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MULTIPLE EXISTENCE OF SOLUTIONS FOR A NONHOMOGENEOUS ELLPITIC PROBLEMS ON R^N

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Abstract. Let $N \geq 3$, $2^* = 2N/(N-2)$ and $p \in (2, 2^*)$. Our purpose in this paper is to consider multiple existence of solutions of problem

$$-\Delta u - \frac{\mu}{|x|^2} + \alpha u \quad = \quad |u|^{p-2} u + \lambda f \qquad\qquad u \in H^1\left(\mathbb{R}^N\right),$$

where $a, \lambda > 0$, $\mu \in \left(0, (N-2)^2/4\right)$ $f \in H^{-1}\left(\mathbb{R}^N\right)$, $f \geq 0$ and $f \not\equiv 0$.

1. Introduction

Let $N \geq 3$, $f \in L^2(\mathbb{R}^N)$ with $f \geq 0$ and $f \not\equiv 0$, and $p \in (2,\ 2^*)$, where $2^* = 2N/(N-2)$. Nonhomogeneous problem

$$\begin{cases} -\Delta u + au &= u^{p-1} + \lambda f \qquad \text{in } \mathbb{R}^N \\ u &> 0 \qquad \text{in } \mathbb{R}^N \\ u &\in H^1(\mathbb{R}^N) \end{cases} \tag{P_0}$$

has been investigated by many authors. Here $a \geq 0$ and $\lambda > 0$. On the multiple existence of solutions, Zhu[7] proved the existence of two positive solutions $u_1, u_2 \in H^1(\mathbb{R}^N)$ of problem (P) for f sufficiently small and having an exponental decay. The first solution u_1 is close to 0 and the second solution u_2 is obtained by mountain path argument. This result was improved in [4] and [5]. In [2] and [6], it was shown that there exists $M > 0$ sych that for each $f \in H^{-1}\left(\mathbb{R}^N\right)$ satisfying $\|f\|_{H^{-1}} < M$, $f \geq 0$, $f \not\equiv 0$, problem (P) possesses at least two positive solutions. That is norm $\|f\|_{H^{-1}}$ determines the mountain pass structure and causes multiple existence of the solutions. It is interesting what the nature of function f affects the multiplicity of the solutions. In [2,4,5,6,8], the authors investigated that some profiles of function f cause multiplicity of the solutions.

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Our purpose in the present paper is to consider the multiplicity of solutions of problem

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} + au = |u|^{p-2} u + \lambda f(\cdot - \eta e) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{P}$$

where $0 < \mu < \overline{\mu} = (N-2)^2/4$, $e \in \mathbb{R}^N$ with $|e| = 1$ and $\eta \in \mathbb{R}$. For problem (P_0), transition of f does not give any effect to the number of solutions, i.e. problem (P_0) is equivarent to problem (P_0) with f replaced $f(\cdot - \eta e)$ for any $e \in \mathbb{R}^N$ with $|e| = 1$ and $\eta \in \mathbb{R}$. We will show, under the presence of Hardy term, the effect of the translation of f and $\|f\|_{H^{-1}(\mathbb{R}^N)}$ on the multiplicity of solutions of (P).

Our main result is as follows:

Theorem 1.1. *There exist $\lambda_0 > 0$ and $\eta_0 > 0$ such that for each $\mu \in (0, \overline{\mu})$, the followings hold;*

(1) *for each $\lambda \in (0, \lambda_0)$ and $\eta \in \mathbb{R}$, problem (P) has at least two solutions;*

(2) *for each $\lambda \in (0, \lambda_0)$ and $\eta \in \mathbb{R} \setminus (-\eta_0, \eta_0)$, problem (P) has at least four solutions.*

2. Preliminaries

We denote by $B_r(x)$ the open ball in $\mathbb{R}^N$ centered at x and radius r. For each $q \in [1, \infty]$, we denote by $|\cdot|_q$ the norm of $L^q(\mathbb{R}^N)$. For simplicity we put $H = H^1(\mathbb{R}^N)$. For $u, v \in H$, we put $\langle u, v\rangle = \int_{\mathbb{R}^N} uv\, dx$. We denote by $\|\cdot\|_0$ the norm of H defined by $\|v\|_0^2 = |\nabla v|_2^2 + a|v|_2$ for $v \in H$. Put $u^+ = \max\{0, u\}$ and $u^- = \min\{0, u\}$ for $u \in H$. We denote by H^+ a subset defined by

$$H^+ = \left\{v \in H : v^+ \not\equiv 0\right\}.$$

We denote by $\nabla F : H \to H$ the gradient of the functional F. First, we retrieve some known results for the homogeneous problem

$$\begin{array}{rcll} -\Delta v + av & = & |v|^{p-1} v & v \in H. \\ v & > & 0 & \text{on } \mathbb{R}^N. \end{array} \tag{1}$$

