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EXISTENCE OF POSITIVE SOLUTIONS FOR THE SECOND ORDER DIFFERENTIAL SYSTEMS WITH STRONGLY COUPLED INTEGRAL BOUNDARY CONDITIONS

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Abstract. This paper concerned the existence of positive solutions to the second order differential systems with strongly coupled integral boundary value conditions. By using Krasnoselskii fixed point theorem, we prove the existence of positive solutions according to the parameters under the proper nonlinear growth conditions.

1. Introduction

In this paper, we study the existence of the following differential system;

$$\begin{cases} u''(t) + \lambda a_1(t) f_1(u(t), v(t)) = 0, \qquad t \in (0,1), \\ v''(t) + \lambda a_2(t) f_2(u(t), v(t)) = 0, \qquad t \in (0,1), \\ u(0) = 0 = v(0), \\ u(1) = \int_0^1 g_1(s)u(s) + g_2(s)v(s)ds, \\ v(1) = \int_0^1 g_3(s)u(s) + g_4(s)v(s)ds \end{cases} \tag{1}$$

where $a_i \in C((0,1),[0,\infty))$, $f_i \in C([0,\infty)^2),[0,\infty))$ and $g_i \in L^1((0,1),[0,\infty))$, for $i \in \{1,2,3,4\}$. We further assume that there exists a closed interval $J \subset (0,1)$ with positive measure such that $a_i(t) > 0$ for all $t \in J$ and $i = 1,2$.
Such differential equations with an integral boundary condition arise in various areas of applied mathematics and physics like heat conduction, chemical engineering, underground water flow, thermo-elasticity, hydro dynamic problems and plasma phenomena. One may refer to [1], [3], [4], [5] and [2] for integral boundary value problems and the references therein. Recently, many works have

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been done for second odrder ordinary differential systems with integral boundary conditions ([6], [7], [8], [9], [10], [11]), but most of papers considered the differntial systems with uncoupled or weakly coupled boundary conditons. For example, in [11], authors considered the following systems with weakly coupled integral boundary conditions,

$$\begin{cases} -x''(t) = f_1(t, x(t), y(t)), \quad t \in (0,1), \\ -y''(t) = f_2(t, x(t), y(t)), \quad t \in (0,1), \\ x(0) = 0 = y(0), \\ x(1) = \int_0^1 y(t)dA(t), \\ y(1) = \int_0^1 x(t)dB(t) \end{cases} \tag{2}$$

They prove the existence of positive solutions for (2) when f_i satisfy some growth conditions which imply the monotonicity of f_i.

In this paper, the problem (1) has more general strongly coupled integral boundary conditions, which makes the operaor T_λ (see Section 2 for definition) complicated and induces substantial difficulties in proving our results. Throughout this paper, we assume the following hypotheses;

$(H0)$ $\int_0^1 s(1-s)a_i(s)ds < \infty$ for $i = 1, 2$.

$(H1)$ $0 < f_{i,0} := \lim\limits_{|u|+|v| \to 0} \dfrac{f_i(u,v)}{u+v} < \infty$ for $i = 1, 2$.

$(H2)$ $0 < f_{i,\infty} := \lim\limits_{|u|+|v| \to \infty} \dfrac{f_i(u,v)}{u+v} < \infty,$ for $i = 1, 2$.

$(H3)$ $0 < \int_0^1 sg_i(s)ds < 1$ for $i = 1, 4$ and

$$(1 - \int_0^1 sg_1(s)ds)(1 - \int_0^1 sg_4(s)ds) - (\int_0^1 sg_2(s)ds)(\int_0^1 sg_3(s)ds) > 0$$

This paper is organized as follows. In Section 2, we present the solution operator to problem (1) and introduce the well-known fixed point theorem which will be used to prove our main result. In Section 3, the main results, Theorem 3.1 and Theorem 3.2, are proven. In Section 4, as applications, the reaults for the existence of radial solutions for the semilinear elliptic systems on exterior domain are given.

2. Preliminaries

In this section, we set up the operator equation for the problem (1). By $(H3)$, let

$$A := \begin{pmatrix} 1 - \int_0^1 sg_1(s)ds & -\int_0^1 sg_2(s)ds \\ -\int_0^1 sg_3(s)ds & 1 - \int_0^1 sg_4(s)ds \end{pmatrix},$$

then $\det A \neq 0$ and let

$$A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Here, we note that from $(H3)$, $a_{ij} > 0$ for all $i, j \in \{1, 2\}$. Let us denote $X := C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ where X is the usual Banach space with the norm $\|(u,v)\| = \|u\|_\infty + \|v\|_\infty$.

