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SIF AND FINITE ELEMENT SOLUTIONS FOR CORNER SINGULARITIES

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Abstract. In [7, 8] they introduced a new finite element method for accurate numerical solutions of Poisson equations with corner singularities. They consider the Poisson equations with homogeneous boundary conditions, compute the finite element solutions using standard FEM and use the extraction formula to compute the stress intensity factor(s), then they posed new PDE with a regular solution by imposing the nonhomogeneous boundary condition using the computed stress intensity factor(s), which converges with optimal speed. From the solution they could get an accurate solution just by adding the singular part.

Their algorithm involves an iteration and the iteration number depends on the acuracy of stress intensity factors, which is usually obtained by extraction formula which use the finite element solutions computed by standard Finite Element Method.

In this paper we investigate the dependence of the iteration number on the convergence of stress intensity factors and give a way to reduce the iteration number, together with some numerical experiments.

1. Introduction

We start with outlines of the algorithm introduced in [7, 8]. First we let Ω be an open, bounded polygonal domain in $\mathbb{R}^2$ and let Γ_D and Γ_N be a partition of the boundary of Ω such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$, and consider the following Poisson equation with mixed boundary conditions:

$$\left\{\begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} &=& 0 & \text{on } \Gamma_N, \end{array}\right. \tag{1}$$

where $f \in L^2(\Omega)$ and Δ stands for the Laplacian operator. Here, ν denote the outward unit vector normal to the boundary.

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For simplicity, we assume that there is only one corner with the inner angle w : $\frac{3\pi}{2} < \omega < 2\pi$ and it satisfies D/N boundary condition as in **Figure 1**.

In this case we have two singular functions s_1 and s_3 and their dual singular functions s_{-1} and s_{-3};

$$s_j = s_j(r,\theta) = r^{\frac{j\pi}{2\omega}} \sin\frac{j\pi\theta}{2\omega}, \quad s_{-j} = s_{-j}(r,\theta) = r^{-\frac{j\pi}{2\omega}} \sin\frac{j\pi\theta}{2\omega}, \ (j = 1, 3) \quad (2)$$

for the model problem (1) and the unique solution $u \in H^1_D(\Omega)$ has the representation (see [3, 4]):

$$u = w + \lambda_1 \eta s_1 + \lambda_3 \eta s_3, \quad (3)$$

where $w \in H^2(\Omega) \cap H^1_D(\Omega)$, and η is a smooth cut-off function which equals one identically in a neighborhood of the origin and the support of η is small enough so that the function ηs vanishes identically on $\partial\Omega\backslash\Gamma_0$, where Γ_0 is the union of two adjacent boundary lines at the corner. (Here, (r, θ) is the polar coordinate.)

Since we are considering the mixed boundary condition at the corner the coefficient, λ_j, can be computed by the following extraction formula (see [3]):

$$\lambda_j = \frac{2}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{2}{j\pi}\int_\Omega u\Delta(\eta s_{-j})dx, \quad (j = 1, 3) \quad (4)$$

and called ‘stress intensity factors’. Note that both s_j and s_{-j} are harmonic functions in Ω.

In [7, 8] they posed the following algorithm:

A1.: Find a solution $u^{(0)}$ of (1) using the standard finite element method.

A2.: Compute the stress intensity factors $\lambda_1^{(0)}$ and $\lambda_3^{(0)}$ from (4) with $u = u^{(0)}$.

A3.: For $i = 1, 2, \cdots, M$;

A3-1.: Solve, for $w^{(i)}$,

$$\begin{cases} -\Delta w^{(i)} &= f & \text{in } \Omega, \\ w^{(i)} &= -\lambda_1^{(i-1)}s_1 - \lambda_3^{(i-1)}s_3 & \text{on } \Gamma_D, \\ \frac{\partial w^{(i)}}{\partial \nu} &= 0 & \text{on } \Gamma_N. \end{cases} \quad (5)$$

A3-2.: Let $u^{(i)} = w^{(i)} + \lambda_1^{(i-1)}s_1 + \lambda_3^{(i-1)}s_3$.