We denote by $I^0 : H \to \mathbb{R}$ the functional associated with the problem (1), i.e.,

$$I^0(v) = \frac{1}{2}\|v\|_0^2 - \frac{1}{p}\left|v^+\right|_p^p \qquad \text{for all } v \in H.$$

We put

$$\mathcal{M}^0 = \left\{v \in H^1(\mathbb{R}) \setminus \{0\} : \|v\|_0^2 = \left|v^+\right|_p^p\right\}$$

and

$$c^0 = \inf\left\{I^0(v) : v \in \mathcal{M}^0\right\}.$$

It is known that problem (1) has a radial solution $U_0 \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$. The solution U_0 is the unique positive solution up to translation on $\mathbb{R}^N$. Moreover U_0 is the least energy solution of (1), i.e., $I^0(U_0) = c^0$. We put $S^0 = |U_0|_p^p$.

That is $S^0 = \frac{2p}{p-1}c^0$. Let $x \in \mathbb{R}^N$. We denote by U_x the function defined by $U_x(\cdot) = U_0(\cdot - x)$. Then each U_x is a solution of problem (1).

Let $\mu \in (0, \overline{\mu})$ and put

$$\|v\|_\mu^2 = |\nabla v|_2^2 - \mu \int \frac{|v|^2}{|x|^2} \qquad \text{for } v \in H^1(\mathbb{R}^N).$$

Then $\|\cdot\|_\mu$ is an equivarent norm with $\|\cdot\|_0$(cf. [7]). It is known the homogeneous problem

$$(2) \qquad -\Delta u - \mu \frac{u}{|x|^2} + au = |u|^{p-2} u, \qquad u \in H$$
$$u > 0 \qquad \text{on } \mathbb{R}^N$$

has a unique solution V_0. The associated functional I^μ of (2) is

$$I^\mu(v) = \frac{1}{2}\|v\|_\mu^2 - \frac{1}{p}\left|v^+\right|_p^p, \qquad v \in H.$$

We put

$$\mathcal{M}^\mu = \left\{ v \in H^1(\mathbb{R}) \setminus \{0\} : \|v\|_\mu^2 = \left|v^+\right|_p^p \right\}.$$

It is obvious that $c_\mu < c_0$, where

$$c_\mu = I^\mu(V_0) = \inf\left\{ I^\mu(v) : v \in \mathcal{M}^\mu \right\}.$$

.

Next we define the functional associated with the problem (P) by

$$I_{\lambda,\eta}^\mu(v) = \frac{1}{2}\|v\|_\mu^2 - \frac{1}{p}\left|v^+\right|_p^p - \lambda \langle v, f(\cdot - \eta)\rangle, \qquad \text{for } v \in H.$$

We also set

$$\mathcal{M}_{\lambda,\eta}^\mu = \left\{ v \in H^1(\mathbb{R}) \setminus \{0\} : \|v\|_\mu^2 = \left|v^+\right|_p^p + \lambda \langle v, f(\cdot - \eta)\rangle \right\}.$$

One can see that each nontrivial critical point is contained in $\mathcal{M}_{\lambda,\eta}^\mu$. Similarily, we put

$$I_\lambda^0(v) = \frac{1}{2}\|v\|_0^2 - \frac{1}{p}\left|v^+\right|_p^p - \lambda \langle v, f\rangle, \qquad \text{for } v \in H$$

and

$$\mathcal{M}_\lambda^0 = \left\{ v \in H^1(\mathbb{R}) \setminus \{0\} : \|v\|_0^2 = \left|v^+\right|_p^p + \lambda \langle v, f\rangle \right\}.$$

One can see that each critical point $u \in H^1(\mathbb{R}^N)$ of I_λ^0 is a solution of problem

$$\begin{cases} -\Delta u + au = |u|^{p-2} u + \lambda f & \text{in } \mathbb{R}^N, \\ u > 0 & \\ u \in H & \end{cases}$$

For each $\lambda > 0$ and $v \in H^+$, we put

$$g_{\lambda,v}(t) = \frac{d}{dt} I_\lambda^0(tv) = t^2 \|v\|_0^2 - t^p |v|_p^p - t\langle f, v\rangle, t > 0. \tag{3}$$

Then there exists $\overline{\lambda} > 0$ such that if $\lambda \in (0, \overline{\lambda})$ and $v \in H^+$, there exist $t_{\lambda,v,-}$, $t_{\lambda,v,+} > 0$ such that $0 < t_{\lambda,v,-} < t_{\lambda,v,+}$ and $g_{\lambda,v}(t_{\lambda,v,-}) = g_{\lambda,v}(t_{\lambda,v,+}) = 0$. That is $\mathcal{M}^0_\lambda = \mathcal{M}^0_{\lambda,-} \cup \mathcal{M}^0_{\lambda,+}$, where

$$\mathcal{M}^0_{\lambda,-} = \left\{t_{\lambda,v,-}v : v \in H^+\right\} \text{ and } \mathcal{M}^0_{\lambda,+} = \left\{t_{\lambda,v,+}v : v \in H^+\right\}.$$

