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AN IMPULSIVE STAGE-STRUCTURED OPTIMAL CONTROL PROBLEM AND OPTIMAL HARVEST STRATEGY OF PACIFIC COD, *GADUS MICROCEPHALUS*, IN THE SOUTH KOREA

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Abstract. We consider an optimal control problem for an impulsive stage-structured model involving ordinary differential equations with impulsive values of initial conditions in the next year. The main goal is to maximize a profit of the catch of Pacific cod in the South Korea through optimal harvest strategy as a control of adult cod. We established necessary conditions for the optimal harvest control using idea of Pontryagin's maximum principle. The optimal harvest strategy is to numerically solve the equation by using an iterative method with the Runge-Kutta method. Finally, we compare a monthly average of fishing mortality of Pacific cod from 2013 to 2017 with monthly fishing mortality for result obtained optimal harvest strategy.

1. Introduction

According to the National Institute of Fisheries Science, the catch of the coastal water fisheries in the South Korea has been the lowest in 44 years. For reasonable fisheries management, we have turned to a management plan that restricts the amount of catches, such as Total allowable catch strategy. Recently, advanced countries on fisheries research have been using bio-economic models to present directions of fisheries management. Therefore, it is necessary to establish a theoretical and methodological framework of fisheries policy management by making practical application to the fisheries management field.

Owing to the small economic value of juvenile population for some creatures, only adult population is usually harvested and utilized. Song and Chen[1] considered stage-structured and harvest model of single species having harvest effort is constant or variable or periodic. Jerry and Rassi[2], and Smith[3, 4] examined a structured fishing models, basically displaying the two stages of ages of a fish population, which are in our case juveniles and adults. Chakraborty et al.[5]

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consider a prey-predator model with stage structure of size for juvenile and adult. In these paper, the rate of transformation of the adult is proportional to the density of existing juvenile with proportionality constant. But it is difficult to use general juvenile-adult model for fish species that spawn only once each year because individual of the juvenile stage transform directly to the adult stage. Therefore, we consider an impulsive stage-structured model involving ordinary differential equations and impulse values for initial conditions can solve this problem.

There is little fisheries research about an optimal harvest strategy for an impulsive stage-structured model. In related to optimal control of impulsive model. Recently, Miller and Rubinovich [6] studied optimal impulse control problems with a restricted number of impulses and Ding[7] set up the mathematical framework with the optimal control of hybrid ODE system when applied to optimal control of treatment to maximize disease-free ticks and minimize infected ticks. Here we focus on the monthly pattern of price for Pacific cod to find the best strategy to maximize profit of the catch. The idea of Pontryagin's Maximum Principle[8] was required to find the necessary conditions for numerical simulating optimal harvest strategy.

The objective of this paper is to propose an optimal harvest strategy maximizing profit of the catch for Pacific cod having a different price per month. Here we consider an impulsive stage-structured model including juvenile and adult stage. Secondly, we find a necessary condition of optimal harvest strategy. Finally, we compare a monthly average of fishing mortality of Pacific cod from 2013 to 2017 and obtained monthly fishing mortality for optimal harvest strategy.

2. Optimal control problem

In the model, we divide to stages into juvenile and adult individuals according to maturity and catchable stock. Here, our state variables are defined as follows:

- $B_{1,n}(t)$: biomass density of juvenile cod in n-th year,
- $B_{2,n}(t)$: biomass density of adult cod in n-th year.

Because the functional form of the harvest is generally considered using the phrase catch-per-unit-effort(CPUE) hypothesis[9] to describe an assumption, the harvest control is expressed as a product of fishing effort and catchability coefficient(α). Here we propose an optimal control problem of the pacific cod with an impulsive stage-structured model.

$$\begin{cases} \dfrac{dB_{1,n}(t)}{dt} = (g_1 - M_1)B_{1,n}(t), \\ \dfrac{dB_{2,n}(t)}{dt} = (g_2 - (M_2 + \alpha u_n(t))B_{2,n}(t) - D_M B_{2,n}^2(t), \\ B_{1,n+1}(n+1) = f_R \dfrac{W_1}{W_2} B_{1,n}(n+1), \\ B_{2,n+1}(n+1) = B_{1,n}(n+1) + B_{2,n}(n+1), \end{cases} \tag{1}$$

where $n = 0, 1, \cdots, N$. $\alpha u_n(t)B_{2,n}(t)$ represents the harvest of Pacific cod as a control variable $u_n(t)$ represents the fishing effort at time $t \in [n, n+1]$. g_i and M_i are the growth rate and natural mortality of juvenile and adult stage, respectively. D_M and f_R are density dependent mortality and effective fecundity of adult stage. W_1 and W_2 are average of weights of the individual for the juvenile and adult states having a non-zero constant derived by von bertalanffy growth equation. Variables of the optimal control model(1) are listed and defined in Table 1. The goal is to show a time-dependent harvest strategy to maximize the profits of the catch. In order to derive the optimal harvest strategy, we need to find a necessary condition with respect to the control $u_n(t)$.