A3-3.: Compute $\lambda_j^{(i)}$ by

$$\lambda_j^{(i)} = \frac{2}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{2}{j\pi}\int_\Omega u^{(i)}\Delta(\eta s_{-j})dx, \quad j = 1, 3. \quad (6)$$

Using this algorithm, they got efficient results in computing the numerical solutions for Poisson equations with singularities. As we can see in **Step A3**, the algorithm involves an iteration, which depends on the accuracy of the values given in **Step A2**. In this paper we consider a Poisson problem with very strong singularity and test the dependency of the accuracy of $u^{(i)}$ to the accuracy of $\lambda_j^{(i)}$.

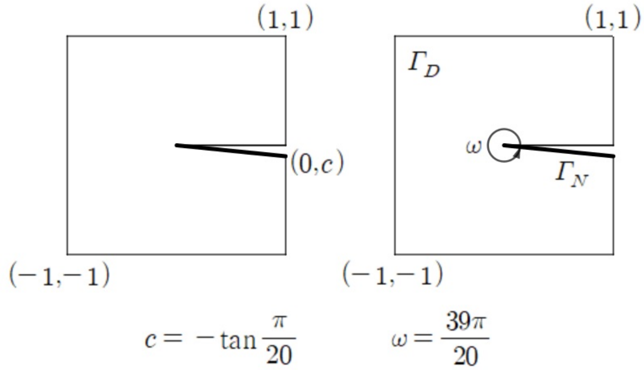

FIGURE 1. *An almostcrack domain Ω and its mixed boundary boundary condition at concave corner*

The computations will be done by using FreeFEM++ code ([5]).

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively, although we will omit Ω if there is no chance of misunderstanding. $H^1_D(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$.

2. Stress intensity factors and algorithms

In this section we will recall the cornerstone of the algorithm given in [7, 8].

We need a cut-off function to derive the singular behavior of the problem. We set

$$B(r_1; r_2) = \{(r,\theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1) = B(0; r_1),$$

and define a smooth enough cut-off function of r as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in} \quad B(\tfrac{1}{2}\rho), \\ \tfrac{1}{16}\{8 - 15p(r) + 10p(r)^3 - 3p(r)^5\} & \text{in} \quad \overline{B}(\tfrac{1}{2}\rho;\rho), \\ 0 & \text{in} \quad \Omega \backslash \overline{B}(\rho), \end{cases} \tag{7}$$

with $p(r) = 4r/\rho - 3$. Here, ρ is a parameter which will be determined so that the singular part $\eta_\rho s_j$ has the same boundary condition as the solution u of the model problem, where s_j is the singular function which is given in (2). Note $\eta_\rho(r)$ is C^2.

The solution of the Poisson equation on the polygonal domain is well known ([1, 4]). Given $f \in L^2(\Omega)$, since we assumed that there is only one reentrant corner with inner angle $\frac{3\pi}{2} < \omega < 2\pi$ and the boundary conditions change form Dirichlet to Neumann at the corner, then there exists a unique solution u and in addition there exists unique numbers λ_1 and λ_3 such that

$$u - \lambda_1 s_1 - \lambda_3 s_3 \in H^2(\Omega). \tag{8}$$

By using the cut-off function $\eta = \eta_\rho$, we may write

$$u = w + \lambda_1 \eta s_1 + \lambda_3 \eta s_3, \tag{9}$$

with $w \in H^2(\Omega) \cap H^1_D(\Omega)$.

The constants λ_j are referred as stress intensity factors and computed by the following formula ([3]);

Lemma 2.1. *The stress intensity factors λ_j can be expressed in terms of u and f by the following extraction formula:*

$$\lambda_j = \frac{2}{j\pi} \int_\Omega f \eta s_{-j} dx + \frac{2}{j\pi} \int_\Omega u \Delta(\eta s_{-j}) dx, \quad j = 1, 3. \tag{10}$$

The idea of the algorithm is the following:
Assume that (1) has a solution u as in (9) and the stress intensity factor λ_j is known, then the following boundary value problem:

$$\begin{cases} -\Delta w &= f & \text{in } \Omega, \\ w &= -\lambda_1 s_1 - \lambda_3 s_3 & \text{on } \Gamma_D, \\ \frac{\partial w}{\partial \nu} &= -\lambda_1 \frac{\partial s_1}{\partial \nu} - \lambda_3 \frac{\partial s_3}{\partial \nu} & \text{on } \Gamma_N \end{cases} \tag{11}$$

has a regular solution.