One can see $I^0_\lambda(t_{\lambda,v,+}v) = \max_{t>0} I^0_\lambda(tv)$ and $I^0_\lambda(t_{\lambda,v,-}v) = \min_{0<t<t_{\lambda,v,+}} I^0_\lambda(tv)$. We also have

$$c^0_{\lambda,-} < 0 < c^0_{\lambda,+} < c^0_{\lambda,-} + c_0, \tag{4}$$

where

$$c^0_{\lambda,+} = \inf\left\{I^0_\lambda(v) : v \in \mathcal{M}^0_{\lambda,+}\right\} \text{ and } c^0_{\lambda,-} = \inf\left\{I^0_\lambda(v) : v \in \mathcal{M}^0_{\lambda,-}\right\}$$

(cf.[2], [6]).

It follows from [5], [4] that problem (3) has solutions $u_{\lambda,+} \in \mathcal{M}^0_{\lambda,+}$, $u_{\lambda,-} \in \mathcal{M}^0_{\lambda,-}$ such that

$$I^0_\lambda(u_{\lambda,+}) = c^0_{\lambda,+} \text{ and } I^0_\lambda(v) = c^0_{\lambda,-}.$$

Lemma 2.1.

$$\lim_{\lambda\to 0} \sup\left\{|v|_p : v \in \mathcal{M}^0_{\lambda,-}\right\} = 0$$

Proof. Let $v \in \mathcal{M}^0$. Then by [5], [4] we have that for $\lambda \in (0, \overline{\lambda})$, there exists $t > 0$ such that $tv \in \mathcal{M}^0_{\lambda,-}$. From the equation

$$t^2 \|v\|^2_\mu = t^p \left|v^+\right|^p_p + t\lambda\langle f, v\rangle,$$

one can see that $t = t_{\lambda,v,-} \to 0$ as $\lambda \to 0$. Since

$$\left|t\left(1 - t^{p-2}\right)\right| = \lambda\frac{|\langle f, v\rangle|}{\|v\|^2_0} \le \lambda\frac{\|f\|_{H^{-1}}}{\|v\|_0},$$

we have

$$|tv|_p \le 2\lambda\frac{\|f\|_{H^{-1}}|v|_p}{\|v\|_0} \le 2C\lambda\|f\|_{H^{-1}} \text{ for } \lambda \text{ sufficiently small,}$$

where $C > 0$ is a constant such that $|w|_p \le C\|w\|_0$ for $w \in H$. Since $v \in H^+$ is arbitrary, the assertion follows. □

For each $\mu \in (0, \overline{\mu}), \lambda > 0, \eta > 0$ and $v \in H^+$, we put

$$g_{\mu,\lambda,\eta,v}(t) = \frac{d}{dt} I^\mu_{\lambda,\eta}(tv) = t^2\|v\|^2_0 - t^p|v|^p_p - t\langle f(\cdot - \eta e), v\rangle, t > 0. \tag{5}$$

Then there exists $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0)$ and $v \in H^+$, there exist $t_{\mu,\lambda,\eta,v,-}$, $t_{\mu,\lambda,\eta,v,+} > 0$ (simply we write $t_{v,-}, t_{v,+}$) such that $0 < t_{v,-} < t_{v,+}$ and $g_{\mu,\lambda,\eta,v}(t_{v,-}) = g_{\mu,\lambda,\eta,v}(t_{v,+}) = 0$. That is $\mathcal{M}^\mu_{\lambda,\eta} = \mathcal{M}^\mu_{\lambda,\eta,-} \cup \mathcal{M}^\mu_{\lambda,\eta,+}$, where

$$\mathcal{M}^\mu_{\lambda,\eta,-} = \left\{t_{v,-}v : v \in H^+\right\} \text{ and } \mathcal{M}^\mu_{\lambda,\eta,+} = \left\{t_{v,+}v : v \in H^+\right\}.$$

By a slight modification of the arguments in [2] and [6], we can prove the follow theorem:

Theorem 2.2. *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \overline{\mu})$,*

$$0 < c^{\mu}_{\lambda,\eta,+} < c^{\mu}_{\lambda,\eta,-} + c_0, \tag{6}$$

where

$$c^{\mu}_{\lambda,\eta,+} = \inf \left\{ I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+} \right\} \text{ and } c^{\mu}_{\lambda,\eta,-} = \inf \left\{ I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,-} \right\}.$$