We define A_λ and B_λ from X to $C([0,1], \mathbb{R})$ by

$$A_\lambda(u,v)(t) := \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds$$

$$B_\lambda(u,v)(t) := \lambda \int_0^1 H_2(t,s)a_2(s)f_2(u(s),v(s)) + tK_2(s)a_1(s)f_1(u(s),v(s))ds$$

where

$$\begin{aligned} H_1(t,s) &= G(t,s) + t\int_0^1 G(\tau,s)(a_{11}g_1(\tau) + a_{12}g_3(\tau))d\tau, \\ H_2(t,s) &= G(t,s) + t\int_0^1 G(\tau,s)(a_{21}g_2(\tau) + a_{22}g_4(\tau))d\tau, \\ K_1(s) &= \int_0^1 G(\tau,s)(a_{11}g_2(\tau) + a_{12}g_4(\tau))d\tau, \\ K_2(s) &= \int_0^1 G(\tau,s)(a_{21}g_1(\tau) + a_{22}g_3(\tau))d\tau, \end{aligned}$$

and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1 \end{cases}$$

Now we define

$$T_\lambda(u,v)(t) := (A_\lambda(u,v)(t), B_\lambda(u,v)(t)).$$

Then $T_\lambda : X \to X$ is well defined and notice that the problem (1) is equivalent to the following operator equation;

$$(u,v) = T_\lambda(u,v) \text{ on } X.$$

Let $\mathcal{P} = \{(u,v) \in X : u(t) \ge 0, v(t) \ge 0 \text{ for all } t \in [0,1]\}$. Then $\mathcal{P}$ is a cone in X. It is clear that $T_\lambda(\mathcal{P}) \subset \mathcal{P}$ and T_λ is completely continuous on X, by standard argument.

We recall $J \subset (0,1)$ is a nondegenerate closed interval such that $a_i(t) > 0$ for all $t \in J$ and $i = 1, 2$. Let $\gamma = \min\{j_*, 1 - j^*\} > 0$ where $j_* = \inf J$ and $j^* = \sup J$. Here we define $\mathcal{K}$ by

$$\mathcal{K} = \{(w_1, w_2) \in \mathcal{P} : \min_J w_i(t) \ge \gamma\|w_i\|_\infty, \text{ for } i = 1, 2\}.$$

Then $\mathcal{K}$ is cone and we have the following lemma.

Remark 1. It is easy to check that

$$G(t,s) \leq s(1-s),\ t,s \in (0,1). \tag{3}$$

and

$$G(t,s) \geq \gamma s(1-s),\ t \in J,\ s \in (0,1). \tag{4}$$

Lemma 2.1. *For a given cone $\mathcal{P}$ in X, it holds that*

$$T_\lambda(\mathcal{P}) \subset \mathcal{K}.$$

Proof. For given $(u,v) \in \mathcal{P}$, from (3), we first find for $t \in [0,1]$,

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\ &\leq \lambda \int_0^1 h_1(s)a_1(s)f_1(u(s),v(s)) + K_1(s)a_2(s)f_2(u(s),v(s))ds \end{aligned}$$

where

$$h_1(s) = s(1-s) + \int_0^1 G(\tau,s)(a_{11}g_1(\tau) + a_{12}g_3(\tau))d\tau.$$

Thus, we obtain

$$\|A_\lambda(u,v)\|_\infty \leq \lambda \int_0^1 h_1(s)a_1(s)f_1(u(s),v(s)) + K_1(s)a_2(s)f_2(u(s),v(s))ds \tag{5}$$

Similarly, we have

$$\|B_\lambda(u,v)\|_\infty \leq \lambda \int_0^1 h_2(s)a_2(s)f_2(u(s),v(s)) + K_2(s)a_1(s)f_1(u(s),v(s))ds \tag{6}$$

where

$$h_2(s) = s(1-s) + \int_0^1 G(\tau,s)(a_{21}g_2(\tau) + a_{22}g_4(\tau))d\tau,$$

Then by (4) and (5), we find that for all $t \in J$,

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\ &\geq \lambda\gamma \left(\int_0^1 h_1(s)a_1(s)f_1(u(s),v(s)) + K_1(s)a_2(s)f_2(u(s),v(s))ds \right) \\ &\geq \gamma \|A_\lambda(u,v)\|_\infty \end{aligned}$$

From the same argument we also have that $B_\lambda(u,v)(t) \geq \gamma\|B_\lambda(u,v)\|_\infty$ for all $t \in J$ by using (6). $\square$

To prove our main result, we use the following fixed point theorem in a cone due to Guo and Lakshmikantham [12].

