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2-TYPE HYPERSURFACES SATISFYING $\langle \Delta x, x - x_0 \rangle = \text{const.}$

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Abstract. Let M be a connected n-dimensional submanifold of a Euclidean space E^{n+k} equipped with the induced metric and Δ its Laplacian. If the position vector x of M is decomposed as a sum of three vectors $x = x_1 + x_2 + x_0$ where two vectors x_1 and x_2 are non-constant eigenvectors of the Laplacian , i.e., $\Delta x_i = \lambda_i x_i, i = 1, 2$ ($\lambda_i \in R$) and x_0 is a constant vector, then, M is called a 2-type submanifold. In this paper we proved that a connected 2-type hypersurface M in E^{n+1} whose postion vector x satisfies $\langle \Delta x, x - x_0 \rangle = c$ for a constant c, where $\langle \ , \ \rangle$ is the usual inner product in E^{n+1}, is of null 2-type and has constant mean curvature and scalar curvature.

1. Introduction

Let M be an n-dimensional submanifold of the $(n+k)$-dimensionl Euclidean space E^{n+k}, equipped with the induced metric. Denote by Δ the Laplacian of M. If the position vector x of M in E^{n+k} can be decomposed as a finite sum of non-constant eigenvectors of Δ, we shall say that M is of finite-type. More precisely, M is said to be of q-type if the position vector x of M can be expressed as in the following form:

$$x = x_0 + x_{i_1} + \cdots + x_{i_q},$$

where x_0 is a constant vector, and $x_{i_j} (j = 1, \cdots, q)$ are non-constant vectors in E^{n+k} such that $\Delta x_{i_j} = \lambda_{i_j} x_{i_j}$, $\lambda_{i_j} \in R$, $\lambda_{i_1} < \cdots < \lambda_{i_q}$. The notion of finite-type submanifolds was introduced by B.-Y. Chen [1]. Many results concerning this subject are obtained during last three decades. One of the interesting research areas on this subject is a classification of 2-type submanifolds and sevaral authors obtained important results ([2][5][6]). The only known examples of finite-type hypersurface are minimal hypersurfaces, hyperspheres, and a spherical cylinders. One can observe that the position vector x of every known finite-type hypersurface satisfies the condition

$$\langle \Delta x, x - x_0 \rangle = c \tag{1}$$

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for a fixed constant vector x_0 and a constant c, where $\langle\ ,\ \rangle$ means the usual inner product in Euclidean space. In [3], the author and H. Jo studied a connected 2-type surface M in E^3 satisfying the condition (1) whose postion vector x is decomposed as $x = x_0 + x_1 + x_2$, $\Delta x_i = \lambda_i x_i, i = 1, 2$ and showed M is an open part of a circular cylinder. In this paper, we will study a connected 2-type hypersurface M whose postion vector x satisfying the condition (1) and will show that such a hypersurface M is of null 2-type (i.e., one of λ_i's is zero.) and has constant mean curvature and scalar curvature. Moreover we will show that its support function $\langle x - x_0, e_{n+1}\rangle$, where e_{n+1} is a unit normal to M, is constant.

2. Preliminaries

Consider a hypersurface M of E^{n+1} and denote $\bar{\nabla}$ and ∇ the usaual Riemannian connection of E^{n+1} and the induced connection on M, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \bar{\nabla}_X \xi &= -A_\xi X + D_X \xi \end{aligned} \qquad (2)$$

for vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form, D the normal connnection, and A the shape operator of M. For each normal vector ξ at a point $p \in M$, the shape operator A_ξ is a self adjoint operator of the tangent space T_pM at p. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y\rangle = \langle h(X, Y), \xi\rangle, \qquad (3)$$

where $\langle\ ,\ \rangle$ is the usaual inner product in E^{n+1}. Let v be an E^{n+1}-valued smooth function on M, and let $\{e_1, e_2, \cdots, e_n\}$ be a local orthornomal frame field of M. We define

$$\Delta v = \sum_{i=1}^{n} (\bar{\nabla}_{e_i} \bar{\nabla}_{e_i} v - \bar{\nabla}_{\nabla_{e_i} e_i} v).$$