The proof of Theorem 2.2 is almost same as the proof for (4). Then we omit the proof. By (6), we have that for $\lambda \in (0, \lambda_0)$, $\mu \in (0, \overline{\mu})$ and $\eta \in \mathbb{R}$, there exist solutions $u_{\mu,\lambda,\eta,-}, u_{\mu,\lambda,\eta,+} \in H$ of (P) such that $I^{\mu}_{\lambda,\eta}(u_{\mu,\lambda,\eta,-}) = c^{\mu}_{\lambda,\eta,-}$ and $I^{\mu}_{\lambda,\eta}(u_{\mu,\lambda,\eta,+}) = c^{\mu}_{\lambda,\eta,+}$. We also have that (6) implies

Lemma 2.3. *For $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \overline{\mu})$,*

$$\lim_{\eta \to \infty} c^{\mu}_{\lambda,\eta,-} = c^0_{\lambda,-} \qquad \text{and} \qquad \lim_{\eta \to \infty} c^{\mu}_{\lambda,\eta,+} = \min \left\{ c^0_{\lambda,+}, c^{\mu} + c^0_{\lambda,-} \right\}.$$

Proof. Let $\lambda \in (0, \lambda_0)$ and $\mu \in (0, \overline{\mu})$. Let $\eta > 0$ and $v_\eta \in H$ be a solution of (P). Then for given $\varphi \in C^1_0(\mathbb{R}^N)$,

$$\lim_{\eta \to \infty} \left\langle -\Delta v_\eta - \frac{\mu}{|x|^2} v_\eta - |v_\eta|^{p-2} v_\eta - f(\cdot - \eta e), \varphi \right\rangle = \lim_{\eta \to \infty} \left\langle -\Delta v_\eta - \frac{\mu}{|x|^2} v_\eta - |v_\eta|^{p-2} v_\eta, \varphi \right\rangle$$

and

$$\lim_{\eta \to \infty} \left\langle -\Delta v_\eta - \frac{\mu}{|x|^2} v_\eta - |v_\eta|^{p-2} v_\eta - f(\cdot - \eta e), \varphi(\cdot - \eta e) \right\rangle$$
$$= \lim_{\eta \to \infty} \left\langle -\Delta v_\eta - |v_\eta|^{p-2} v_\eta - f(\cdot - \eta e), \varphi(\cdot - \eta e) \right\rangle,$$

we find that v_η has the form $v_\eta = v_{\eta,1} + v_{\eta,2}$, where $\lim_{\eta \longrightarrow \infty} \nabla I^{\mu}(v_{\eta,1}) = 0$ and $\lim_{\eta \longrightarrow \infty} \nabla I^0_{\lambda}(v_{\eta,2}(\cdot - \eta e)) = 0$. Suppose that $I^{\mu}_{\lambda,\eta}(v_\eta) = c^{\mu}_{\lambda,\eta,-}$ for $\eta > 0$. By the minimality of $c^{\mu}_{\lambda,\eta,-}$, we have, noting that $I^{\mu}(v_{\eta,1}) \geq 0$, that $\lim_{\eta \longrightarrow \infty} \nabla I^{\mu}_{\lambda,\eta}(v_\eta) = \lim_{\eta \longrightarrow \infty} \nabla I^{\mu}_{\lambda,\eta}(v_{\eta,2}) = c^0_{\lambda,-}$. If $I^{\mu}_{\lambda,\eta}(v_\eta) = c^{\mu}_{\lambda,\eta,+}$ for $\eta > 0$, we find that (i) $\lim_{\eta \longrightarrow \infty} I^{\mu}(v_{\eta,1}) = 0$ and $\lim_{\eta \longrightarrow \infty} \nabla I^{\mu}_{\lambda,\eta}(v_{\eta,2}) = c^0_{\lambda,+}$, or (ii) $\lim_{\eta \longrightarrow \infty} I^{\mu}(v_{\eta,1}) = c^{\mu}$ and $\lim_{\eta \longrightarrow \infty} \nabla I^{\mu}_{\lambda,\eta}(v_{\eta,2}) = c^0_{\lambda,-}$. Then by the definition of $c^{\mu}_{\lambda,\eta}$, the assertion follows. □

3. Proof of Theorem

Throughout this section, we fix $\mu \in (0, \overline{\mu}.)$. Let $R_0 > 0$ such that

$$\int_{B_{R_0}(0)} |U_0|^p > \frac{2}{3} S^0 \text{ and } \int_{B_{R_0}(0)} |V_\mu|^p > \frac{2}{3} \int |V_\mu|^p. \tag{7}$$

For each $v \in L^p\left(\mathbb{R}^N\right)$, we set

$$\widehat{v}(x) = \int_{B_{R_0}(x)} |v|^p \qquad \text{for } x \in \mathbb{R}^N$$

and

$$\Omega(v) = \left\{ x \in \mathbb{R}^N : \widehat{v}(x) - \frac{|\widehat{v}|_\infty}{2} > 0 \right\}. \tag{8}$$

We also set

$$\beta(v) = \frac{\int_{\Omega(v)} x \left(\widehat{v}(x) - \frac{|\widehat{v}|_\infty}{2} \right)}{\int_{\Omega(v)} \left(\widehat{v}(x) - \frac{|\widehat{v}|_\infty}{2} \right)} \text{ for } v \in L^p\left(\mathbb{R}^N\right).$$

The mapping β is called generalized barycenter, which was introduced in [3](cf. also [1]).