Theorem 2.2. *(Fixed point theorem)*
Let X is a real banach space, $\mathcal{K}$ is a cone of X. Assume that Ω_1 , Ω_2 are openset of X with $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$ and let $T : \mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow \mathcal{K}$ be completely continuous and satisfying either

(i) $\|Tu\| \leq \|u\|, u \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, u \in \mathcal{K} \cap \partial\Omega_2$

or

(ii) $\|Tu\| \geq \|u\|, u \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, u \in \mathcal{K} \cap \partial\Omega_2$

Then T has a fixed point in $\mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$

3. Main Result

In this section, we establish the existence results for positive solutions of (1).

Theorem 3.1. *Assume $(H0) \sim (H3)$ and $\lambda_* < \lambda^*$ when*

$$\lambda^* = \min \left\{ \frac{1}{f_{1,0} \int_0^1 h_1(s)a_1(s)ds + f_{2,0} \int_0^1 K_1(s)a_2(s)ds}, \frac{1}{f_{2,0} \int_0^1 h_2(s)a_2(s)ds + f_{1,0} \int_0^1 K_2(s)a_1(s)ds} \right\}$$

$$\lambda_* = \max \left\{ \frac{1}{\gamma^2(f_{1,\infty} \int_J h_1(s)a_1(s)ds + f_{2,\infty} \int_0^1 K_1(s)a_2(s)ds)}, \frac{1}{\gamma^2(f_{2,\infty} \int_J h_2(s)a_2(s)ds + f_{1,\infty} \int_0^1 K_2(s)a_1(s)ds)} \right\}$$

where

$$\begin{aligned} h_1(s) &= G(s,s) + \int_0^1 G(\tau,s)(a_{11}g_1(\tau) + a_{12}g_3(\tau))d\tau, \\ h_2(s) &= G(s,s) + t\int_0^1 G(\tau,s)(a_{21}g_2(\tau) + a_{22}g_4(\tau))d\tau. \end{aligned}$$

Then for all λ satisfying

$$\frac{1}{2}\lambda_* < \lambda < \frac{1}{2}\lambda^*,$$

there exist at least one positive solution of (1).

Proof. Let λ be given as hypothesis. Choose $\epsilon > 0$ as

$$\begin{aligned} \lambda &< \frac{1}{2((f_{1,0}+\epsilon) \int_0^1 h_1(s)a_1(s)ds + (f_{2,0}+\epsilon) \int_0^1 K_1(s)a_2(s)ds)} \quad \text{and} \\ \lambda &< \frac{1}{2((f_{2,0}+\epsilon) \int_0^1 h_2(s)a_2(s)ds + (f_{1,0}+\epsilon) \int_0^1 K_2(s)a_1(s)ds)} \end{aligned}$$

$$\lambda > \frac{1}{2\gamma^2((f_{1,\infty}-\epsilon)\int_J h_1(s)a_1(s)ds + (f_{2,\infty}-\epsilon)\int_0^1 K_1(s)a_2(s)ds)} \quad \text{and}$$
$$\lambda > \frac{1}{2\gamma^2((f_{2,\infty}-\epsilon)\int_J h_2(s)a_2(s)ds + (f_{1,\infty}-\epsilon)\int_0^1 K_2(s)a_1(s)ds)}$$

From $(H1)$, there exists R_1 such that $f_i(u,v) \le (f_{i,0}+\epsilon)(u+v)$ when $0 < |u|+|v| \le R_1$. Define $\Omega_1 = \{(u,v) \in X | \|(u,v)\| < R_1\}$. Then if $(u,v) \in \mathcal{K} \cap \partial\Omega_1$, then $u(s)+v(s) \le \|u\|_\infty + \|v\|_\infty = \|(u,v)\| = R_1$ for all $s \in [0,1]$ and for $t \in [0,1]$,

