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STABILITY OF MIXED TYPE FUNCTIONAL EQUATIONS WITH INVOLUTION IN NON-ARCHIMEDEAN SPACES

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Abstract. In this paper, we consider the generalized Hyers-Ulam stability for the following additive-quadratic functional equation with involution $f(x+2y) - f(2x+y) + f(x+y) + f(\sigma(x)+y) + f(x) - 4f(y) - f(\sigma(y)) = 0$ in non-Archimedean spaces.

1. Introduction and Preliminaries

In 1940, Ulam [13] posed the following problem concerning the stability of functional equations: *Let G_1 be a group and let G_2 a meric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?*

In 1941, Hyers [8] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [10] generalized the result of Hyers. Rassias [10] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$$

for some $\epsilon(\ge 0)$, $p(< 1)$ and for all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. The paper of Rassias [10] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassis approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1}$$

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is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [11] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [5] proved the generalized Hyers-Ulam stability for the quadratic functional equation.

For an additive mapping $\sigma : X \longrightarrow X$ with $\sigma(\sigma(x)) = x$ for all $x \in X$, σ is called *an involution* of X. Let $\sigma : X \longrightarrow X$ be an involution. Then the functional equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) \tag{2}$$

is called *an additive functional equation with involution* and a solution of (2) is called *an additive mapping with involution*. And the functional equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y) \tag{3}$$

is called *the quadratic functional equation with involution* and a solution of (3) is called *a quadratic mapping with involution*. The functional equation (3) has been studied by Stetkær [2, 3, 9, 12].

In this paper, using fixed point method, we prove the generalized Hyers-Ulam stability of the following functional equation with involution

$$f(x+2y)-f(2x+y)+f(x+y)+f(\sigma(x)+y)+f(x)-4f(y)-f(\sigma(y)) = 0. \tag{4}$$

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that for any $r, s \in K$, the following conditions hold: (i) $|r| = 0$ if and only if $r = 0$, (ii) $|rs| = |r||s|$, and (iii) $|r + s| \le |r| + |s|$.

A field K is called *a valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. If the triangle inequality is replaced by $|r + s| \le max\{|r|, |s|\}$ for all $r, s \in K$, then the valuation $|\cdot|$ is called a *non-Archimedean valuation* and the field with a non-Archimedean valuation is called *non-Archimedean field*. If $|\cdot|$ is a non-Archimedean valuation on K, then clearly, $|1| = |-1|$ and $|n| \le 1$ for all $n \in \mathbb{N}$.

Definition 1. Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is called *a non-Archimedean norm* if satisfies the following conditions:
(a) $\|x\| = 0$ if and only if $x = 0$,
(b) $\|rx\| = |r|\|x\|$, and
(c) the strong triangle inequality (ultrametric) holds, that is,

$$||x + y|| \le max\{\|x\|, \|y\|\}$$

for all $x, y \in X$ and all $r \in K$.

If $\|\cdot\|$ is a non-Archimedean norm, then $(X, \|\cdot\|)$ is called *a non-Archimedean normed space*. Let $(X, \|\cdot\|)$ be a non-Archimedean normed space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In that case, x is called *the limit of the sequence* $\{x_n\}$, and one denotes it by $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ is said to be *a Cauchy sequence* if $\lim_{n\to\infty} \|x_{n+p} - x_n\| = 0$ for all $p \in \mathbb{N}$. Since

$$\|x_n - x_m\| \leq max\{\|x_{j+1} - x_j\| \mid m \leq j \leq n-1\} \quad (n > m),$$

a sequence $\{x_n\}$ is Cauchy in $(X, \|\cdot\|)$ if and only if $\{x_{n+1} - x_n\}$ converges to zero in $(X, \|\cdot\|)$. By *a complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

Theorem 1.1. [6] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point x^* of J ;*
(3) *x^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, we assume that X is a non-Archimedean normed space and Y is a complete non-Archimedean normed space.

2. The generalized Hyers-Ulam stability for (4)

Using the fixed point methods, we will prove the generalized Hyers-Ulam stability of the cubic functional equation (4) with involution σ in non-Archimedean normed spaces.