It is well known that the position vector x and the mean curvature vector H of M in E^{n+1} satisfy

$$\Delta x = H. \qquad (4)$$

Let e_{n+1} be a local unit normal vector to M. Since the mean curvature vector H is normal to M, we have $H = \langle H, e_{n+1}\rangle e_{n+1}$. The function $\langle H, e_{n+1}\rangle$ is called mean curvature function and it will be denoted by α. The general basic formula of ΔH derived in [1] plays an important role in the study of low type. In particular, if M is a hypersurface in E^{n+1}, it reduces to

$$\Delta H = (\Delta\alpha - \alpha||A_{e_{n+1}}||^2)e_{n+1} - 2A_{e_{n+1}}(\text{grad}\alpha) - \alpha\text{grad}\alpha. \qquad (5)$$

3. 2-type hypersurface in E^{n+1} satisfying $\langle \Delta x, x - x_0 \rangle =$ const.

Let M be a connected 2-type hypersurface in E^{n+1}. Then its position vector x is expressed in the form

$$x = x_0 + x_1 + x_2,$$

where x_0 is a constant vector, and $x_i (i = 1, 2)$ are nonconstant vectors in E^{n+1} such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. By (4) we have $\Delta x = H = \lambda_1 x_1 + \lambda_2 x_2$ and $\Delta^2 x = \Delta H = \lambda_1^2 x_1 + \lambda_2^2 x_2$. Thus

$$\Delta^2 x = (\lambda_1 + \lambda_2)\Delta x - \lambda_1 \lambda_2 (x - x_0). \tag{6}$$

By comparing the tangential part of both (5) and (6), we find

$$\lambda_1 \lambda_2 (x - x_0)^T = 2A_{e_{n+1}}(\mathrm{grad}\alpha) + \alpha \mathrm{grad}\alpha, \tag{7}$$

where $(x - x_0)^T$ means the tangential part of the vector $x - x_0$. Now suppose that M satisfies $\langle \Delta x, x - x_0 \rangle = c$ for a constant c. We have the following lemma.

Lemma 3.1. *Let M be a hypersurface of the Euclidean sapce E^{n+1} satifying the condition $\langle \Delta x, x - x_0 \rangle = c$ for a constant vector x_0 and a constant c. Then we get the following:*

$$\langle \Delta^2 x, x - x_0 \rangle = \langle \Delta x, \Delta x \rangle. \tag{8}$$

Proof. Let $\{e_1, e_2, \cdots, e_n\}$ be a local orthonormal frame of M. Since

$$\begin{aligned}
\Delta \langle \Delta x, x - x_0 \rangle &= \sum_{i=1}^{n} e_i e_i \langle \Delta x, x - x_0 \rangle - \sum_{i=1}^{n} \nabla_{e_i} e_i \langle \Delta x, x - x_0 \rangle \\
&= \sum_{i=1}^{n} e_i (\langle \bar{\nabla}_{e_i}(\Delta x), x - x_0 \rangle + \langle \Delta x, e_i \rangle) \\
&\quad - \sum_{i=1}^{n} (\langle \bar{\nabla}_{\nabla_{e_i} e_i}(\Delta x), x - x_0 \rangle + \langle \Delta x, \nabla_{e_i} e_i \rangle) \\
&= \sum_{i=1}^{n} e_i \langle \bar{\nabla}_{e_i}(\Delta x), x - x_0 \rangle - \sum_{i=1}^{2} \langle \bar{\nabla}_{\nabla_{e_i} e_i}(\Delta x), x - x_0 \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(\langle \bar{\nabla}_{e_i}\bar{\nabla}_{e_i}(\Delta x), x-x_0\rangle + \langle \bar{\nabla}_{e_i}(\Delta x), e_i\rangle) \\
&\quad -\sum_{i=1}^{n}\langle \bar{\nabla}_{\nabla_{e_i}e_i}(\Delta x), x-x_0\rangle \\
&= \langle \Delta(\Delta x), x-x_0\rangle + \sum_{i=1}^{n}\langle \bar{\nabla}_{e_i}(\Delta x), e_i\rangle \\
&= \langle \Delta^2 x, x-x_0\rangle + \sum_{i=1}^{n}\langle D_{e_i}(\Delta x) - A_{\Delta x}e_i, e_i\rangle \text{ (by (2))} \\
&= \langle \Delta^2 x, x-x_0\rangle - \sum_{i=1}^{n}\langle A_{\Delta x}e_i, e_i\rangle \\
&= \langle \Delta^2 x, x-x_0\rangle - \sum_{i=1}^{n}\langle \Delta x, h(e_i,e_i)\rangle \text{ (by (3))} \\
&= \langle \Delta^2 x, x-x_0\rangle - \langle \Delta x, \Delta x\rangle
\end{aligned}
$$