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\ &\le \lambda \int_0^1 H_1(t,s)a_1(s)(f_{1,0}+\epsilon)(u(s)+v(s)) \\ &\qquad\qquad + K_1(s)a_2(s)(f_{2,0}+\epsilon)(u(s)+v(s))ds \\ &\le \lambda \int_0^1 h_1(s)a_1(s)(f_{1,0}+\epsilon) + K_1(s)a_2(s)(f_{2,0}+\epsilon)ds(\|u\|_\infty + \|v\|_\infty) \\ &\le \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Similarly, we can get

$$B_\lambda(u,v)(t) \le \frac{1}{2}\|(u,v)\| \text{ for } t \in [0,1]$$

Thus $\|T_\lambda(u,v)\|_\infty \le \|A_\lambda(u,v)\|_\infty + \|B_\lambda(u,v)\|_\infty \le \|(u,v)\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$. Next, from $(H2)$, there exist $\overline{R_2} > 0$ such that $f_i(u,v) \ge (f_{i,\infty}-\epsilon)(u+v)$ when $|u|+|v| \ge \overline{R_2}$. Let $R_2 = \max\{2R_1, \frac{1}{\gamma}\overline{R_2}\}$ and let $\Omega_2 = \{(u,v) \in X | \|(u,v)\| < R_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and if $(u,v) \in \mathcal{K} \cap \partial\Omega_2$, then we know that

$$\min_{t \in J}(u(t)+v(t)) \ge \gamma(\|u\|_\infty + \|v\|_\infty) = \gamma R_2 \ge \overline{R_2}.$$

For $t \in J$,

$$\begin{aligned} A_\lambda(u,v)(t) &= \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\ &\ge \lambda \int_J H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\ &\ge \lambda \int_J H_1(t,s)a_1(s)(f_{1,\infty}-\epsilon)(u(s)+v(s)) \\ &\qquad\qquad + \gamma K_1(s)a_2(s)(f_{2,\infty}-\epsilon)(u(s)+v(s))ds \\ &\ge \lambda\gamma^2 \left(\int_J h_1(s)a_1(s)(f_{1,\infty}-\epsilon) + \gamma K_1(s)a_2(s)(f_{2,\infty}-\epsilon)ds\right) \|(u,v)\| \\ &\ge \frac{1}{2}\|(u,v)\|. \end{aligned}$$

Similarly, we can get

$$B_\lambda(u,v)(t) \geq \frac{1}{2}\|(u,v)\| \text{ for } t \in J$$

Thus $\|T_\lambda(u,v)\|_\infty = \|A_\lambda(u,v)\|_\infty + \|B_\lambda(u,v)\|_\infty \geq \|(u,v)\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$. By Theorem 2.2, T_λ has a fixed point $(u,v) \in \mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$. □

The following result looks similar with Theorem 3.1, but the proof is different because of the growth conditions.

Theorem 3.2. *Assume* $(H0) \sim (H3)$ *and* $\underline{\lambda} < \overline{\lambda}$ *when*

$$\overline{\lambda} = \min \Big\{\frac{1}{f_{1,\infty}\int_0^1 h_1(s)a_1(s)ds + f_{2,\infty}\int_0^1 K_1(s)a_2(s)ds}, \frac{1}{f_{2,\infty}\int_0^1 h_2(s)a_2(s)ds + f_{1,\infty}\int_0^1 K_2(s)a_1(s)ds}\Big\}$$

$$\underline{\lambda} = \max \Big\{\frac{1}{\gamma^2(f_{1,0}\int_J h_1(s)a_1(s)ds + f_{2,0}\int_0^1 K_1(s)a_2(s)ds)}, \frac{1}{\gamma^2(f_{2,0}\int_J h_2(s)a_2(s)ds + f_{1,0}\int_0^1 K_2(s)a_1(s)ds)}\Big\}$$

where h_1 and h_2 are the same in the Theorem 3.1. Then for all λ satisfying

$$\frac{1}{2}\underline{\lambda} < \lambda < \frac{1}{2}\overline{\lambda},$$

there exist at least one positive solution of (1).