We define the control set as

$$\begin{aligned} U_H &= \big\{(u_0(t), u_1(t), \cdots, u_N(t)) : u_n \text{ is Lebesgue measurable}, \\ &\quad t \in [n, n+1] \text{ for } n = 0, \cdots, N\big\}. \end{aligned}$$

We want to maximize the following objective functional that mean the profit of the catch.

$$\begin{aligned} \mathcal{J}(u) &= \sum_{n=0}^{N} \int_{n}^{n+1} C_1(t)\alpha u_n(t)B_{2,n}(t) - C_2 u_n^2(t)dt \\ &\quad + C_3(B_{1,N}(N+1) + B_{2,N}(N+1)), \end{aligned} \tag{2}$$

subject to model (1). $C_1(t)$ represents the positive step function with respect to monthly price unit biomass and C_2 is a cost unit effort having non-zero constant.

Theorem 2.1. *Let $B^*_{1,n}(t)$ and $B^*_{2,n}(t)$ be optimal state solutions with associated optimal control u^*_n subject to model (1) for $n = 0, \cdots, N$. Then, there exists adjoint variables $\lambda_{i,n}(t)$, for $i = 1, 2$ satisfying*

$$\begin{aligned} \lambda'_{1,n}(t) &= (M_1 - g_1)\lambda_{1,n}(t), \\ \lambda'_{2,n}(t) &= -C_1(t)\alpha u_n(t) + (M_2 - g_2 + \alpha u_n(t) - 2D_M B_{2,n}(t))\lambda_{2,n}(t), \end{aligned}$$

for $t \in [n, n+1]$, $n = 0, \cdots, N$ and with transversality conditions

$$\begin{aligned} \lambda_{1,n}((n+1)) &= \lambda_{2,n+1}(n+1), \\ \lambda_{2.n}((n+1)) &= f_R \frac{W_1}{W_2}\lambda_{1,n+1}(n+1) + \lambda_{2,n+1}(n+1), \end{aligned}$$

for $n = 0, \cdots, N-1$ and

$$\begin{aligned} \lambda_{1,N}((N+1)) &= C_3, \\ \lambda_{2,N}((N+1)) &= C_3. \end{aligned}$$

Furthermore, the optimal controls u^*_n satisfy

$$u^*_n(t) = \alpha \frac{B^*_{2,n}(t)(C_1(t) + \lambda_{2,n}(t))}{2C_2},$$

for $n = 0, \cdots, N$.

Proof. Denote $X_{i,n}(B_{i,n}(t), u_n(t)) := B'_{i,n}(t)$ and $Y_n(B_{2,n}(t), u_n(t)) := C_1(t)\alpha u_n(t)B_{2,n}(t) - C_2u_n^2(t)$. Let $\lambda_{i,n}(t)$ be a piecewise differentiable function on $[n, n+1]$ to be determined. By the Fundamental Theorem of Calculus,

$$\sum_{n=0}^{N}\bigg(\int_n^{n+1}\frac{d}{dt}\Big[\sum_{i=1}^{2}\lambda_{i,n}(t)B_{i,n}(t)\Big]dt + \sum_{i=1}^{2}\lambda_{i,n}(n)B_{i,n}(n) - \lambda_{i,n}(n+1)B_{i,n}(n+1)\bigg) = 0.$$