and $\langle \Delta x, x-x_0\rangle = c$, we have

$$\langle \Delta^2 x, x-x_0\rangle = \langle \Delta x, \Delta x\rangle.$$

□

From (6), (8) and $\langle \Delta x, x-x_0\rangle = c$, we get

$$(\lambda_1+\lambda_2)c - \lambda_1\lambda_2\langle x-x_0, x-x_0\rangle - \alpha^2 = 0.$$

Differentiating both sides of the above equation in the direction of a tangent vector X on M, we find

$$-2\lambda_1\lambda_2\langle x-x_0, X\rangle - 2\alpha X(\alpha) = 0$$

or

$$X(\alpha) = -\frac{\lambda_1\lambda_2}{\alpha}\langle X, (x-x_0)^T\rangle.$$

This implies that

$$\text{grad}\alpha = -\frac{\lambda_1\lambda_2}{\alpha}(x-x_0)^T. \tag{9}$$

Proposition 3.2. *Let M be a connnected 2-type hypersurface in E^{n+1} whose position vector x is expressed as $x = x_0+x_1+x_2$, where x_0 is a constant vector, and $x_i (i=1,2)$ are nonconstant vectors in E^{n+1} such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. Then M satisfies $\langle \Delta x, x-x_0\rangle = c$ for a constant c if and only if M is a null 2-type hypersurface with cosnstant mean curvature and constant scalar curvature and its support finction $\langle x-x_0, e_{n+1}\rangle$, where e_{n+1} is a unit normal to M, is constant.*

Proof. First we will show that the necessary condition holds. Suppose that α is nonconstant. Since grad$\alpha \neq 0$, by (9) M is not of null 2-type. Substituting (9) into (7) we get

$$A_{e_n+1}(x-x_0)^T = -\alpha(x-x_0)^T,$$

which implies that gradα is a principal vector of the shape operator $A_{e_{n+1}}$ and the corresponding principal curvature is $-\alpha$. Since

$$\langle \Delta x, x-x_0\rangle = \langle \alpha e_{n+1}, x-x_0\rangle = c,$$

we have

$$\langle e_{n+1}, x-x_0\rangle = \frac{c}{\alpha}.$$

Differentiating both sides of the above eqaution in the direction of the tangent vector fields e_1 which is parallel to gradα, we find

$$\langle \alpha e_1, x-x_0\rangle = -\frac{e_1(\alpha)c}{\alpha^2},$$

or

$$\langle \alpha e_1, (x-x_0)^T\rangle = -\frac{e_1(\alpha)c}{\alpha^2}.$$

From (9) we know that $(x-x_0)^T = -\frac{\alpha}{\lambda_1\lambda_2}$grad$\alpha$. Since grad$\alpha = e_1(\alpha)e_1$, from the above equation, we get

$$-\frac{e_1(\alpha)\alpha^2}{\lambda_1\lambda_2} = -\frac{e_1(\alpha)c}{\alpha^2}$$

or

$$e_1(\alpha)\frac{\alpha^4 - c\lambda_1\lambda_2}{\lambda_1\lambda_2\alpha^2} = 0,$$

