski space. One might say that the Berwald space in question is modeled on a single Minkowski space.

Let $\sigma : [0, r] \to X$ be a piecewise C^∞ curve, then the integral length of σ is given by $L_F(\sigma) := \int_0^r F(\sigma, \dot\sigma)dt$. For $x_1, x_2 \in X$, denote by $\Lambda(x_1, x_2)$ the set of all piecewise C^∞ curves $\sigma : [0, r] \to M$ such that $\sigma(0) = x_1, \sigma(r) =$

x_2. Define the *distance function* $d_F : X \times X \to [0, +\infty)$ by $d_F(x_1, x_2) = \inf_{\sigma \in \Lambda(x_1,x_2)} L_F(\sigma)$. Clearly, $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$. Moreover, d_F satisfies the triangle inequality, but $d_F(x_1, x_2)$ can be different from $d_F(x_2, x_1)$. The open forward metric ball with center $x_0 \in X$ and radius $\rho > 0$ is defined by $B(x_0, \rho) = \{x \in X : d_F(x_0, x) < \rho\}$. Note that usually $\{x \in X : d_F(x, x_0) < \rho\}$ is different from $B(x_0, \rho)$. In particular, when $(X, F) = (\mathbb{R}^N, F)$ is a Minkowski space, one can obtain $d_F(x_1, x_2) = F(x_2 - x_1)$.

A smooth curve $\sigma : [0, l) \to X$ is called a *geodesic* if it locally minimizes d_F and has a constant speed (i.e., $F(\sigma, \dot{\sigma})$ is constant). The geodesic (Euler-Lagrange) equation can be written down in terms of covariant derivative along σ (see [8] for the details). The Finsler manifold (X, F) is complete if any geodesic $\sigma : [0, l) \to X$ can be extended to a geodesic $\sigma : \mathbb{R} \to X$.

The *polar transform* (or the *dual norm*) of F is defined by

$$F^*(x, \alpha) := \sup_{v \in T_x X \setminus \{0\}} \frac{\alpha(v)}{F(x, v)}$$

for every $(x, \alpha) \in T^*X$, where, as usual, T^*X is the dual space of TX. For every $x \in X$, the function $F^*(x, \cdot)$ is a Minkowski norm on the cotangent space T_x^*X. In particular, if $(\mathbb{R}^n, F)$ is a Minkowski space, so is $(\mathbb{R}^n, F^*)$. For $u(x) = d_F(x_0, x)$ with some fixed $x_0 \in X$, one can easily see that $F^*(x, Du(x)) = 1$ for a.e. $x \in X$.

Let $v \in T_x X$. Define a map $R^v : T_x X \to T_x X$ by $R^v(u) = R(U, V)V$, where $U = (v; u) \in \pi^* TX$, $V = (v; v) \in \pi^* TX$. In fact, R^v is called the curvature tensor (see [8] for the details). Let σ_v be the geodesic such that $\sigma_v(0) = x$ and $\dot{\sigma}_v(0) = v$. A vector field J along σ_v is a *Jacobi field* if it satisfies the Jacobi equation

$$D_{\dot{\sigma}_v} D_{\dot{\sigma}_v} J + R^{\dot{\sigma}_v}(J, \dot{\sigma}_v)\dot{\sigma}_v = 0, \tag{4.1}$$

where $D_{\dot{\sigma}_v}$ is the covariant derivative with respect to the vector $\dot{\sigma}_v$. Let $u, v \in T_x X$ be two linearly independent vectors and $\mathcal{S} = \mathrm{span}\{u, v\} \subset T_x X$. The *flag curvature* of the *flag* $\{\mathcal{S}; v\}$ is defined by

$$K(\mathcal{S}; v) = \frac{g_v(R^v(U, V)V, U)}{g_v(V, V)g_v(U, U) - g_v(U, V)^2},$$

where g_v denotes the inner product induced by $g_{ij}(x, v)$. If (X, F) is a Riemannian manifold, then the flag curvature reduces to the well-known sectional curvature, which only depends on $\mathcal{S}$ (not on the choice of $v \in \mathcal{S}$).