Here we note that the input function f is the same as in (1).

The following theorems show (11) has a regular solution. The proofs of the following two theorems are very similar to those in [6], although the singular function s is different. We just state them for the completeness without proofs.

Theorem 2.2. *If* (1) *has a solution u as in* (9) *with the stress intensity factor $\lambda_j (j = 1, 3)$, then* (11) *has a unique solution w in $H^2(\Omega)$.*

Theorem 2.3. *If λ_j is the stress intensity factors given by* (10) *with the solution u in* (1) *and w is the solution of* (11)*, then $u = w + \lambda_1 s_1 + \lambda_3 s_3$ is the unique solution of* (1)*.*

Motivated by these theorems, they posed the following algorithm:

A1.: Find a solution $u^{(0)}$ of (1) and

A2.: compute the stress intensity factors $\lambda_j^{(0)}, j = 1, 3$, from (10).

A3.: For $i = 1, 2, \cdots, N$;

A3-1.: Solve, for $w^{(i)}$,

$$\begin{cases} -\Delta w^{(i)} &= f & \text{in } \Omega, \\ w^{(i)} &= -\lambda_1^{(i-1)} s_1 - \lambda_3^{(i-1)} s_3 & \text{on } \Gamma_D, \\ \frac{\partial w^{(i)}}{\partial \nu} &= -\lambda_1^{(i-1)} \frac{\partial s_1}{\partial \nu} - \lambda_3^{(i-1)} \frac{\partial s_3}{\partial \nu} & \text{on } \Gamma_N. \end{cases} \tag{12}$$

A3-2.: Let $u^{(i)} = w^{(i)} + \lambda_1^{(i-1)} s_1 + \lambda_3^{(i-1)} s_3$.

A3-3.: Compute $\lambda_j^{(i)}$ by

$$\lambda_j^{(i)} = \frac{2}{j\pi} \int_\Omega f\eta s_{-j} dx + \frac{2}{j\pi} \int_\Omega u^{(i)} \Delta(\eta s_{-j}) dx, \quad j = 1, 3. \tag{13}$$

The number of iterations in the loop of **A3** is the issue of this paper. It is known that $N = 1$ is enough for the cases D/N or N/D with $\frac{\pi}{2} < \omega \leq \frac{3\pi}{2}$ and cases D/D or N/N with any concave angle. We need $N = 2$ for the more singular cases D/N or N/D with $\frac{3\pi}{2} < \omega < 2\pi$ ([7, 8]).

To get the purpose of this paper, we propose a modified algorithm;

MA1.: Choose two approximations $\lambda_1^{(0)}$ and $\lambda_3^{(0)}$ of the stress intensity factors λ_1 and λ_3.

MA2.: For $i = 1, 2, \cdots, N$;

MA2-1.: Solve, for $w^{(i)}$,

$$\begin{cases} -\Delta w^{(i)} & = & f & \text{in } \Omega, \\ w^{(i)} & = & -\lambda_1^{(i-1)} s_1 - \lambda_3^{(i-1)} s_3 & \text{on } \Gamma_D, \\ \frac{\partial w^{(i)}}{\partial \nu} & = & -\lambda_1^{(i-1)} \frac{\partial s_1}{\partial \nu} - \lambda_3^{(i-1)} \frac{\partial s_3}{\partial \nu} & \text{on } \Gamma_N. \end{cases} \tag{14}$$

MA2-2.: Let $u^{(i)} = w^{(i)} + \lambda_1^{(i-1)} s_1 + \lambda_3^{(i-1)} s_3$.

MA2-3.: Compute $\lambda_j^{(i)}$ by

$$\lambda_j^{(i)} = \frac{2}{j\pi} \int_\Omega f\eta s_{-j} dx + \frac{2}{j\pi} \int_\Omega u^{(i)} \Delta(\eta s_{-j}) dx, \quad j = 1, 3. \tag{15}$$