which implies that α is constant. This is a contradiction. So the mean curvature α is constant. Subsequently, from (9), we know that M is of null 2-type. Without loss of generalty, we may assume that $\lambda_2 = 0$ and $\Delta x = \Delta x_1 = \lambda_1 x_1$. Since $\Delta x = \alpha e_{n+1}$, we find $\alpha^2 = \lambda_1^2||x_1||^2$ or $||x_1||^2 = \frac{\alpha^2}{\lambda_1^2}$. From $\Delta H = \lambda_1^2 x_1$ and (7) we know that $\lambda_1 = -||A_{e_{n+1}}||^2$. So we can conclude that the scalar curvature of M is constant. From (8), we can see that

$$\langle \lambda_1^2 x_1, x_1 + x_2\rangle = \langle \Delta x, x-x_0\rangle = \langle \Delta x, \Delta x\rangle = \langle \lambda_1 x_1, \lambda_1 x_1\rangle.$$

So we have $\langle x_1, x_2\rangle = 0$, which means that x_2 is tangential. Therefore the support fuction $\langle x-x_0, e_{n+1}\rangle$ is equal to the constant $\frac{\alpha}{\lambda_1}$. So the necessary part is proven. The sufficient condition can be easily proven. □

Corollary 3.3. [3] *Let M be a connected 2-type surface in E^3 whose position vector x is expressed as $x = x_0 + x_1 + x_2$, where x_0 is a constant vector, and x_i $(i = 1, 2)$ are nonconstant vectors in E^3 such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. Assume that $\langle \Delta x, x-x_0\rangle = c$ holds for a constant c. Then M is an open part of a circular cylinder.*

Proof. By Proposition 3.2 , the mean curvature and the Gauss curvature of M is consatnt. Thus M is an open part of a plane or a sphere or a circular cylinder. But both of plane and sphere are of 1-type. So M is an open part of a circular cylinder. □

Corollary 3.4. *Let M be a connected 2-type hypersurface in E^4 whose position vector x is expressed as $x = x_0 + x_1 + x_2$, where x_0 is a constant vector, and $x_i (i = 1, 2)$ are nonconstant vectors in E^4 such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. Assume that $\langle \Delta x, x - x_0 \rangle = c$ holds for a constant c. Then M is an open part of a spherical cylinder.*

Proof. By Proposition 3.2 M is of null 2-type. So we may assume that $\lambda_2 = 0$. Also we can see that x_1 is normal to M and x_2 is tangential to M and the support function $\langle x - x_0, e_4 \rangle$, where e_4 is a unit normal vector field to M, is constant. Differentiating this in the direction of arbitrary tangent vector X, we get

$$\langle x - x_0, -A_{e_4} X \rangle = \langle x_2, -A_{e_4} X \rangle = -\langle A_{e_4} x_2, X \rangle = 0.$$

This implies that $A_{e_4} x_2 = 0$. If the set $\{p \in M | x_2(p) = 0\}$ has a nonempty interior, then M is locally 1-type, which is a contradiction. Thus we can say that the set $\{p \in M | x_2(p) = 0\}$ has a empty interior and 0 is a principal curvature of M. By Proposition 3.2 the mean curvature and the scalar curvture M is constant. So every principal curvature of M is constant. Therefore M is an open part of a spherical cylinder. □

Corollary 3.5. *Let M be a complete oriented 2-type hypersurface in E^{n+1} whose position vector x is expressed as $x = x_0 + x_1 + x_2$, where x_0 is a constant vector, and $x_i (i = 1, 2)$ are nonconstant vectors in E^{n+1} such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in R$, $\lambda_1 \neq \lambda_2$. Assume that $\langle \Delta x, x - x_0 \rangle = c$ holds for a constant c. Then M is a spherical cylinder.*

Proof. In [4] it was shown that a connected and oriented complete hypersurface with constant support function in Euclidean space is either hyperplane or a hypersphere or a spherical cylinder. By Proposition 3.2 M is of null 2-type and its support function is constant. So we can conclude that M is a spherical cylinder. □

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