By Lemma 2.1, we can choose $\lambda_0 > 0$ so small that that

$$\int |u_{\lambda,-}|^p < \frac{1}{3} \int |V_\mu|^p \qquad \text{for } \lambda \in (0, \lambda_0). \tag{9}$$

Lemma 3.1. (1) *There exists $R_1 > 0$ such that*

$$\beta\left(s_\eta V_\mu + u_{\lambda,-}(\cdot - \eta e)\right) \subset B_{R_1}(0) \tag{10}$$

and

$$\beta\left(t_\eta\left(u_{\lambda,+}(\cdot - \eta e)\right)\right) \subset B_{R_1}(\eta e), \tag{11}$$

where $s_\eta > 0$ such that $s_\eta\left(V_\mu + u_{\lambda,-}(\cdot - \eta e)\right) \in \mathcal{M}^\mu_{\lambda,\eta,+}$ for each $\eta > 0$, and $t_\eta > 0$ such that $t_\eta u_{\lambda,+}(\cdot - \eta e) \in \mathcal{M}^\mu_{\lambda,\eta,+}$ for $\eta > 0$.

(2) $\lim_{\eta \longrightarrow \infty} I^\mu_{\lambda,\eta}\left(s_\eta\left(V_\mu + u_{\lambda,-}(\cdot - \eta e)\right)\right) = c^\mu + c^0_{\lambda,-}$ *and* $\lim_{\eta \longrightarrow \infty} I^\mu_{\lambda,\eta}\left(t_\eta u_{\lambda,+}(\cdot + \eta e)\right) = c^0_{\lambda,+}$.

Proof. Let $\lambda \in (0, \lambda_0)$. For simplicity, we put $v_\eta = u_{\lambda,-}(\cdot - \eta e)$ for $\eta > 0$. Then

$$\|s_\eta V_\mu + v_\eta\|^2_\eta = \left|s_\eta V_\mu + v_\eta\right|^p_p + \lambda \langle f(\cdot - \eta e), s_\eta V_\mu + v_\eta \rangle \text{ for } \eta > 0.$$

Noting that $\inf_{\eta>0} \langle f(\cdot - \eta e), v_\mu \rangle > 0$,one can see $0 < \inf_{\eta>0} s_\eta < \sup_{\eta>0} s_\eta < \infty$. Then since $\|s_\eta V_\mu + v_\eta\|^2_\mu - \|s_\eta V_\mu\|^2_\mu - \|v_\eta\|^2_0 \to 0$, $\left|s_\eta V_\mu + v_\eta\right|^p_p - |s_\eta V_\mu|^p_p - \left|v_\eta\right|^p_p \to 0$ and $\langle f(\cdot - \eta e), V_\mu \rangle \to 0$, as $\eta \to \infty$,

$$\begin{aligned} &\left(\|s_\eta V_\mu\|^2_\mu + \|v_\eta\|^2_0\right) - \left(|s_\eta V_\mu|^p_p + \left|v_\eta\right|^p_p\right) - \lambda \langle f(\cdot - \eta e), v_\eta \rangle \\ &= s^2_\eta \|V_\mu\|^2_\mu - s^p_\eta |V_\mu|^p_p \to 0 \text{ as } \eta \to \infty. \end{aligned}$$

This implies $s_\eta \to 1$, as $\eta \to \infty$. Then by (8)

$$\left|(s_\eta \widehat{V_\mu + v_\eta})(x)\right|_\infty \geq \int_{B_{R_0}(0)} |s_\eta V_\mu + v_\eta|^p > \frac{2}{3} \int |V_\mu|^p$$

for η sufficiently large. On the other hand, we have by (9) that

$$(s_\eta \widehat{V_\mu + v}_\eta)(x) = \int_{B_{R_0}(x)} |s_\eta V_\mu + v_\eta|^p < \frac{1}{3}\int |V_\mu|^p$$

for all $\eta > 0$ and $|x|$ sufficiently large. That is there exists $R > 0$ such that

$$\Omega(s_\eta V_\mu + v_\eta) \subset B_R(0) \qquad \text{for } \eta \text{ sufficiently large.}$$

Then from the definition of β,

$$\beta(s_\eta V_\mu + v_\eta) \subset B_R(0) \qquad \text{for } \eta \text{ sufficiently large.}$$

Therefor by taking $R_1 > 0$ large, we obtain (10). By a parallel argument as above, we have $t_\eta \to 1$ as $\eta \to \infty$. Let $R > 0$ such that $\beta(u_{\lambda,+}) \subset B_R(0)$. Then we find

$$\beta(t_\eta u_{\lambda,+}(\cdot - \eta e)) \subset B_R(\eta e) \qquad \text{for } \eta \text{ sufficiently large.}$$