3. Finite element approximation

In this section we present the standard finite element approximation for the algorithms considered in the previous section. First we assume the P_1 finite element spaces V_k nested, i.e.,

$$V_1 \subset V_2 \subset \cdots \subset H_D^1(\Omega),$$

whose mesh sizes $h_k = \max_{T \in \mathcal{T}_k} \text{diam} T$ are related by

$$h_k = 2h_{k+1} \quad \text{for} \quad k = 1, 2, 3, \cdots.$$

Here $\mathcal{T}_k$ is a partition of the domain Ω into triangular finite elements; i.e., $\Omega = \cup_{K \in \mathcal{T}_k} K$, and V_k is a continuous piecewise linear finite element space; i.e.,

$$V_k = \{\phi_h \in C^0(\Omega) : \phi_h|_K \in P_1(K)\ \forall K \in \mathcal{T}_k, \phi_h = 0 \text{ on } \Gamma_{\mathrm{D}}\} \subset \mathrm{H}_{\mathrm{D}}^1(\Omega),$$

where $P_1(K)$ is the space of linear functions on K.

Now, the approximated solution $u_k \in V_k$ of the algorithm in [7, 8] comes as in the following:

FEA1.: find $u_k^{(0)} \in V_k$ such that

$$(\nabla u_k^{(0)}, \nabla v) = (f, v), \quad \forall\ v \in V_k. \tag{16}$$

FEA2.: Then, compute $\lambda_{j,k}^{(0)}$ by

$$\lambda_{j,k}^{(0)} = \frac{2}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{2}{j\pi}\int_\Omega u_k^{(0)}\Delta(\eta s_{-j})dx, \quad j=1,3. \tag{17}$$

FEA3.: Do the followings, for $i = 1, 2, \cdots, N$;

FEA3-1.: find $w_k^{(i)}$ such that $w_k^{(i)} + \lambda_{1,k}^{(i-1)}s_1 + \lambda_{3,k}^{(i-1)}s_3 \in V_k$ and

$$(\nabla w_k^{(i)}, \nabla v) = (f, v) - \lambda_1(\frac{\partial s_1}{\partial \nu}, v)|_{\Gamma_N} - \lambda_3(\frac{\partial s_3}{\partial \nu}, v)|_{\Gamma_N}, \quad \forall\, v \in V_k. \tag{18}$$

FEA3-2.: Set $u_k^{(i)} = w_k^{(i)} + \lambda_{1,k}^{(i-1)}s_1 + \lambda_{3,k}^{(i-1)}s_3$.

FEA3-3.: Compute $\lambda_{j,k}^{(i)}$ by

$$\lambda_{j,k}^{(i)} = \frac{2}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{2}{j\pi}\int_\Omega u_{j,k}^{(i)}\Delta(\eta s_{-j})dx, \quad j=1,3. \tag{19}$$

Now our modified finite element approximation is the following:

MFEA1.: Choose two approximations $\lambda_1^{(0)}$ and $\lambda_3^{(0)}$ of the stress intensity factors λ_1 and λ_3.

MFEA2.: For $i = 1, 2, \cdots, N$;

MA2-1.: Solve, for $w^{(i)}$,

$$\begin{cases} -\Delta w^{(i)} &= f & \text{in } \Omega, \\ w^{(i)} &= -\lambda_1^{(i-1)}s_1 - \lambda_3^{(i-1)}s_3 & \text{on } \Gamma_D, \\ \frac{\partial w^{(i)}}{\partial \nu} &= -\lambda_1^{(i-1)}\frac{\partial s_1}{\partial \nu} - \lambda_3^{(i-1)}\frac{\partial s_3}{\partial \nu} & \text{on } \Gamma_N. \end{cases} \tag{20}$$

MA2-2.: Let $u^{(i)} = w^{(i)} + \lambda_1^{(i-1)}s_1 + \lambda_3^{(i-1)}s_3$.

MA2-3.: Compute $\lambda_j^{(i)}$ by

$$\lambda_j^{(i)} = \frac{2}{j\pi}\int_\Omega f\eta s_{-j}dx + \frac{2}{j\pi}\int_\Omega u^{(i)}\Delta(\eta s_{-j})dx, \quad j=1,3. \tag{21}$$

Now we are ready to investigate the dependence of the iteration numbers on the convergence of stress intensity factors and give a way to reduce the iteration number. This will be done by choosing special sequences of stress infactor factors for the mixed boundary poisson problem with bad singularity.

4. Numerical results and conclusion

As a model problem we consider a Poisson problem with the mixed boundary condition, on a concave corner with an inner angle $\omega = \frac{39\pi}{20}$.