Proof. Let λ be given as hypothesis. Choose $\epsilon > 0$ as

$$\lambda < \frac{1}{2((f_{1,\infty}+\epsilon)\int_0^1 h_1(s)a_1(s)ds + (f_{2,\infty}+\epsilon)\int_0^1 K_2(s)a_2(s)ds)} \quad \text{and}$$

$$\lambda < \frac{1}{2((f_{2,\infty}+\epsilon)\int_0^1 h_2(s)a_2(s)ds + (f_{1,\infty}+\epsilon)\int_0^1 K_1(s)a_1(s)ds)}$$

$$\lambda > \frac{1}{2\gamma^2((f_{1,0}-\epsilon)\int_J h_1(s)a_1(s)ds + (f_{2,0}-\epsilon)\int_0^1 K_2(s)a_2(s)ds)} \quad \text{and}$$

$$\lambda > \frac{1}{2\gamma^2((f_{2,0}-\epsilon)\int_J h_2(s)a_2(s)ds + (f_{1,0}-\epsilon)\int_0^1 K_1(s)a_1(s)ds)}$$

There exists R_1 such that $f_i(u,v) \geq (f_{i,0}-\epsilon)(u+v)$ when $0 < |u| + |v| \leq R_1$. Define $\Omega_1 = \{(u,v) \in X | \|(u,v)\| < R_1\}$. Then for $(u,v) \in \mathcal{K} \cap \partial\Omega_1$, by using

$$\min_{t \in J}(u(t) + v(t)) \geq \gamma(\|u\|_\infty + \|v\|_\infty),$$

we have for $t \in J$

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\
&\geq \lambda \int_J H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\
&\geq \lambda \int_J H_1(t,s)a_1(s)(f_{1,0}-\epsilon)(u(s)+v(s)) \\
&\qquad\qquad + \gamma K_1(s)a_2(s)(f_{2,0}-\epsilon)(u(s)+v(s))ds \\
&\geq \lambda\gamma^2 \left(\int_J h_1(s)a_1(s)(f_{1,0}-\epsilon) + \gamma K_1(s)a_2(s)(f_{2,0}-\epsilon)ds \right) \|(u,v)\| \\
&\geq \frac{1}{2}\|(u,v)\|.
\end{aligned}$$

Similarly, we can get

$$B_\lambda(u,v)(t) \geq \frac{1}{2}\|(u,v)\| \text{ for } t \in J.$$

Thus $\|T_\lambda(u,v)\|_\infty \geq \|A_\lambda(u,v)\|_\infty + \|B_\lambda(u,v)\|_\infty \geq \|(u,v)\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$. Next, if we define the function $\overline{f}_i \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ for $i = 1, 2$ by $\overline{f}_i(u,v) = \max_{(x,y)\in[0,u]\times[0,v]} f(x,y)$, then it is easy to check that $f_i(u,v) \leq \overline{f}_i(u,v)$ for all $(u,v) \in \mathbb{R}_+^2$, $\overline{f}_i$ are monotone increasing and

$$\lim_{u+v\to\infty} \frac{\overline{f}_i(u,v)}{u+v} = f_{i,\infty}. \tag{7}$$

From (7), there exist $\overline{R_2} > 0$ such that $\overline{f}_i(u,v) \leq (f_{i,\infty}+\epsilon)(u+v)$ when $|u|+|v| \geq \overline{R_2}$. Let $R_2 = \max\{2R_1, \overline{R_2}\}$ and $\Omega_2 = \{(u,v) \in X | \|(u,v)\| < R_2\}$, then $\overline{\Omega_1} \subset \Omega_2$ and if $(u,v) \in \mathcal{K} \cap \partial\Omega_2$, then$\|(u,v)\| = \|u\|_\infty + \|v\|_\infty = R_2 \geq \overline{R_2}$, and for $t \in [0,1]$,

$$\begin{aligned}
A_\lambda(u,v)(t) &= \lambda \int_0^1 H_1(t,s)a_1(s)f_1(u(s),v(s)) + tK_1(s)a_2(s)f_2(u(s),v(s))ds \\
&\leq \lambda \int_0^1 h_1(s)a_1(s)\overline{f}_1(u(s),v(s)) + K_1(s)a_2(s)\overline{f}_2(u(s),v(s))ds \\
&\leq \lambda \int_0^1 h_1(s)a_1(s)\overline{f}_1(\|u\|_\infty,\|v\|_\infty) + K_1(s)a_2(s)\overline{f}_2(\|u\|_\infty,\|v\|_\infty)ds \\
&\leq \lambda \left(\int_0^1 h_1(s)a_1(s)(f_{1,\infty}+\epsilon) + K_1(s)a_2(s)(f_{2,\infty}+\epsilon)ds \right) \|(u,v)\| \\
&\leq \frac{1}{2}\|(u,v)\|.
\end{aligned}$$