We add zero terms to the objective functional and then,

$$\begin{aligned}
\mathcal{J}(u) &= \sum_{n=0}^{N}\bigg(\int_n^{n+1^-} C_1(t)\alpha u_n(t)B_{2,n}(t) - C_2u_n(t)^2dt\bigg) \\
&\quad + C_3\Big(B_{1,N}(N+1) + B_{2,N}(N+1)\Big) \\
&= \sum_{n=0}^{N}\bigg(\int_n^{n+1} Y_n(B_{2,n}(t), u_n(t)) + \frac{d}{dt}\Big[\sum_{i=1}^{2}\lambda_{i,n}(t)B_{i,n}(t)\Big]dt\bigg) \\
&\quad + \sum_{n=0}^{N}\bigg(\sum_{i=1}^{2}\lambda_{i,n}(n)B_{i,n}(n) - \lambda_{i,n}(n+1)B_{i,n}(n+1)\bigg) \\
&\quad + C_3\Big(B_{1,N}(N+1) + B_{2,N}(N+1)\Big) \\
&= \sum_{n=0}^{N}\bigg(\int_n^{n+1} Y_n(B_{2,n}(t), u_n(t)) + \sum_{i=1}^{2}\Big[\lambda'_{i,n}(t)B_{i,n}(t) + \lambda_{i,n}(t)B'_{i,n}(t)\Big]dt\bigg) \\
&\quad + \sum_{n=0}^{N}\bigg(\sum_{i=1}^{2}\lambda_{i,n}(n)B_{i,n}(n) - \lambda_{i,n}(n+1)B_{i,n}(n+1)\bigg) \\
&\quad + C_3\Big(B_{1,N}(N+1) + B_{2,N}(N+1)\Big).
\end{aligned}$$

We first consider the boundary terms for biomass.

$$\begin{aligned}
&\sum_{n=0}^{N}\bigg[\sum_{i=1}^{2}\bigg(-\lambda_{i,n}(n+1)B_{i,n}(n+1) + \lambda_{i,n}(n)B_{i,n}(n)\bigg)\bigg] \\
= &\sum_{i=1}^{2}\bigg[\Big(-\lambda_{i,0}(1)B_{i,0}(1) + \lambda_{i,0}(0)B_{i,0}(0)\Big) + \Big(-\lambda_{i,1}(2)B_{i,1}(2) + \lambda_{i,1}(1)B_{i,1}(1)\Big) \\
&+ \cdots + \Big(-\lambda_{i,N}(N+1)B_{i,N}(N+1) + \lambda_{i,N}(N)B_{i,N}(N)\Big)\bigg].
\end{aligned} \tag{3}$$

For $n = 0, \cdots, N-1$, we choose

$$\begin{aligned}
\lambda_{1,n}(n+1) &= \lambda_{2,n+1}(n+1), \\
\lambda_{2,n}(n+1) &= f_R\,\frac{W_1}{W_2}\,\lambda_{1,n+1}(n+1) + \lambda_{2,n+1}(n+1).
\end{aligned}$$

Then, the boundary terms (3) become

$$\sum_{i=1}^{2}\Big(\lambda_{i,0}(0)B_{i,0}(0)-\lambda_{i,N}(N+1)B_{i,N}(N+1)\Big).$$

We rewrite the objective functional as

$$\begin{aligned}\mathcal{J}(u) \quad = \quad & \sum_{n=0}^{N}\Bigg(\int_{n}^{n+1} Y_n(B_{2,n}(t),u_n(t))+\sum_{i=1}^{2}\lambda'_{i,n}(t)B_{i,n}(t)\\ & +\sum_{i=1}^{2}\lambda_i(t)X_{i,n}(B_{i,n}(t),u_n(t))dt\Bigg)\\ & +\sum_{i=1}^{2}\Big(\lambda_{i,0}(0)B_{i,0}(0)-\lambda_{i,N}(N+1)B_{i,N}(N+1)\Big)\\ & + C_3\Big(B_{1,N}(N+1)+B_{2,N}(N+1)\Big).\end{aligned}$$

Let $u^* \in U_H$ be the optimal control and B_i^* be the corresponding optimal state solutions. Let $u^*+\epsilon h \in U_H$ for $\epsilon > 0$, where $h=(h_0,h_1,\cdots,h_N)$ and $h_n \in L^\infty[n,n+1]$ for $n=0,1,\cdots,N$. In addition, let $B_{i,n}^\epsilon := B_{i,n}(u_n^*+\epsilon h_n)$ be corresponding solutions of the model for $n=0,1,\cdots,N$. Then, we compute the directional derivative of the objective functional $J(u^*)$ with respect to u^* in the direction of h. Since $J(u^*)$ is the maximum value, we have