Choose $v \in T_x X$ such that $F(x, v) = 1$ and let $\{e_i\}_{i=1,\ldots,N}$ with $e_N = v$ be an orthonormal basis of $(T_x X, g_v)$. Let $\mathcal{S}_i = \mathrm{span}\{e_i, v\}$, $i = 1, \ldots, N-1$. Then *Ricci curvature* $\mathrm{Ric} : TX \to \mathbb{R}$ is defined by $\mathrm{Ric}(v) = \sum_{i=1}^{N-1} K(\mathcal{S}_i; v)$. For $c \geq 0$, we set $\mathrm{Ric}(cv); = c^2 \mathrm{Ric}(v)$.

We also need the following definition of N-Ricci curvature associated with an arbitrary measure on X (see also, e.g., [20, 30] for this notion).

Definition 4.1. Let μ be a positive smooth measure on X. Given $v \in T_xX\backslash\{0\}$, let $\sigma : (-\epsilon, \epsilon) \to X$ be the geodesic with $\dot{\sigma} = v$ and decompose μ along σ as $\mu = e^{-\psi}\mathrm{Vol}_{\dot{\sigma}}$, where $\mathrm{Vol}_{\dot{\sigma}}$ is the volume element of the Riemannian structure $\langle , \rangle_{\dot{\sigma}}$. Then, for $N \in [n, \infty]$, the *N-Ricci curvature* Ric_N is defined by

$$\mathrm{Ric}_N(v) = \mathrm{Ric}(v) + (\psi \circ \sigma)''(0) - \frac{(\psi \circ \sigma)'(0)^2}{N - n},$$

where the third term is understood as 0 if $N = \infty$ or if $N = n$ with $(\psi\circ\sigma)'(0) = 0$, and as $-\infty$ if $N = n$ with $(\psi \circ \sigma)'(0) \neq 0$.

By applying the concept of the N-Ricci curvature Ric_N, Ohta [30] proved the following Bishop-Gromov type volume comparison result in the Finsler case.

Theorem 4.2 ([30, Theorem 7.3])**.** *Let (X, F, μ) be a complete n-dimensional Finsler manifold with nonnegative N-Ricci curvature. Then we have*

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq \left(\frac{R}{r}\right)^N \quad \textit{for every } x \in X, \textit{ and } 0 < r < R.$$

Moreover, if equality holds with $N = n$ for all $x \in X$ and $0 < r < R$, then any Jacobi field J along a geodesic σ has the form $J(t) = tP(t)$, where P is a parallel vector field along σ (i.e., $D^{\dot{\sigma}}_{\dot{\sigma}}P \equiv 0$).

4.2. Hardy type inequality on Minkowski space

Let (X, F) be a Finsler manifold and $u \in \mathrm{Lip}_0(M)$. The local Lipschitz constant of u is given by $|Du|(x) = F^*(x, Du(x))$ for a.e. $x \in X$. Therefore, due to density reasons, for any $u \in C_0^\infty(X)$, the Hardy type inequality (1.4) in the Finsler context has the following form

$$\left(\int_X |u(x)|^p d_F(x_0, x)^{\gamma p} d\mu(x)\right)^{\frac{1}{p}} \leq C \left(\int_X F^*(x, Du(x))^q d\mu(x)\right)^{\frac{1}{q}}. \tag{4.2}$$

We will prove that the sharp Hardy-type inequality (i.e., $C = K(n, q, \gamma)$) holds on an arbitrary Minkowski space $(\mathbb{R}^n, F)$ endowed with the Lebesgue measure μ_F normalized so that $\mu_F(B(0, 1)) = w_n$.