A mapping $f : X \longrightarrow Y$ with involution is called *odd(even*, resp.) if for any $x \in X$, $f(\sigma(x)) = -f(x)(f(\sigma(x)) = f(x)$, resp.). For a given mapping $f : X \longrightarrow Y$ with involution, we define operators Df, $D_o f$, and $D_e f$ by

$$Df(x,y) = f(x+2y) - f(2x+y) + f(x+y) + f(\sigma(x)+y) + f(x) - 4f(y) - f(\sigma(y)),$$

$$D_o f(x,y) = f(x+2y) - f(2x+y) + f(x+y) + f(\sigma(x)+y) + f(x) - 3f(y),$$

$$D_e f(x,y) = f(x+2y) - f(2x+y) + f(x+y) + f(\sigma(x)+y) + f(x) - 5f(y).$$

Lemma 2.1. *Let $f : X \longrightarrow Y$ be a mapping. Then we have the following :*
(a) *Suppose that f is an odd mapping satisfying*

$$D_o f(x,y) = 0 \tag{5}$$

for all $x, y \in X$. Then f is an additive mapping with involution.
(b) *Suppose that f is an even mapping satisfying*

$$D_e f(x,y) = 0 \tag{6}$$

for all $x, y \in X$. Then f is a quadratic mapping with involution.

Proof. (a) Interchanging x and y in (5), we have

$$f(2x+y) - f(x+2y) + f(x+y) - f(\sigma(x)+y) + f(y) - 3f(x) = 0 \tag{7}$$

for all $x, y \in X$ and by (5) and (7), we have

$$f(x+y) = f(x) + f(y) \tag{8}$$

for all $x, y \in X$. Letting $y = \sigma(y)$ in (8), we get

$$f(x+\sigma(y)) = f(x) + f(\sigma(y)) = f(x) - f(y) \tag{9}$$

for all $x, y \in X$, By (8) and (9), one has the result.

(b) Similar to (a), we have (b). □

Theorem 2.2. *Assume that $\phi : X^2 \longrightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\phi(2x, 2y) \leq |2|L\phi(x, y), \ \phi(x+\sigma(x), y+\sigma(y)) \leq |2|L\phi(x, y) \tag{10}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an odd mapping such that $f(0) = 0$ and

$$\|Df(x, y)\| \leq \phi(x, y) \tag{11}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ with involution such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2|(1-L)}\phi_0(x) \tag{12}$$

for all $x \in X$, where $\phi_0(x) = \max\left\{\phi(x, 0), \phi(x, x), \frac{1}{|2|L}\phi(x+\sigma(x), 0)\right\}$

Proof. Since f is an odd mapping, (11) is replaced by

$$\|D_o f(x, y)\| \leq \phi(x, y) \tag{13}$$

for all $x, y \in X$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g, h) = \inf\left\{c \in [0, \infty) \mid \|g(x) - h(x)\| \leq c\phi_0(x), \forall x \in X\right\}$. Then (S, d) is a complete metric space([9]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = \dfrac{1}{2}\{g(2x) + g(x+\sigma(x))\}$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some non-negative real number c. Then by (10), we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| &\leq \frac{1}{|2|}\max\{\|g(2x) - h(2x)\|, \|g(x+\sigma(x)) - h(x+\sigma(x))\|\} \\ &\leq cL\max\left\{\phi_0(x), \frac{1}{|2|L}\phi_0(x+\sigma(x))\right\} \\ &\leq cL\phi_0(x)\end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$ and so J is a strictly contractive mapping. Now, putting $y = 0$ in (13), we get

$$\|f(2x) - 2f(x)\| \leq \phi(x, 0) \tag{14}$$

for all $x \in X$ and putting $y = x$ in (13), by (14), we get

$$\|f(x+\sigma(x))\| \le \max\{\phi(x,0), \phi(x,x)\} \tag{15}$$

for all $x \in X$. By (14) and (15), we have

$$\|Jf(x) - f(x)\| \le \frac{1}{|2|}\max\{\phi(x,0), \phi(x,x)\} \le \frac{1}{|2|}\phi_0(x)$$

for all $x \in X$ and we have

$$d(Jf, f) \le \frac{1}{|2|} < \infty. \tag{16}$$

By Theorem 1.1, there exists a mapping $A : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, A) \to 0$ as $n \to \infty$.