Therefore again by taking $R_1 > 0$ large, we obtain (11). □

Lemma 3.2. *For each* $\lambda \in (0, \lambda_0)$,

$$\left\{\beta(v) : v \in \mathcal{M}^\mu_{\lambda,\eta}, I^\mu_{\lambda,\eta}(v) < c^0 + c^\mu_{\lambda,\eta,-}\right\} = \mathbb{R}^N.$$

Proof. Let $\lambda \in (0, \lambda_0)$. To prove the assertion, it is sufficient to show

$$\sup_{t>0}\left\{I^\mu_{\lambda,\eta}(tU_x + u_{\mu,\lambda,\eta,-})\right\} < c^0 + c^\mu_{\lambda,\eta,-} \qquad \text{for all } x \in \mathbb{R}^N. \tag{12}$$

Let $x \in \mathbb{R}^N$. For simplicity, we put $U = U_x$, $u = u_{\mu,\lambda,\eta,-}$ and $f_\eta = f(\cdot - \eta e)$. From the definition of $I^\mu_{\lambda,\eta}$,

$$\begin{aligned} I^\mu_{\lambda,\eta}(tU + u) &= \frac{1}{2}\|tU + u\|^2_\mu - \frac{1}{p}(tU + u)^p - \langle f_\eta, tU + u\rangle \\ &\le \frac{1}{2}\|tU\|^2_0 + \frac{1}{2}\|u\|^2_\mu + \langle \nabla tU, \nabla u\rangle + a\langle tU, u\rangle - \frac{\mu}{|x|^2}\langle tU, u\rangle \\ &\quad - \frac{1}{p}(tU + u)^p - \langle f_\eta, tU + u\rangle. \end{aligned}$$

Then noting that

$$\begin{aligned} \langle \nabla tU, \nabla u\rangle + a\langle tU, u\rangle - \frac{\mu}{|x|^2}\langle tU, u\rangle - \langle f_\eta, tU\rangle &= \left\langle u^{p-1}, tU\right\rangle \\ &\le \int_{\{u>tU\}} tUu^{p-1} + \int_{\{u\le tU\}} (tU)^{p-1}u, \end{aligned}$$

$$\frac{1}{2}\|tU\|^2_0 - \frac{1}{p}(tU)^p \le c^0 \text{ and } \frac{1}{2}\|u\|^2_\mu - \frac{1}{p}u^p - \langle f_\mu, u\rangle = c^\mu_{\lambda,\eta,-},$$

we have

$$(13)\ I^{\mu}_{\lambda,\eta}(tU+u) \le c^0 + c^{\mu}_{\lambda,\eta,-} + \int \left(\frac{1}{p}(tU)^p + \frac{1}{p}u^p\right) + \int_{\{u>tU\}} tUu^{p-1} + \int_{\{u\le tU\}} (tU)^{p-1}u - \frac{1}{p}\int (tU+u)^p .$$

Noting $\frac{1}{p} < \frac{p-1}{2}$, we have by Taylar expansion that

$$(14)\qquad \int_{\{u>tU\}} \left(\frac{1}{p}u^p + \frac{1}{p}(tU)^p + tUu^{p-1} - \frac{1}{p}(tU+u)^p\right) = \int_{\{u>tU\}} \left(\frac{1}{p}(tU)^p - \frac{p-1}{2}(\theta U + (1-\theta)u)^{p-2}(tU)^2\right) < 0$$

and

$$(15)\qquad \int_{\{u<tU\}} \left(\frac{1}{p}u^p + \frac{1}{p}(tU)^p + (tU)^{p-1}u - \frac{1}{p}(tU+u)^p\right) = \int_{\{u>tU\}} \left(\frac{1}{p}u^p - \frac{p-1}{2}(\theta' U + (1-\theta')u)^{p-2}u^2\right) < 0,$$

where $0<\theta,\theta'<1$. Then combining (13), (14) and (15), we obtain (12). □

Lemma 3.3. *Let* $\lambda \in (0,\lambda_\mu)$. *Then*

$$c^{\infty}_{\lambda} = \liminf_{R\to\infty}\left\{I^{\mu}_{\lambda,4Re}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v) \in B_{3R}(4Re)\backslash B_{2R}(4Re)\right\} = c^0 + c^0_{\lambda,-}.$$