$$\begin{aligned}0 \quad \geq \quad & \lim_{\epsilon\to 0^+}\frac{J(u^*+\epsilon h)-J(u^*)}{\epsilon}\\ = \quad & \lim_{\epsilon\to 0^+}\frac{1}{\epsilon}\sum_{n=0}^{N}\int_{n}^{n+1}\Bigg[\Big(Y_n(B_{2,n}^\epsilon(t),u_n^*(t)+\epsilon h_n)-Y_n(B_{2,n}^*(t),u_n^*(t))\Big)\\ & +\sum_{i=1}^{2}\lambda_{i,n}(t)\Big(X_{i,n}(B_{i,n}^\epsilon(t),u_n^*(t)+\epsilon h_n)-X_{i,n}(B_{i,n}^*(t),u_n^*(t))\Big)\\ & +\sum_{i=1}^{2}\lambda'_{i,n}(t)\Big(B_{i,n}^\epsilon(t)-B_{i,n}^*(t))\Big)\Bigg]dt\\ & +\sum_{i=1}^{2}\Bigg[\lambda_{i,0}(0)\Big(B_{i,0}^\epsilon(0)-B_{i,0}^*(0))\Big)\\ & -\lambda_{i,N}(N+1)\Big(B_{i,N}^\epsilon(N+1)-B_{i,N}^*(N+1)\Big)\Bigg]\\ & + C_3\Big(\sum_{i=1}^{2}B_{i,N}^\epsilon(N+1)-B_{i,N}^*(N+1)\Big).\end{aligned}$$

We denote $\dfrac{B^{\epsilon}_{i,n} - B^*_{i,n}}{\epsilon} \to \psi_{i,n}(t)$ as $\epsilon \to 0$ and $\psi_{i,n}(0) = 0$. By using the chain rule,

$$B^{\epsilon}_{i,n}(u^*_n + \epsilon h_n) - B^*_{i,n}u^*_n = (B^{\epsilon}_{i,n} - B^*_{i,n})(u^*_n + \epsilon h_n) + B^*_{i,n}(u^*_n + \epsilon h_n - u^*_n).$$

Then, we have

$$\begin{aligned} 0 \;\geq\; & \sum_{n=0}^{N} \int_n^{n+1^-} \Big[(Y_n)_{B_{i,n}}(B^*_{i,n}, u^*_n)\psi_{i,n}(t) + (Y_n)_{u_n}(B^*_{i,n}, u^*_n)h_n(t) \\ & + \sum_{i=1}^{2} \lambda_{i,n}(t)\Big((X_{i,n})_{B_{i,n}}\psi_{i,n}(t) - (X_{i,n})_{u_n}h_n(t)\Big) \\ & + \sum_{i=1}^{2} \lambda'_{i,n}(t)\psi_{i,n}(t)\Big]dt \\ & + \sum_{i=1}^{2}\Big(\lambda_{i,0}(0)\psi_{i,n}(0) - \lambda_{i,N}(N+1)\psi_{i,n}(N+1)\Big) \\ & + C_3\sum_{i=1}^{2}\psi_{i,n}(N+1). \end{aligned}$$

By combining the terms involving $\psi_{i,n}$ and h_n, we have

$$\begin{aligned} 0 \;\geq\; & \sum_{n=0}^{N} \int_n^{n+1^-} \Big[((Y_n)_{B_{i,n}}(B^*_{i,n}, u^*_n) + \sum_{i=1}^{2}\lambda_{i,n}(t)(X_{i,n})_{B_{i,n}} \\ & + \sum_{i=1}^{2}\lambda'_{i,n}(t))\psi_{i,n}(t) + ((Y_n)_{u_n}(B^*_{i,n}, u^*_n) + \sum_{i=1}^{2}\lambda_{i,n}(t)(X_{i,n})_{u_n})h_n(t)\Big]dt \\ & + \sum_{i=1}^{2}\Big(C_3 - \lambda_{i,N}(N+1)\Big)\psi_{i,n}(N+1). \end{aligned}$$

Thus, we choose $\lambda_{i,n}$ to solve

$$\lambda'_{i,n}(t) = -(Y_n)_{B_{i,n}}(B_{i,n}, u_n)(t) - \lambda_{i,n}(t)(X_{i,n})_{B_{i,n}}(t).$$

Therefore, we obtain

$$\begin{aligned} \lambda'_{1,n}(t) &= (M_1 - g_1)\lambda_{1,n}(t), \\ \lambda'_{2,n}(t) &= -C_1(t)\alpha u_n(t) + (M_2 - g_2 + \alpha u_n(t) - 2D_M B_{2,n}(t))\lambda_{2,n}(t), \\ \lambda_{1,N}(N+1) &= C_3, \\ \lambda_{2,N}(N+1) &= C_3, \end{aligned}$$