Example 1. Consider a Poisson equation (1) on a domain $\Omega = ((-1,-1)\times(1,1))\setminus\left\{(x,y) : 0 \le x \le 1, -\tan(\frac{\pi}{20})\,x \le y \le 0\right\}$ as in Figure 1. Note that the inner angle $\omega = \frac{39\pi}{20}$, and the singular functions are given by

$$s_1 = s_1(r,\theta) = r^{\frac{10}{39}} \sin\frac{10\theta}{39} \quad \text{and} \quad s_3 = s_3(r,\theta) = r^{\frac{10}{13}} \sin\frac{10\theta}{13}.$$

Let $f = -\Delta(\eta_{0.75}s_1) - \Delta(\eta_{0.75}s_3)$ with the exact solution $u_{\text{exact}} = \eta_{0.75}s_1 + \eta_{0.75}s_3$. So, the exact stress intensity factors are $\lambda_1 = \lambda_3 = 1$.

We choose approximations of $\lambda_1^{(0)}$ and $\lambda_3^{(0)}$ in **MFEA1.** as follows;
Let $\lambda_1^{(0)} = 1 - 3h^\alpha$ and $\lambda_3^{(0)} = 1 + 3h^\alpha$ with $\alpha = \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}$. So, α plays a role as a convergence rate of approximated stress intensity factors.

Remark : If we are considering the Poisson problem defined on L-shape domain with Mixed boundary condition, the stress intensity factors computed form the standard finite element solution u_h of (1) converge with convergence rate $2 \cdot \frac{\pi}{2\omega} = \frac{2}{3}$ since $\omega = \frac{3\pi}{2}$.(See [8])

We list the computational results in Table 1-6, by **MFEA.** algorithms for each $\alpha's$.

h	$\lambda_1^{(0)}$	$\lambda_3^{(0)}$	$\|\|E\|\|_{L^2}$		$\lambda_1^{(1)}$	$\lambda_3^{(1)}$
1/4	-0.50000	2.50000	2.66070E-01	ratio	0.34173	0.37073
1/8	-0.06066	2.06066	1.11536E-01	1.25430	0.73983	0.81557
1/16	0.25000	1.75000	4.75939E-02	1.22866	0.98754	0.95405
1/32	0.46967	1.53033	2.16401E-02	1.13707	1.01088	0.98859
1/64	0.62500	1.37500	1.03324E-02	1.06653	1.01272	0.99712
1/128	0.73484	1.26517	4.98632E-03	1.05113	1.00784	0.99933

TABLE 1. The case $\alpha = \frac{1}{2}$: The L^2-norm errors of u_h with the convergence ratios and the values of $\lambda_j^{(1)}$, $j = 1, 3$

h	$\lambda_1^{(0)}$	$\lambda_3^{(0)}$	$\|\|E\|\|_{L^2}$		$\lambda_1^{(1)}$	$\lambda_3^{(1)}$
1/4	-0.06066	2.06066	2.17383E-01	ratio	0.24643	0.36003
1/8	0.36933	1.63067	7.85979E-02	1.46768	0.65732	0.81936
1/16	0.62500	1.37500	2.71654E-02	1.53272	0.94535	0.95413
1/32	0.77702	1.22298	9.80635E-03	1.46998	0.98716	0.98864
1/64	0.86742	1.13258	3.82088E-03	1.35981	0.99995	0.99711
1/128	0.92117	1.07883	1.51319E-03	1.33631	1.00106	0.99932

TABLE 2. The case $\alpha = \frac{3}{4}$: The L^2-norm errors of u_h with the convergence ratios and the values of $\lambda_j^{(1)}$, $j = 1, 3$

h	$\lambda_1^{(0)}$	$\lambda_3^{(0)}$	$\|\|E\|\|_{L^2}$		$\lambda_1^{(1)}$	$\lambda_3^{(1)}$
1/4	0.25000	1.75000	1.86838E-01	ratio	0.17904	0.35246
1/8	0.62500	1.37500	6.18121E-02	1.59583	0.60826	0.82161
1/16	0.81250	1.18750	1.83723E-02	1.75036	0.92426	0.95417
1/32	0.90625	1.09375	5.33574E-03	1.78377	0.97719	0.98866
1/64	0.95313	1.04688	1.64710E-03	1.69576	0.99543	0.99711
1/128	0.97656	1.02344	5.10949E-04	1.68868	0.99905	0.99932