Now, we claim that the following equality holds:

$$(J^n f)(x) = \frac{1}{2^n}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\} \tag{17}$$

for all $x \in X$ and $n \in \mathbb{N}$. It is clear for $n = 1$. Suppose that (17) holds for some $n(n \ge 2)$. Then we get

$$\begin{aligned}(J^{n+1} f)(x) &= J\Big[\frac{1}{2^n}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\}\Big] \\ &= \frac{1}{2^n}\{Jf(2^n x) + (2^n - 1)Jf(2^{n-1}(x + \sigma(x)))\} \\ &= \frac{1}{2^{n+1}}\{f(2^{n+1}x) + f(2^n(x + \sigma(x))) + 2(2^n - 1)f(2^n(x + \sigma(x)))\}.\end{aligned}$$

By induction, (17) holds. Since $d(J^n f, A) \to 0$ as $n \to \infty$, we have

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\} \tag{18}$$

for all $x \in X$.

Since $|2^n - 1| \le 1$, by (10) and (13), we get

$$\begin{aligned}&\|J^n D_o f(x,y)\| \\ &\le \max\Big\{\frac{1}{|2|^n}|\|D_o f(2^n x, 2^n y)\|, \frac{|2^n - 1|}{|2|^n}\|D_o f(2^{n-1}(x+\sigma(x)), 2^{n-1}(y+\sigma(y)))\|\Big\} \\ &\le \max\Big\{\frac{1}{|2|^n}\phi(2^n x, 2^n y), \frac{|2^n - 1|}{|2|^n}\phi(2^{n-1}(x+\sigma(x)), 2^{n-1}(y+\sigma(y)))\Big\} \\ &\le \max\Big\{L^n\phi(x,y), \frac{L^{n-1}}{|2|}\phi(x+\sigma(x)), y+\sigma(y)))\Big\} \\ &\le L^n\phi(x,y)\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we have $D_o A(x,y) = 0$ for all $x, y \in X$ and since f is odd, by (18), A is odd. By Lemma 2.1, A is an additive mapping with involution. By (4) in Theorem 1.1 and (16), we have (12).

Assume that $A_1 : X \longrightarrow Y$ is another additive mapping with (12). We know that A_1 is a fixed point of J. Due to (3) in Theorem 1.1, we get $A = A_1$. This proves the uniqueness of A. □

Theorem 2.3. *Assume that $\phi : X^2 \longrightarrow [0, \infty)$ is a mapping and there exists a real number L with $0 < L < 1$ such that*

$$\phi(2x, 2y) \leq |2|^2 L\phi(x, y), \ \ \phi(x + \sigma(x), y + \sigma(y)) \leq |2|^2 L\phi(x, y) \tag{19}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an even mapping such that $f(0) = 0$ and (11). Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ with involution such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|2|^2(1 - L)}\phi_1(x) \tag{20}$$

for all $x \in X$, where $\phi_1(x) = \max\left\{\phi(x, 0), \phi(x, x), \frac{1}{|4|L}\phi(x + \sigma(x), 0)\right\}$

Proof. Since f is an even mapping, (11) is replaced by

$$\|D_e f(x, y)\| \leq \phi(x, y) \tag{21}$$

for all $x, y \in X$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d in S defined by $d(g, h) = \inf\{c \in [0, \infty) | \ \|g(x) - h(x)\| \leq c\phi_1(x), \ \forall x \in X\}$. Then (S, d) is a complete metric space([9]). Define a mapping $T : S \longrightarrow S$ by

$$Tg(x) = \frac{1}{4}\{g(2x) + g(x + \sigma(x))\}$$

for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some non-negative real number c. Then by (19), we have

$$\begin{aligned}\|Tg(x) - Th(x)\| &\leq \frac{1}{|2|^2}\max\{\|g(2x) - h(2x)\|, \|g(x + \sigma(x)) - h(x + \sigma(x))\|\} \\ &\leq cL\phi_1(x)\end{aligned}$$

for all $x \in X$. Hence we have $d(Tg, Th) \leq Ld(g, h)$ for any $g, h \in S$ and so T is a strictly contractive mapping. Now, putting $y = 0$ in (13), we get

$$\|f(2x) - 4f(x)\| \leq \phi(x, 0)$$

for all $x \in X$ and putting $y = x$ in (13), we get

$$\|f(2x) + f(x + \sigma(x)) - 4f(x)\| \leq \phi(x, x)$$

for all $x \in X$. Hence we have

$$\|f(x + \sigma(x))\| \leq \max\{\phi(x, 0), \phi(x, x)\}$$

for all $x \in X$ and since $|4|L < 1$,

$$\|Tf(x) - f(x)\| \leq \frac{1}{|4|}\max\{\phi(x, 0), \phi(x, x)\} \leq \frac{1}{|4|}\phi_1(x)$$