Proof. By Lemma 3.2, we have $c^{\infty}_{\lambda} \le c^0 + c^0_{\lambda,-}$. We will show $c^{\infty}_{\lambda} \ge c^0 + c^0_{\lambda,-}$. Let $\{R_n\}\subset \mathbb{R}$ and $\{u_n\}\subset \mathcal{M}^{\mu}_{\lambda,\eta,+}$ be sequences such that $\lim_{n\longrightarrow\infty} R_n = \infty$, $\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(u_n) = c^{\infty}$ and $\beta(u_n) \in B_{3R_n}(4R_n e)\backslash B_{2R_n}(4R_n e)$. Then by the concentrate compactness lemma, we have that there exist sequences $\{v_n\},\{w_n\}\subset H$ such that $\lim_{n\longrightarrow\infty}\|u_n - v_n - w_n\|_\mu = 0$, $\liminf_{n\to\infty}\int |v_n|^p > 0$ and $\lim_{n\longrightarrow\infty} dist\,(\text{supp } v_n, \text{supp } w_n) = \infty$. It then follows that

$$\lim_{n\longrightarrow\infty} \nabla I^{\mu}_{\lambda,4R_n}(v_n) = \lim_{n\longrightarrow\infty} \nabla I^{\mu}_{\lambda,4R_n}(w_n) = 0.$$

We may assume $\lim_{n\longrightarrow\infty} dist\,(\text{supp } v_n, 4R_n e) = \infty$. Then noting that $\lim_{n\longrightarrow\infty}\langle f(\cdot - 4R_n e), v_n\rangle = 0$, we have

$$\liminf_{n\to\infty} I^{\mu}_{\lambda,4R_n}(v_n) = \liminf_{n\to\infty} I^{\mu}(v_n) \ge c^{\mu}.$$

If $\liminf_{n\to\infty}\int_{B_R(0)}|v_n| > 0$ for some $R>0$, then by subtracting subsequences we have

$$v_n \to V_\mu \text{ as } n\to\infty \text{ in } L^p\left(\mathbb{R}^N\right) \qquad (16)$$

and then $\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(v_n) = c^{\mu}$. If $\liminf_{n\to\infty}\int_{B_R(0)}|v_n| = 0$ for any $R>0$, then again by subtracting subsequences we have that there exists a sequence $\{x_n\}\subset\mathbb{R}^N$ such that $\lim_{n\longrightarrow\infty}|x_n| = \infty$ and

$$v_n - U_{x_n} \to 0 \text{ as } n\to\infty \text{ in } L^p\left(\mathbb{R}^N\right) \tag{17}$$

and then $\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,\eta}(v_n) = c^0$. On the other hand, we have $\liminf_{n\to\infty} I^0_{\lambda}(w_n) \geq c^0_{\lambda,-}$.

Case 1. $\liminf_{n\to\infty}\int_{B_R(4R_ne)}|w_n| > 0$ for some $R>0$. In this case, by subtracting subsequences, we have $w_n - u_{\lambda,-}(\cdot - 4R_ne) \to 0$ as $n\to\infty$ in $L^p\left(\mathbb{R}^N\right)$ and then $\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(w_n) = c^0_{\lambda,-}$. If $\lim_{n\longrightarrow\infty} v_n = V_{\mu}$, we have by (7), (9) and (8) that there exists $R>0$ such that $\Omega(u_n)\subset B_R(0)$ for n sufficiently large. Thus we find $\beta(u_n)\subset B_R(0)$ for n sufficiently large. This is a contradiction. Therefore (17) holds and then

$$\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(u_n) = \lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(v_n) + \lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(w_n) = c^0 + c^0_{\lambda,-}.$$

Case 2. $\liminf_{n\to\infty}\int_{B_R(4R_ne)}|w_n| = 0$ for any $R>0$. If (16) holds, then by the definition, $\liminf_{n\to\infty}\int_{B_R(0)}|w_n| = 0$ holds for all $R>0$. Therefore by subtracting subsequences we have that there exists a sequence $\{y_n\}\subset\mathbb{R}^N$ such that $\lim_{n\longrightarrow\infty}|y_n| = \infty$ and

$$w_n - U_{y_n} \to 0 \text{ as } n\to\infty \text{ in } L^p\left(\mathbb{R}^N\right). \tag{18}$$

That is $\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(u_n) = \lim_{n\longrightarrow\infty} I^{\mu}(v_n) + \lim_{n\longrightarrow\infty} I^0(w_n) = c^{\mu} + c^0$. Since $c^{\mu} + c^0 > c^0 + c^0_{\lambda,-}$, this is a contradiction. Next, we assume that (17) holds. Then by a parallel argument as above, we obtain $\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,4R_n}(u_n) = \lim_{n\longrightarrow\infty} I^{\mu}(v_n) + \lim_{n\longrightarrow\infty} I^{\mu}(w_n) \geq c^0 + c^{\mu}$. This contradicts to the assumption. Thus the assertion follows.. □

Lemma 3.4. *For each* $\eta > 0$,

$$c^{\infty}_{\eta} = \liminf_{R\to\infty}\left\{I^{\mu}_{\lambda,\eta,}(v) : v\in\mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v)\in\mathbb{R}^N\backslash B_R(0)\right\} \geq c^0 + c^{\mu}_{\lambda,\eta,-}. \tag{19}$$