Let $\Omega \subset \mathbb{R}^n$ be a measurable set. Denote by $\Omega^\star$ the *anisotropic symmetrization* of Ω, i.e., it is the open ball with center 0 such that $\mu_F(\Omega^\star) = \mu_F(\Omega)$. For a function $u : \mathbb{R}^n \to \mathbb{R}$, $u^\star(x) := \sup\{c \in \mathbb{R} : x \in \{x \in \mathbb{R}^n : u(x) > c\}^\star\}$ is the *anisotropic* (*decreasing*) *symmetrization* of u. Due to Alvino-Ferone-Lions-Trombetti [2, Theorem 3.1] and Van Schaftingen [33, Proposition 2.28], we have following.

(1) *anisotropic Pólya-Szegö inequality*: for all $u \in C_0^\infty(\mathbb{R}^n)_+$ and $q \geq 1$,

$$\int_{\mathbb{R}^n} F^*(Du^\star(x))^q d\mu(x) \leq \int_{\mathbb{R}^n} F^*(Du(x))^q d\mu(x); \tag{4.3}$$

(2) *anisotropic Hardy-Littlewood inequality*: for all $u \in C_0^\infty(\mathbb{R}^n)_+$, $q \geq 1$ and $\gamma \in [-1, 0]$,

$$\int_{\mathbb{R}^n} u(x)^p F(x)^{\gamma p} dm \leq \int_{\mathbb{R}^n} u^\star(x)^p F(x)^{\gamma p} d\mu(x), \tag{4.4}$$

where $C_0^\infty(\mathbb{R}^n)_+ := \{u \in C_0^\infty(\mathbb{R}^n) : u \geq 0\}$.

Applying the above two functional inequalities, we can obtain the following.

Theorem 4.3. *Let* $(\mathbb{R}^n, F)$ *be a Minkowski space with* $n \geq 3$, $x_0 \in \mathbb{R}^n$, *and* $1 < q < n$, $q < p \leq \frac{nq}{n-q}$, $\gamma = -1 + n\left(\frac{1}{q} - \frac{1}{p}\right)$. *Then the sharp Hardy type inequality*

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} |u(x)|^p d_F(x_0, x)^{\gamma p} d\mu_F(x)\right)^{\frac{1}{p}} \\ &\leq K(n, q, \gamma) \left(\int_{\mathbb{R}^n} F^*(x, Du(x))^q d\mu_F(x)\right)^{\frac{1}{q}} \end{aligned} \tag{4.5}$$

holds on $(\mathbb{R}^n, F, \mu_F)$. *Moreover, the constant* $K(n, q, \gamma)$ *is optimal and a family of extremals is given by*

$$u_\lambda(x) = \left(\lambda + d_F(x_0, x)^{\frac{q+\gamma p}{q-1}}\right)^{\frac{q-n}{q+\gamma p}}, \qquad \lambda > 0.$$

Proof. Without loss of generality, we may assume that $x_0 = 0$. Then $d_F(x_0, x) = F(x - x_0) = F(x)$. Consider the constant

$$C_a = \inf_{u \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} F^*(x, Du(x))^q d\mu_F(x)\right)^{\frac{1}{q}}}{\left(\int_{\mathbb{R}^n} |u(x)|^p F(x)^{\gamma p} d\mu_F(x)\right)^{\frac{1}{p}}}.$$

We claim that $C_a = K(n, q, \gamma)^{-1}$. In fact, by the reversibility of F, it is sufficient to consider only non-negative functions in the above equality. By (4.3) and (4.4), we can obtain

$$C_a = \inf_{u \in C_0^\infty(\mathbb{R}^N)_+ \setminus \{0\}} \frac{\left(\int_{\mathbb{R}^n} F^*(x, Du^\star(x))^q d\mu_F(x)\right)^{\frac{1}{q}}}{\left(\int_{\mathbb{R}^n} |u^\star(x)|^p F(x)^{\gamma p} d\mu_F(x)\right)^{\frac{1}{p}}}.$$