TABLE 3. The case $\alpha = 1$: The L^2-norm errors of u_h with the convergence ratios and the values of $\lambda_j^{(1)}$, $j = 1, 3$

h	$\lambda_1^{(0)}$	$\lambda_3^{(0)}$	$\|\|E\|\|_{L^2}$		$\lambda_1^{(1)}$	$\lambda_3^{(1)}$
1/4	0.46967	1.53033	1.68306E-01	ratio	0.13139	0.34710
1/8	0.77702	1.22298	5.38757E-02	1.64338	0.57909	0.82295
1/16	0.90625	1.09375	1.51040E-02	1.83470	0.91371	0.95419
1/32	0.96058	1.03942	3.96232E-03	1.93051	0.97300	0.98867
1/64	0.98343	1.01657	1.04768E-03	1.91915	0.99383	0.99711
1/128	0.99303	1.00697	2.67995E-04	1.96692	0.99845	0.99932

TABLE 4. The case $\alpha = \frac{5}{4}$: The L^2-norm errors of u_h with the convergence ratios and the values of $\lambda_j^{(1)}$, $j = 1, 3$

h	$\lambda_1^{(0)}$	$\lambda_3^{(0)}$	$\|\|E\|\|_{L^2}$		$\lambda_1^{(1)}$	$\lambda_3^{(1)}$
1/4	0.62500	1.37500	1.57312E-01	ratio	0.09769	0.34332
1/8	0.86742	1.13258	5.02744E-02	1.64573	0.56174	0.82375
1/16	0.95313	1.04688	1.40045E-02	1.84393	0.90843	0.95420
1/32	0.98343	1.01657	3.62005E-03	1.95181	0.97123	0.98868
1/64	0.99414	1.00586	9.21911E-04	1.97331	0.99327	0.99711
1/128	0.99793	1.00207	2.28341E-04	2.01344	0.99827	0.99932

TABLE 5. The case $\alpha = \frac{3}{2}$: The L^2-norm errors of u_h with the convergence ratios and the values of $\lambda_j^{(1)}$, $j = 1, 3$

Recall that, in [8], they suggested an efficient algorithm to get accurate numerical solutions for (1), which contains one or two iteration(s). Moreover, the iteration number depends on the accuracy of $\lambda_j^{(0)}, j = 1, 3$.

By the above six numerical experiments, we can get several important remarks and conclusion as in followings;

h	$\lambda_1^{(0)}$	$\lambda_3^{(0)}$	$\|\|E\|\|_{L^2}$		$\lambda_1^{(1)}$	$\lambda_3^{(1)}$
1/4	0.73484	1.26517	1.50842E-01	ratio	0.07387	0.34064
1/8	0.92117	1.07883	4.86304E-02	1.63311	0.55143	0.82422
1/16	0.97656	1.02344	1.36308E-02	1.83499	0.90580	0.95421
1/32	0.99303	1.00697	3.53610E-03	1.94664	0.97049	0.98868
1/64	0.99793	1.00207	8.95972E-04	1.98063	0.99307	0.99711
1/128	0.99938	1.00062	2.22535E-04	2.00942	0.99822	0.99932

TABLE 6. The case $\alpha = \frac{7}{4}$: The L^2-norm errors of u_h with the convergence ratios and the values of $\lambda_j^{(1)}$, $j = 1, 3$

Remark 1 : In most cases, the stress intensity factors $\lambda_j^{(1)}$ are better than $\lambda_j^{(0)}$, $j = 1, 3$. So we can get more accurate stress intensity factors by applying the algorithm.

Remark 2 : When we use the values $\lambda_j^{(0)}$ in the algorithm, with convergence ratio $\alpha \leq 1$ for the stress intensity factors used in **MA1**, then we may not get the optimal convergence in u_h as we see in Table 1-3.

Remark 3 : The results in Table 5-6 shows that we have almost the same results for both cases with $\alpha = 3/2$ and $\alpha = 7/4$. That means that we cannot get any better results if the ratio of convergenc reaches to some degree.

Finally, we have the following regarding the convergence factors.

Conclusion : If we can find relatively accurate stress intensity factors with convergence rate approximately larger than $5/4$, then the iteration number $M = 1$ is enough for the algorithm in [7, 8].

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