for all $x \in X$. Thus we have

$$d(Tf, f) \leq \frac{1}{|2|^2} < \infty. \tag{22}$$

By Theorem 1.1, there exists a mapping $Q : X \longrightarrow Y$ which is a fixed point of T such that $d(T^n f, Q) \to 0$ as $n \to \infty$. Similar to Theorem 2.2, we can show that

$$(T^n f)(x) = \frac{1}{2^{2n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}(x + \sigma(x)))\}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $d(T^n f, Q) \to 0$ as $n \to \infty$,

$$Q(x) = \lim_{n\to\infty} \frac{1}{2^{2n}}\{f(2^n x) + (2^n - 1)f(2^{n-1}\big(x + \sigma(x)))\}. \tag{23}$$

Since $|2^n - 1| \leq 1$, by (19) and (21), we get

$$\begin{aligned}
&\|J^n D_e f(x, y)\| \\
&\leq \max\Big\{\frac{1}{|2|^{2n}}|\|D_e f(2^n x, 2^n y)\|, \frac{|2^n - 1|}{|2|^{2n}}\|D_e f(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y)))\|\Big\} \\
&\leq \max\Big\{\frac{1}{|2|^{2n}}\phi(2^n x, 2^n y), \frac{|2^n - 1|}{|2|^{2n}}\phi(2^{n-1}(x + \sigma(x)), 2^{n-1}(y + \sigma(y)))\Big\} \\
&\leq \max\Big\{L^n\phi(x, y), \frac{L^{n-1}}{|2|^2}\phi(x + \sigma(x)), y + \sigma(y)))\Big\} \\
&\leq L^n\phi(x, y)
\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we have $D_e Q(x, y) = 0$ for all $x, y \in X$ and since f is even, by (23), Q is even. By Lemma 2.1, Q is a quadratic mapping with involution. By (4) in Theorem 1.1 and (22), Q satisfies (20).

Assume that $Q_1 : X \longrightarrow Y$ is another quadratic mapping with (20). We know that Q_1 is a fixed point of J. Due to (3) in Theorem 1.1, we get $Q = Q_1$. This proves the uniqueness of Q. □

From now on, for any mapping $f : X \longrightarrow Y$, we deonte

$$f_o(x) = \frac{f(x) - f(\sigma(x))}{2}, \quad f_e(x) = \frac{f(x) + f(\sigma(x))}{2}.$$

Then f_o is an odd mapping and f_e is an even mapping. Hence by Theorem 2.2 and Theorem 2.3, we have the following theorem.

Theorem 2.4. *Assume that $\phi : X^2 \longrightarrow [0, \infty)$ is a mapping with (19). Let $f : X \longrightarrow Y$ be a mapping satisfying* (11) *and* $f(0) = 0$. *Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ with involution such that*

$$\|f(x) - F(x)\| \leq \frac{1}{|2|^3(1 - L)}\Phi(x) \tag{24}$$

for all $x \in X$, where

$$\Phi(x) = \max\Big\{\phi(x,0), \phi(\sigma(x),0), \phi(x,x), \phi(\sigma(x),\sigma(x)), \frac{1}{|4|L}\phi(x+\sigma(x),0)\Big\}.$$

Proof. By (11), we have

$$\|Df_o(x,y)\| \le \frac{1}{|2|}\max\{\phi(x,y), \phi(\sigma(x),\sigma(y))\}.$$

and since $|2|^2 \le |2|$, by (19), $\max\{\phi(x,y), \phi(\sigma(x),\sigma(y))\}$ satisfies (10). By Theorem 2.2, there exists a unique additive mapping $A : X \longrightarrow Y$ with involution such that

$$\|f_o(x) - A(x)\| \le \frac{1}{|2|^2(1-L)}\Psi(x) \tag{25}$$

for all $x \in X$, where

$$\Psi(x) = \max\Big\{\phi(x,0), \phi(\sigma(x),0), \phi(x,x), \phi(\sigma(x),\sigma(x)), \frac{1}{|2|L}\phi(x+\sigma(x),0)\Big\}.$$

By (11), we have

$$\|Df_e(x,y)\| \le \frac{1}{|2|}\max\{\phi(x,y), \phi(\sigma(x),\sigma(y))\}$$

Then by Theorem 2.3, there exists a unique quadratic mapping $Q : X \longrightarrow Y$ with involution such that

$$\|f_e(x) - Q(x)\| \le \frac{1}{|2|^3(1-L)}\Phi(x) \tag{26}$$

for all $x \in X$. Let $F = A + Q$. Then F is an additive-quadratic mapping with involution and by (25) and (26), we have

$$\|F(x) - f(x)\| \le \max\{\|A(x) - f_o(x)\|, \|Q(x) - f_e(x)\|\}$$

for all $x \in X$. Thus F satisfies (24).