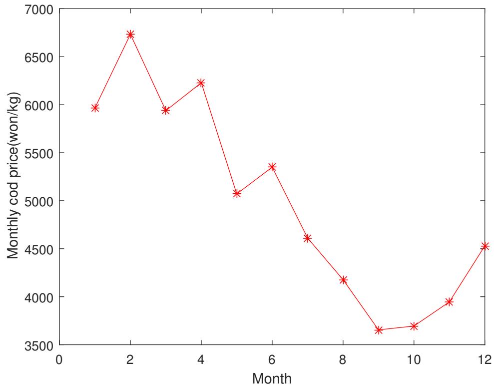


FIGURE 1. Average monthly price of Pacific cod from 2013 to 2017 in South Korea

for $t \in [n, n+1]$, where $n = 0, 1, \cdots, N$. Hence, we obtain

$$\begin{aligned} 0 &\geq \sum_{n=0}^{N} \int_{n}^{n+1^-} \left[((Y_n)_{u_n}(B^*_{i,n}, u^*_n) + \sum_{i=1}^{2} \lambda_{i,n}(t)(X_{i,n})_{u_n}) \right] h_n(t)dt \\ &= \sum_{n=0}^{N} \int_{n}^{n+1^-} \left[C_1(t)\alpha B^*_{2,n}(t) - 2C_2 u^*_n(t) + \alpha B^*_{2,n}(t)\lambda_{2,n}(t) \right] h_n(t)dt. \end{aligned}$$

Since h_n is arbitrary for $n = 0, 1, \cdots, N$, which implies that

$$u^*_n(t) = \frac{C_1(t)\alpha B^*_{2,n}(t) + \alpha B^*_{2,n}(t)\lambda_{2,n}(t)}{2C_2}.$$

□

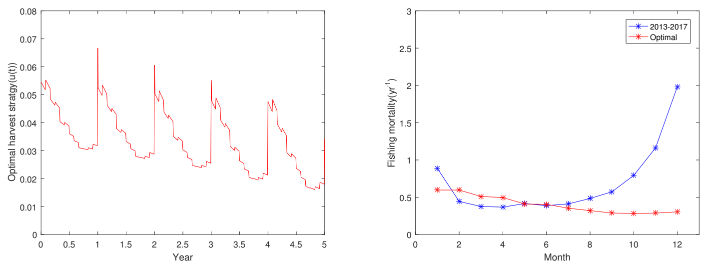
3. Numerical Simulation

We obtain parameters with respect to growth rate, natural morality, and effective fecundity of Pacific cod in Jung et al.[10]. Since they estimated parameters of Pacific cod by age, we adjusted the value of parameters of juvenile and adult stage as a mean value. The data for the monthly catch and price of Pacific cod have been obtained from Ministry of Oceans and Fisheries. Figure 1 shows an average monthly price of Pacific cod from 2013 to 2017 in South Korea. We assumed that the density dependent mortality(D_M) and catchability coefficient(α) are 10^{-6} and 1, respectively. Table 1 is the list of all parameters used in our numerical simulation.

In order to simulate optimal harvest strategy, we assumed that a cost unit effort(C_2) and weight of terminal term of objective functional are 2.74 billion won and 1. Figure 2(a) shows the optimal harvest control from 2013 to 2017

TABLE 1. Parameters used in optimal control model (1)

Parameter	Description	value
g_1	average growth rate of juvenile	$0.4246yr^{-1}$
g_2	average growth rate of adult	$0.0463yr^{-1}$
M_1	natural mortality of juvenile	$0.3788yr^{-1}$
M_2	natural mortality of adult	$0.0343yr^{-1}$
D_M	density dependent mortality	10^{-6}
f_R	average effective fecundity of adult	1820
W_1	average weight of an individual of juvenile	$15.1g$
W_2	average weight of an individual of adult	$1.467kg$



(a) (b)

FIGURE 2. (a) Optimal harvest control from 2013 to 2017 when our initial condition $(B_{1,0}(0), B_{2,0}(0)) = (7960, 28250)$ metric tons. (b) Monthly average fishing mortality and monthly average optimal harvest strategy of Pacific cod from 2013 to 2017.

when our initial condition $(B_{1,0}(0), B_{2,0}(0)) = (7960, 28250)$ metric tons. Optimal harvest strategy is almost the same pattern every year. It is best to fishing of the Pacific cod in January and to reduce fishing continuously until just before December. Figure 2(b) show estimated monthly average fishing mortality and monthly average optimal harvest strategy of Pacific cod from 2013 to 2017. This result represent the opposite result about estimated monthly average optimal harvest strategy and estimated monthly average fishing mortality using real catch data, which is the largest catch in December. Although it is the theoretical harvest strategy, it is necessary to change the present fishing strategy that has the opposite result from the optimal harvest strategy. In this paper, although we have proposed an optimal harvest strategy for Pacific cod only, we believe that our approach and method can be successfully extended to other fish species.

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