Similarly, we can get

$$B_\lambda(u,v)(t) \leq \frac{1}{2}\|(u,v)\| \text{ for all } t \in [0,1].$$

Thus $\|T_\lambda(u,v)\|_\infty \le \|A_\lambda(u,v)\|_\infty + \|B_\lambda(u,v)\|_\infty \le \|(u,v)\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$. By Theorem 2.2, T_λ has a fixed point $(u,v) \in \mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$. $\square$

4. Application

In this section, we consider the existence of positive radial solutions to the following integral boundary value system on an exterior domain:

$$\begin{cases} \Delta u + \lambda k_1(|x|) f_1(u(x), v(x)) = 0, & x \in \Omega_e, \\ \Delta v + \lambda k_2(|x|) f_2(u(x), v(x)) = 0, & x \in \Omega_e, \\ u(x) \to 0, \quad v(x) \to 0, & \text{if } \|x\| \to \infty, \\ u(x) = \int_{\Omega_e} l_1(|y|)u(y) + l_2(|y|)v(y)dy, & \text{if } \|x\| = r_0, \\ v(x) = \int_{\Omega_e} l_3(|y|)u(y) + l_4(|y|)v(y)dy, & \text{if } \|x\| = r_0, \end{cases} \tag{8}$$

where $\Omega_e = \{x \in \mathbb{R}^N : \|x\|_\infty \ge r_0 \text{ for } r_0 > 0, N \ge 3\}$, $k_i \in C((r_0,\infty),(0,\infty))$, $f_i \in C([0,\infty) \times [0,\infty)), [0,\infty))$, and $l_i \in L^1((r_0,\infty)), [0,\infty))$. We further assume that there exists an interval $I \subset (r_0,\infty)$ with positive measure such that $k_i(r) > 0$ for all $r \in I$ and $i = 1, 2$.

By the change of variables $r = |x|$ and $t = (\frac{r}{r_0})^{2-N}$, (8) can be transformed into (1) with

$$a_i(t) = \left(\frac{1}{N-2}\right)^2 r_0^2 t^{\frac{-2(N-1)}{N-2}} k_i\left(r_0 t^{\frac{-1}{N-2}}\right),$$
$$g_i(t) = w_N \left(\frac{1}{N-2}\right) r_0^N t^{\frac{-2(N-1)}{N-2}} l_i\left(r_0 t^{\frac{-1}{N-2}}\right),$$

and w_N is the surface area of unit sphere in $\mathbb{R}^N$. Hence the existence of positive solutions for the system (1) guarantees the existence of positive radial solutions for (8). Thus we consider the following assumptions;

$(H0')$ $\int_{r_0}^\infty r k_i(r)dr < \infty$ for $i = 1, 2$.

$(H3')$ $0 < w_N r_0^{N-2} \int_{r_0}^\infty r l_i(r)dr < 1$ for $i = 1, 4$ and

$$(w_N^{-1} r_0^{2-N} - \int_{r_0}^\infty r l_1(r)dr)(w_N^{-1} r_0^{2-N} - \int_{r_0}^\infty r l_4(r)dr) \\ -(\int_{r_0}^\infty r l_2(r)dr)(\int_{r_0}^\infty r l_3(r)dr) > 0$$

It is easy to check that $(H0')$ and $(H3')$ imply $(H0)$ and $(H3)$. Thus we can apply Theorem 3.1 and Theorem 3.2 to obtain the following results.

Corollary 4.1. *Assume* $(H0')$, $(H1)$, $(H2)$, *and* $(H3')$. *If* $\lambda_* < \lambda^*$ *when* λ^* *and* λ_* *are the ones defined in Theorem 3.1, then the problem* (8) *has at least one positive radial solution for* $\lambda \in (\frac{1}{2}\lambda_*, \frac{1}{2}\lambda^*)$.

Corollary 4.2. *Assume* $(H0')$, $(H1)$, $(H2)$, *and* $(H3')$. *If* $\underline{\lambda} < \overline{\lambda}$ *when* $\overline{\lambda}$ *and* $\underline{\lambda}$ *are the ones defined in Theorem 3.2, then the problem* (8) *has at least one positive radial solution for* $\lambda \in (\frac{1}{2}\underline{\lambda}, \frac{1}{2}\overline{\lambda})$.

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