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AN EXTRAPOLATED HIGHER ORDER CHARACTERISTIC FINITE ELEMENT METHOD FOR NONLINEAR SOBOLEV EQUATIONS

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Abstract. In this paper, we introduce an extrapolated higher order characteristic finite element method to approximate solutions of nonlinear Sobolev equations with a convection term and we establish the higher order of convergence in the temporal and the spatial directions with respect to L^2 norm.

1. Introduction

In this paper, we will consider the following nonlinear Sobolev equation with a convection term:

$$\begin{aligned} c(u)u_t + \boldsymbol{d}(u)\cdot\nabla u - \nabla\cdot(a(u)\nabla u) - \nabla\cdot(b(u)\nabla u_t)& \\ = f(\boldsymbol{x},t,u), &\qquad \text{in } \Omega\times(0,T], \\ u(\boldsymbol{x},t) = \boldsymbol{0}, &\qquad \text{on } \partial\Omega\times(0,T], \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), &\qquad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^m$, $1\le m\le 3$, is a bounded convex domain with boundary $\partial\Omega$ and $c, \boldsymbol{d}, a, b$ and f are known functions. For the theoretical results of the existence, uniqueness, and regularity of Sobolev equations and their physical applications, refer to [2, 3, 4, 23, 26] and the papers cited therein.

When Sobolev equations have no convection term, a lot of numerical techniques such as classical finite element methods [1, 6, 10, 11, 12], least-squares methods [9, 20, 22, 22, 28], mixed finite element methods [8], discontinuous finite element methods [13, 14, 24, 25] are used to define their approximate solutions. However, when there is a convection term to describe the convection dominated diffusion, we generally use a characteristic method to consider both the time derivative term and the convection term effectively. Especially, this technique is very effective for convection dominated diffusion problems as shown in [5, 7].

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Gu in [7] and Shi et al [22] introduce a characteristic finite element method and establish the higher order convergence in the spatial variable and the first order convergence in the temporal variable for approximate solutions for a Sobolev equation. However, the first order convergence in the temporal variable worsens the higher order convergence in the spatial variable. So, Ohm and Shin in [15, 17, 18] introduce a Crank-Nicolson or an extrapolated Crank-Nicolson characteristic finite element method and a higher order characteristic finite element method for a Sobolev equation to obtain the higher order of convergence in both the spatial direction and the temporal direction in L^2 normed space when the given functions $c(\cdot)$ and $\boldsymbol{d}(\cdot)$ depend only on $\boldsymbol{x}$. And Ohm and Shin [16, 19] introduce a Crank-Nicolson or an extrapolated Crank-Nicolson characteristic finite element method for a nonlinear Sobolev equation with a convection term to establish the higher order of convergence which extend their previous results to the nonlinear Sobolev equation.

In this paper, we will introduce an extrapolated higher order characteristic finite element method to construct approximate solutions of a nonlinear Sobolev equation with a convection term and establish the higher order of convergence in the temporal direction as well as in the spatial direction in L^2 normed space. These results also extend our previous work in [18] to the nonlinear Sobolev equation. The outline of our paper is organized as follows. We state some smoothness assumptions for $u(\boldsymbol{x},t)$, the conditions for the given functions, and basic notations in Section 2. In Section 3, we introduce finite element spaces with basic approximation properties and some elliptic projection. In Section 4, we construct an extrapolated higher order characteristic finite element approximation of $u(\boldsymbol{x},t)$ and obtain the higher order of convergence in L^2 and H^1 normed spaces.

2. Assumptions and notations

Throughout this paper, let $W^{s,p}(\Omega)$ denote the Sobolev space equipped with its norm $\|\cdot\|_{s,p}$ for an $s \geq 0$ and $1 \leq p \leq \infty$. For the sake of our convenience and simplicity, instead of $W^{s,2}(\Omega)$ and $H^0(\Omega)$, we use the notation $H^s(\Omega)$ and $L^2(\Omega)$, respectively. Similarly, we use $\|\cdot\|$, $\|\cdot\|_\infty$, and $\|\cdot\|_s$, instead of $\|\cdot\|_{0,2}$, $\|\cdot\|_{0,\infty}$, and $\|\cdot\|_{s,2}$, respectively. Let $H_0^1(\Omega) = \{w \in H^1(\Omega) \mid w(\boldsymbol{x}) = 0 \text{ on } \partial\Omega\}$ and $\boldsymbol{H}^s(\Omega) = \{\boldsymbol{w} = (w_1, w_2, \ldots, w_m) \mid w_i \in H^s(\Omega), 1 \leq i \leq m\}$ be the Sobolev space equipped with its norm $\|\boldsymbol{w}\|_s^2 = \sum\limits_{i=1}^{m} \|w_i\|_s^2$. For a given Banach space X and $t_1, t_2 \in [0,T]$, we introduce Sobolev spaces with the corresponding norms:

$$W^{s,p}(t_1,t_2;X) = \Big\{w(\boldsymbol{x},t) \mid \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X \in L^p(t_1,t_2), 0 \leq \beta \leq s\Big\},$$

where

$$\|w\|_{W^{s,p}(t_1,t_2;X)} = \begin{cases} \left(\sum_{\beta=0}^{s} \int_{t_1}^{t_2} \| \frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X^p dt \right)^{1/p}, & 1 \le p < \infty, \\ max_{0\le\beta\le s} \ \text{esssup}_{t\in(t_1,t_2)} \| \frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X, & p=\infty. \end{cases}$$

We will denote $L^p(X)$ and $W^{s,p}(X)$, instead of $W^{0,p}(0,T;X)$ and $W^{s,p}(0,T;X)$, respectively.

Assume that $c(p), \boldsymbol{d}(p) = (d_1(p), d_2(p), \dots, d_m(p))^T$, $a(p), b(p)$ and $f(\boldsymbol{x},t,p)$ satisfy

(A1) There exist constants $c_*, c^*, d^*, a_*, a^*,\ b_*$, and b^* such that $0 < c_* \le c(p) \le c^*,\ 0 < |\boldsymbol{d}(p)| \le d^*,\ 0 < a_* \le a(p) \le a^*,\ 0 < b_* \le b(p) \le b^*$, for all $p \in \mathbb{R}$, where $|\boldsymbol{d}(p)| = \sum_{i=1}^{m} d_i^2(p)$.

(A2) $\frac{d^j}{dp^j}a(p), \frac{d^j}{dp^j}b(p)$, for $j = 1,2,3,\ \frac{d}{dp}c(p)$, and $\frac{d}{dp}d_i(p)$ are bounded continuous functions.

(A3) $f(\boldsymbol{x},t,p)$ is locally Lipschitz continuous in the third variable p, i.e. if $|\ p^* - p\ | \le \tilde{K}$ then $|f(\boldsymbol{x},t,p^*) - f(\boldsymbol{x},t,p)| \le K(p,\tilde{K})|\ p^* - p\ |$.

By following the idea in [5], let $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x},t)$ be the unit vector for a given $(\boldsymbol{x},t)$ such that $\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{c(u)}{\psi(u)}\frac{\partial u}{\partial t} + \frac{\boldsymbol{d}(u)}{\psi(u)} \cdot \nabla u$, where $\psi(u) = [c(u)^2 + |\boldsymbol{d}(u)|^2]^{\frac{1}{2}}$. Then the nonlinear Sobolev equation (1.1) can be rewritten as follows

$$\begin{aligned} \psi(u)\frac{\partial u}{\partial \boldsymbol{\nu}} - \nabla\cdot(a(u)\nabla u) - \nabla\cdot(b(u)\nabla u_t) &= f(\boldsymbol{x},t,u), && \text{in } \Omega\times(0,T], \\ u(\boldsymbol{x},t) &= 0, && \text{on } \partial\Omega\times(0,T], \\ u(\boldsymbol{x},0) &= u_0(\boldsymbol{x}), && \text{in } \Omega. \end{aligned} \tag{2.1}$$

And the variational form of the equation (2.1) is given as follows: Find $u(\boldsymbol{x},t) \in H_0^1(\Omega)$ such that

$$\begin{aligned} &(\psi(u)\frac{\partial u}{\partial \boldsymbol{\nu}}, \tau) + (a(u)\nabla u, \nabla\tau) + (b(u)\nabla u_t, \nabla\tau) \\ &\qquad = (f(x,t,u),\tau), \qquad \forall \tau \in H_0^1(\Omega), \\ &u(\boldsymbol{x},0) = u_0(\boldsymbol{x}). \end{aligned} \tag{2.2}$$