We may assume that $u^\star \in C_0^1(\mathbb{R}^n)_+$, otherwise a density argument applies. Then there exists a non-increasing function $h : [0, +\infty) \to [0, +\infty)$ of class C^1 such that $u^\star(x) = h(F(x))$. Furthermore, it is not hard to get

$$F^*(Du^\star(x)) = F^*(h'(F(x)))DF(x) = -h'(F(x))F^*(DF(x)) = -h'(F(x)).$$

By a direct computation, we can obtain

$$\frac{\left(\int_{\mathbb{R}^n} F^*(x, Du^\star(x))^q d\mu_F(x)\right)^{\frac{1}{q}}}{\left(\int_{\mathbb{R}^n} |u^\star(x)|^p F(x)^{\gamma p} d\mu_F(x)\right)^{\frac{1}{p}}} = \alpha_n^{\frac{1}{q} - \frac{1}{p}} \frac{\left(\int_0^\infty h'(\rho)^q \rho^{n-1} d\rho\right)^{\frac{1}{q}}}{\left(\int_0^\infty h(\rho)^p \rho^{n-1+\gamma p} d\rho\right)^{\frac{1}{p}}}, \tag{4.6}$$

where $\alpha_n = n w_n$ is the area of the unit sphere in $\mathbb{R}^n$. On the other hand, using the approaches of proofs of the Hardy inequality in the Euclidean case (see

[11, 23, 24]), one can see that the minimizing expression is precisely the right hand side of (4.6). Therefore, we have $C_a = K(n,q,\gamma)^{-1}$, which proves our claim. Moreover, by the standard Euler-Lagrange method, a class of minimizers h_λ for (4.6) is $h_\lambda(\rho) = \left(\lambda + \rho^{\frac{q+\gamma p}{q-1}}\right)^{\frac{q-n}{q+\gamma p}}$, $\lambda > 0$. This completes the proof of Theorem 4.3. □

4.3. Proofs of Theorems 1.5 and 1.7

Proof of Theorem 1.5. Since (X,F) is complete, we know that (X, d_F, μ) is a *proper* metric measure space by applying the Hopf-Rinow theorem. Since the n-Ricci curvature Ric_n is nonnegative, by Theorem 4.2, we can obtain (1.5) with $C_0 = 1$. As pointed out in Remark 1.2(6), one can normalize the fixed positive measure μ such that (1.6) is satisfied. Then by these two facts, as mentioned in Remark 1.2(4), we can get

$$\mu(B(x,\rho)) \le \mu_E(\mathbb{B}_n(\rho)) \quad \text{for all } \rho > 0,\ x \in X.$$

However, since the Hardy type inequality (1.4) is satisfied with the best constant (i.e., $C = K(n,q,\gamma)$), by Theorem 1.3, we have

$$\mu(B(x,\rho)) \ge \mu_E(\mathbb{B}_n(\rho)) \quad \text{for all } \rho > 0,\ x \in X.$$

Therefore, $\mu(B(x,\rho)) = \mu_E(\mathbb{B}_n(\rho))$ for all $\rho > 0$ and $x \in X$. By applying Theorem 4.2 directly, we know that every Jacobi field J along a geodesic σ has the form $J(t) = tP(t)$, where P is a parallel vector field along σ. Together with the Jacobi equation (4.1), it follows that $R^{\dot\sigma}(J,\dot\sigma)\dot\sigma \equiv 0$. Then $K(\mathcal{S};\dot\sigma) \equiv 0$ with $\mathcal{S} = \mathrm{span}(\dot\sigma, P)$. Since σ and J are arbitrary, we know $K \equiv 0$, which equivalently says that the flag curvature of (X,F) is identically zero. □

Proof of Theorem 1.7. On one hand, since on every Berwald space $\mathrm{Ric}_n = \mathrm{Ric}$ holds for the Hausdorff measure μ_F (cf. [32, Propositions 2.6, 2.7]), we can apply Theorem 1.5 to get that the flag curvature of (X,F) is identically zero. On the other hand, every Berwald space with zero flag curvature is necessarily a locally Minkowski space (cf. [8, Section 10.5]). Due to the volume identity $\mu_F(B(x,\rho)) = \mu_E(\mathbb{B}_n(\rho))$ for all $\rho > 0$, we know that (X,F) must be isometric to a Minkowski space. □

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