Proof. Let $\eta>0$. Let $\{R_n\}\subset\mathbb{R}$ and $\{u_n\}\subset\mathcal{M}^{\mu}_{\lambda,\eta,+}$ be sequences such that $\lim_{n\longrightarrow\infty} R_n = \infty$, $\beta(u_n)\in\mathbb{R}^N\backslash B_{R_n}(0)$ and $\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,\eta e}(u_n) = c^{\infty}_{\eta}$. Then by the concentrate compactness lemma, we have that there exist sequences $\{v_n\},\{w_n\}\subset H$ such that $\lim_{n\longrightarrow\infty}\|u_n - v_n - w_n\|_{\mu} = 0$, $\liminf_{n\to\infty}\int|v_n|^p > 0$ and $\lim_{n\longrightarrow\infty} dist(\text{supp } v_n, \text{supp } w_n) = \infty$. It then follows that $\lim_{n\longrightarrow\infty}\nabla I^{\mu}_{\lambda,\eta}(v_n) = \lim_{n\longrightarrow\infty}\nabla I^{\mu}_{\lambda,\eta}(w_n) = 0$. We may assume $\lim_{n\longrightarrow\infty} dist(\text{supp } v_n, 0) = \infty$. Then noting that $\lim_{n\longrightarrow\infty}\langle f(\cdot - \eta e), v_n\rangle = 0$ and $\lim_{n\longrightarrow\infty}(\|v_n\|_{\mu} - \|v_n\|_0) = 0$, we have

$\liminf_{n\to\infty} I^{\mu}_{\lambda,\eta}(v_n) = \liminf_{n\to\infty} I^0(v_n) = c^0$. On the other hand, we have $\liminf_{n\to\infty} I^0_{\lambda}(w_n) \geq c^0_{\lambda,\eta,-}$. Therefore

$$\lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,\eta}(u_n) = \lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,\eta}(v_n) + \lim_{n\longrightarrow\infty} I^{\mu}_{\lambda,\eta}(w_n) \geq c^0 + c^0_{\lambda,\eta,-}.$$

This completes the proof. □

Proof of Theorem First, we choose $c > 0$ such that

$$\max\left\{c^{\mu} + c^0_{\lambda,-}, c^0_{\lambda,+}\right\} < c < c^0 + c^0_{\lambda,-}.$$

Then by Lemma 3.1, Lemma 3.3 and Lemma 3.4, we can choose η so large that

$$\inf\left\{I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v) \subset B_{R_1}(0)\right\} < c, \tag{20}$$

$$\inf\left\{I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v) \subset B_{R_1}(\eta e)\right\} < c, \tag{21}$$

$$\inf\left\{I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v) \subset B_{3\eta/4}(\eta e)\backslash B_{\eta/2}(\eta e)\right\} > c \tag{22}$$

and

$$\inf\left\{I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v) \subset \mathbb{R}^N\backslash B_{2\eta}(0)\right\} > c. \tag{23}$$

Then by $(20), (22)$ and (23), there exists $u_1 \in \mathcal{M}^{\mu}_{\lambda,\eta,+}$ such that

$$I^{\mu}_{\lambda,\eta}(u_1) = \inf\left\{I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v) \subset B_{2\eta}(0)\backslash B_{3\eta/4}(\eta e)\right\}. \tag{24}$$

While by (21) and (22), there exists $u_2 \in \mathcal{M}^{\mu}_{\lambda,\eta,+}$ such that

$$I^{\mu}_{\lambda,\eta}(u_2) = \inf\left\{I^{\mu}_{\lambda,\eta}(v) : v \in \mathcal{M}^{\mu}_{\lambda,\eta,+}, \beta(v) \subset B_{\eta/2}(\eta e)\right\}. \tag{25}$$

Next we set

$$M = \left\{\rho \in C\left([0,1]; \mathcal{M}^{\mu}_{\lambda,\eta,+}\right) : \rho(0) = u_1, \rho(u_2) = u_3\right\}$$

and

$$c_m = \min_{\rho\in M}\max_{t\in[0,1]} I^{\mu}_{\lambda,\eta}(\rho(t)).$$

By Lemma 3.2 and (22), we have $c < c_m < c^0 + c^{\mu}_{\lambda,\eta,-}$. Then noting (19) holds, we have by a mountain pass argument that there exists a critical point $u_3 \in \mathcal{M}^{\mu}_{\lambda,\eta,+}$ of $I^{\mu}_{\lambda,\eta}$ such that $I^{\mu}_{\lambda,\eta}(u_3) = c_m$. On the other hand, we already know by Theorem 2.2 that there exists a solution $u_0 \in \mathcal{M}^{\mu}_{\lambda,\eta,-}$ of (P). Therefore we find problem (P) has at least four solutions $u_0, u_1, u_2, u_3 \in H$ as claimed.

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