3. Finite element spaces and an elliptic projection

For $h > 0$, let $\{S_h^r\}$ be a family of finite dimensional subspaces of $H_0^1(\Omega)$ satisfying the following approximation and inverse properties: for $\phi \in H_0^1(\Omega) \cap W^{s,p}(\Omega)$, there exist a positive constant K_1, independent of h, ϕ, and r, and a sequence $P_h\phi \in S_h^r$ such that for any $0 \le q \le s$ and $1 \le p \le \infty$

$$\|\phi - P_h\phi\|_q \le K_1 h^{\mu-q}\|\phi\|_s,$$

where $\mu = \min(r+1, s)$ and also there exist a positive constant K_2 independent of h and r, such that

$$\|\varphi\|_1 \leq K_2 h^{-1}\|\varphi\| \text{ and } \|\varphi\|_\infty \leq K_2 h^{-\frac{m}{2}}\|\varphi\|, \ \ \forall \varphi \in S_h^r.$$

For the sake of our error analysis, bilinear forms A and B are defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ for a given u as follows:

$$A(u : v, w) = (a(u)\nabla v, \nabla w), \quad B(u : v, w) = (b(u)\nabla v, \nabla w). \tag{3.1}$$

By following the idea in [10, 14] and the assumption (A1), there exists a differentiable function $\tilde{u} : [0, T] \to S_h^r$ satisfying

$$\begin{aligned} A(u : u - \tilde{u}, \chi) + B(u : u_t - \tilde{u}_t, \chi) = 0, &\qquad \forall \chi \in S_h^r, \\ (\tilde{u}(0), \chi) = (u_0, \chi), &\quad \forall \chi \in S_h^r. \end{aligned} \tag{3.2}$$

Now letting $\eta = u - \tilde{u}$, we obtain the following two lemmas whose proofs can be found in [15, 17].

Lemma 3.1. *Let $u_0 \in H^s(\Omega)$, $u_t, u_{tt}, u_{ttt} \in H^s(\Omega)$, and $u_t \in L^2(H^s(\Omega))$. Then there exists a constant K, independent of h, such that*

(**i**) $\|\eta\| + h\|\eta\|_1 \leq Kh^\mu(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s)$,
(**ii**) $\|\eta_t\| + h\|\eta_t\|_1 \leq Kh^\mu(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s)$,
(**iii**) $\|\eta_{tt}\|_1 \leq Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s)$,
(**iv**) $\|\eta_{ttt}\|_1 \leq Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s + \|u_{ttt}\|_s)$,

where $\mu = \min(r+1, s)$ and $s \geq 2$.

Lemma 3.2. *Let $u_0 \in H^s(\Omega)$, $u, u_t, u_{tt}, u_{ttt} \in L^\infty(H^s(\Omega)) \cap L^\infty(W^{1,\infty}(\Omega))$, and $u_t \in L^2(H^s(\Omega))$. If $\mu \geq 1 + \frac{m}{2}$, then there exists a constant K, independent of h, such that*

$$\max\{\|\eta\|_\infty, \ \|\nabla\eta\|_\infty, \ \|\nabla\eta_t\|_\infty, \ \|\nabla\eta_{tt}\|_\infty, \ \|\nabla\eta_{ttt}\|_\infty\} \leq K,$$

where $\mu = \min(r+1, s)$.

Throughout this paper, a generic positive constant K depends on the domain $\Omega, \tilde{K}$, and $u(\boldsymbol{x}, t)$ but is independent of the discretization magnitudes of the spatial and the temporal directions. So any K in the different places does have different values.