Assume that $G : X \longrightarrow Y$ is another additive-quadratic mapping with (24). Then f_o and G_o are additive mappings such that

$$\begin{aligned}\|f_o(x) - G_o(x)\| &\le \frac{1}{|2|}\max\{\|f(x) - G(x)\|, \|f(\sigma(x)) - G(\sigma(x))\|\}\\ &\le \frac{1}{|2|^4(1-L)}\max\{\Phi(x), \Phi(\sigma(x))\}\\ &= \frac{1}{|2|^4(1-L)}\Phi(x)\end{aligned}$$

Due to (3) in Theorem 1.1, we have $F_o = A = G_o$.

Similarly, $F_e = Q = G_e$. Hence $G = G_o + G_e = A + Q = F$ and this proves the uniqueness of F. □

Using Theorem 2.4, we obtain the following corollary concerning the stability of (4).

Corollary 2.5. *Let $\theta \geq 0$ and p be a positive real number with $p > 2$. Let $f : X \longrightarrow Y$ be a mapping satisfying*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{27}$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \leq |2|\|x\|$ for all $x \in X$. Then there exists a unique mapping $F : X \longrightarrow Y$ with involution such that F is a solution of the functional equation (4) and the inequality

$$||f(x) - F(x)|| \leq \frac{\max\left\{2, \frac{1}{|2|^2}\right\}}{|2|(|2|^2 - |2|^p)}\theta||x||^p \tag{28}$$

holds for all $x \in X$.

Proof. Let $\phi(x,y) = \theta(\|x\|^p + \|y\|^p)$. Then ϕ satisfies (19) and since $\|x+\sigma(x)\| \leq |2|\|x\|$, $\|\sigma(x)\| \leq \|x\|$ for all $x \in X$, we have the results. □

By Theorem 2.4, we obtain the following corollary concerning the stability of (4).

Corollary 2.6. *Let $\alpha_i : [0,\infty) \longrightarrow [0,\infty)$ $(i = 1,2,3)$ be increasing mappings satisfying*

(i) $\alpha_i(0) = 0$ and $0 < M = \max\{(\alpha_1(|2|))^2, \alpha_2(|2|), \alpha_3(|2|)\} < |2|^2$,

(ii) $\alpha_i(|2|t) \leq \alpha_i(|2|)\alpha_i(t)$ for all $t \geq 0$.

Let $f : X \longrightarrow Y$ be a mapping such that for some $\theta \geq 0$

$$\|Df(x,y)\| \leq \theta[\alpha_1(\|x\|)\alpha_1(\|y\|) + \alpha_2(\|x\|) + \alpha_3(\|y\|)] \tag{29}$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \leq |2|\|x\|$ for all $x \in X$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ with involution such that

$$||f(x) - F(x)|| \leq \frac{\theta}{|2|^2(|2|^2 - M)}(\alpha_1(\|x\|)^2 + \alpha_2(\|x\|) + \alpha_3(\|x\|))$$

for all $x \in X$.

From Corollary 2.6, we can take $\alpha_1(t) = t^p, \alpha_2(t) = \alpha_3(t) = t^{2p}$ for all $t \geq 0$. Then we have the following example.

Example 1. *Let $\theta \geq 0$ and p a positive real number with $p > 1$. Let $f : X \longrightarrow Y$ be a mapping satisfying*

$$\|Df(x,y)\| \leq \theta(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \tag{30}$$

for all $x, y \in X$. Suppose that $\|x + \sigma(x)\| \leq |2|\|x\|$ for all $x \in X$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ with involution such that

$$||f(x) - F(x)|| \leq \frac{3\theta}{|2|^2(|2|^2 - |2|^{2p})}||x||^{2p}$$

for all $x \in X$.

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