4. The optimal $L^\infty(L^2)$ and $L^\infty(H^1)$ error estimates

Let N be a given positive integer, $\Delta t = T/N$ a time step, and $t^n = n\Delta t$ discrete time for $0 \leq n \leq N$. Denote $u^j = u(x, t^j)$, $u^{j-\frac{1}{2}} = \frac{1}{2}(u^j + u^{j-1})$,

$t^{j-\frac{1}{2}} = \frac{1}{2}(t^j + t^{j-1})$, and $\tilde{\boldsymbol{d}}(\cdot) = \boldsymbol{d}(\cdot)/c(\cdot)$. From (2.2) and the definitions of bilinear forms A and B, we have

$$\begin{aligned}&\Big(\psi(u(t^{\frac{1}{2}}))\frac{\partial u(t^{\frac{1}{2}})}{\partial \boldsymbol{\nu}}, \chi\Big) + A(u(t^{\frac{1}{2}}) : u(t^{\frac{1}{2}}), \chi)\\ &\qquad + B(u(t^{\frac{1}{2}}) : u_t(t^{\frac{1}{2}}), \chi) = (f(\boldsymbol{x}, t^{\frac{1}{2}}, u(t^{\frac{1}{2}})), \chi), \quad \forall \chi \in S_h^r,\end{aligned} \tag{4.1}$$

$$\begin{aligned}&\Big(\psi(u^{n+1})\frac{\partial u^{n+1}}{\partial \boldsymbol{\nu}}, \chi\Big) + A(u^{n+1} : u^{n+1}, \chi)\\ &\qquad + B(u^{n+1} : u_t^{n+1}, \chi) = (f(\boldsymbol{x}, t^{n+1}, u^{n+1}), \chi), \quad \forall \chi \in S_h^r, \quad n \geq 1\end{aligned} \tag{4.2}$$

and so, we get

$$\begin{aligned}&\Big(c(u(t^{\frac{1}{2}}))\frac{\bar{\bar{u}}^1 - \breve{u}^0}{\Delta t}, \chi\Big) + A(u(t^{\frac{1}{2}}) \ : \ u^{\frac{1}{2}}, \chi)\\ &\qquad + B(u(t^{\frac{1}{2}}) \ : \ \frac{u^1 - u^0}{\Delta t}, \chi)\\ &\quad = (f(\boldsymbol{x}, t^{\frac{1}{2}}, u(t^{\frac{1}{2}})), \chi) + Q_1 + Q_2 + Q_3, \ \forall \chi \in S_h^r,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\Big(c(u^{n+1})\frac{\frac{3}{2}u^{n+1} - 2\breve{u}^n + \frac{1}{2}\hat{u}^{n-1}}{\Delta t}, \chi\Big) + A(u^{n+1} \ : \ u^{n+1}, \chi)\\ &\qquad + B(u^{n+1} \ : \ \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t}, \chi)\\ &\quad = (f(\boldsymbol{x}, t^{n+1}, u^{n+1}), \chi) + P_1 + P_2, \ \forall \chi \in S_h^r, \quad n \geq 1\end{aligned} \tag{4.4}$$

where $\bar{\bar{u}}^1 = u(\bar{\bar{\boldsymbol{x}}}, t^1)$, $\breve{u}^0 = u(\breve{\boldsymbol{x}}, t^0)$, $\bar{\bar{\boldsymbol{x}}} = \boldsymbol{x} + \frac{1}{2}\tilde{\boldsymbol{d}}(u(t^{\frac{1}{2}}))\Delta t$, $\breve{\boldsymbol{x}} = \boldsymbol{x} - \frac{1}{2}\tilde{\boldsymbol{d}}(u(t^{\frac{1}{2}}))\Delta t$, $Q_1 = (c(u(t^{\frac{1}{2}}))\frac{\bar{\bar{u}}^1 - \breve{u}^0}{\Delta t} - \psi(u(t^{\frac{1}{2}}))\frac{\partial u(t^{\frac{1}{2}})}{\partial \boldsymbol{\nu}}, \chi)$, $Q_2 = A(u(t^{\frac{1}{2}}) : \ u^{\frac{1}{2}} - u(t^{\frac{1}{2}}), \chi)$, and $Q_3 = B(u(t^{\frac{1}{2}}) : \ \frac{u^1 - u^0}{\Delta t} - u_t(t^{\frac{1}{2}}), \chi)$, $\breve{u}^n = u(\breve{\boldsymbol{x}}, t^n)$, $\hat{u}^{n-1} = u(\hat{\boldsymbol{x}}, t^{n-1})$, $\breve{\boldsymbol{x}} = \boldsymbol{x} - \tilde{\boldsymbol{d}}(u^{n+1})\Delta t$, $\hat{\boldsymbol{x}} = \boldsymbol{x} - 2\tilde{\boldsymbol{d}}(u^{n+1})\Delta t$, $P_1 = (c(u^{n+1})\frac{\frac{3}{2}u^{n+1} - 2\breve{u}^n + \frac{1}{2}\hat{u}^{n-1}}{\Delta t} - \psi(u^{n+1})\frac{\partial u^{n+1}}{\partial \boldsymbol{\nu}}, \chi)$, and $P_2 = B(u^{n+1} : \ \frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t} - u_t^{n+1}, \chi)$. Notice that (4.3) is obtained at $t = t^{\frac{1}{2}}$ by the Crank-Nicolson technique and (4.4) at $t = t^{n+1}$ by the backward three point technique. Both techniques are usually used to get the higher order convergence in the temporal direction.

To avoid the difficulty of solving the system of nonlinear equations, an extrapolated higher order characteristic finite element scheme for (1.1) is given as follows: Find a sequence $\{u_h^n\}_{n=0}^N$ in S_h^r such that

$$\begin{aligned}&\Big(c(Eu_h^n)\frac{\frac{3}{2}u_h^{n+1} - 2\breve{u}_h^n + \frac{1}{2}\hat{u}_h^{n-1}}{\Delta t}, \chi\Big) + A(Eu_h^n : u_h^{n+1}, \chi)\\ &\qquad + B(Eu_h^n : \frac{\frac{3}{2}u_h^{n+1} - 2u_h^n + \frac{1}{2}u_h^{n-1}}{\Delta t}, \chi)\\ &\quad = (f(\boldsymbol{x}, t^{n+1}, Eu_h^n), \chi), \quad \forall \chi \in S_h^r, \ n = 1, \ldots, N-1,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\left(c(u_h^{\frac{1}{2}})\frac{\bar{u}_h^1-\breve{u}_h^0}{\Delta t},\chi\right)+A(u_h^{\frac{1}{2}}:u_h^{\frac{1}{2}},\chi)+B(u_h^{\frac{1}{2}}:\frac{u_h^1-u_h^0}{\Delta t},\chi)\\&\qquad=(f(\boldsymbol{x},t^{\frac{1}{2}},u_h^{\frac{1}{2}}),\chi),\quad\forall\chi\in S_h^r,\end{aligned}\tag{4.6}$$

$$u_h^0(\boldsymbol{x})=\tilde{u}(\boldsymbol{x},0),\tag{4.7}$$

where $Eu_h^n=2u_h^n-u_h^{n-1}$, $\breve{u}_h^n=u_h^n(\breve{\boldsymbol{x}})$, $\hat{u}_h^{n-1}=u_h^{n-1}(\hat{\boldsymbol{x}})$, $\breve{\boldsymbol{x}}=\boldsymbol{x}-\tilde{\boldsymbol{d}}(Eu_h^n)\Delta t$, $\hat{\boldsymbol{x}}=\boldsymbol{x}-2\tilde{\boldsymbol{d}}(Eu_h^n)\Delta t$, $\bar{u}_h^1=u_h^1(\bar{\boldsymbol{x}})$, $\breve{u}_h^0=u_h^0(\breve{\boldsymbol{x}})$, $\bar{\boldsymbol{x}}=\boldsymbol{x}+\frac{1}{2}\tilde{\boldsymbol{d}}(u_h^{\frac{1}{2}})\Delta t$, $\breve{\boldsymbol{x}}=\boldsymbol{x}-\frac{1}{2}\tilde{\boldsymbol{d}}(u_h^{\frac{1}{2}})\Delta t$, and $u_h^{\frac{1}{2}}=\frac{1}{2}(u_h^1+u_h^0)$. Notice that (4.5) and (4.6) are based on the backward three point approximation and the Crank-Nicolson approximation for $\psi(u(t))\frac{\partial u(t)}{\partial\boldsymbol{\nu}}$, respectively.

For our error analysis, we set $\xi^n=u_h^n-\tilde{u}^n$ and $\partial_t\xi^n=\frac{\xi^n-\xi^{n-1}}{\Delta t}$. For $t=t^1$, (4.6) is considered and the following theorem is given in [16].

Theorem 4.1. *Let u and $\{u_h^n\}$ be solutions of (2.2) and (4.5)-(4.7), respectively. In addition to the assumptions of Lemma 3.2, if $\mu\geq 1+\frac{m}{2}$, $u\in L^\infty(H^3(\Omega))$, and $\Delta t=O(h)$, then*

$$\|\nabla\xi^1\|^2+\Delta t(\|\partial_t\xi^1\|^2+\|\nabla\partial_t\xi^1\|^2)\leq K\Delta t(h^{2\mu}+(\Delta t)^4),$$

where $\mu=\min(r+1,s)$.

For $t=t^n$, we will try to prove the following theorem.

Theorem 4.2. *Under the same assumptions of Theorem 4.1, we have*

$$\max_{0\leq n\leq N}\left[\|u^n-u_h^n\|+h\|\nabla(u^n-u_h^n)\|\right]\leq K(h^\mu+(\Delta t)^2),$$

where $\mu=\min(r+1,s)$.

Proof. First, we will prove, by mathematical induction, that there exist $0<\tilde{h}<1$ and $0<\tilde{\Delta t}<1$ such that

$$\|\nabla\xi^n\|^2+\Delta t(\|\partial_t\xi^n\|^2+\|\nabla\partial_t\xi^n\|^2)\leq K(h^{2\mu}+(\Delta t)^4)\tag{4.8}$$

for any $0<h<\tilde{h}$, $0<\Delta t<\tilde{\Delta t}$ and $n=1,2,\ldots,N$. We define $Eu_h^0=0$ for our simplicity and convenience. By Theorem 4.1, (4.8) holds for $n=1$. Now we assume that (4.8) holds for $n\leq l-1$. Notice that $\|\xi^n\|_\infty\leq K,\ \ 0\leq n\leq l-1$.

Using (4.4) and (4.5) with $1 \le n \le l-1$ and $\chi = \partial_t \xi^{n+1}$, we get

$$
\begin{aligned}
&\Big(c(Eu_h^n)\frac{\frac{3}{2}(u^{n+1}-u_h^{n+1})-2(\check{u}^n-\check{u}_h^n)+\frac{1}{2}(\hat{u}^{n-1}-\hat{u}_h^{n-1})}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&+A(Eu_h^n : u^{n+1}-u_h^{n+1},\partial_t\xi^{n+1})\\
&+B(Eu_h^n : \frac{\frac{3}{2}(u^{n+1}-u_h^{n+1})-2(u^n-u_h^n)+\frac{1}{2}(u^{n-1}-u_h^{n-1})}{\Delta t},\partial_t\xi^{n+1})\\
&=\Big((c(Eu_h^n)-c(u^{n+1}))\frac{\frac{3}{2}u^{n+1}-2\check{u}^n+\frac{1}{2}\hat{u}^{n-1}}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&+\Big((a(Eu_h^n)-a(u^{n+1}))\nabla u^{n+1},\nabla\partial_t\xi^{n+1}\Big)\\
&+\Big((b(Eu_h^n)-b(u^{n+1}))\frac{\frac{3}{2}\nabla u^{n+1}-2\nabla u^n+\frac{1}{2}\nabla u^{n-1}}{\Delta t},\nabla\partial_t\xi^{n+1})\\
&+(f(\boldsymbol{x},t^{n+1},u^{n+1})-f(\boldsymbol{x},t^{n+1},Eu_h^n),\partial_t\xi^{n+1})+P_1+P_2.
\end{aligned}
\tag{4.9}
$$

And so, we get

$$
\begin{aligned}
&\Big(c(Eu_h^n)\frac{\frac{3}{2}\xi^{n+1}-2\check{\xi}^n+\frac{1}{2}\hat{\xi}^{n-1}}{\Delta t},\partial_t\xi^{n+1}\Big)+A(Eu_h^n\ :\ \xi^{n+1},\partial_t\xi^{n+1})\\
&+B(Eu_h^n : \frac{\frac{3}{2}\xi^{n+1}-2\xi^n+\frac{1}{2}\xi^{n-1}}{\Delta t},\partial_t\xi^{n+1})\\
=&\Big(c(Eu_h^n)\frac{\frac{3}{2}\eta^{n+1}-2\check{\eta}^n+\frac{1}{2}\hat{\eta}^{n-1}}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&-\Big(c(Eu_h^n)\frac{2(\check{u}^n-\check{u}^n)}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&+\Big(c(Eu_h^n)\frac{\frac{1}{2}(\hat{u}^{n-1}-\hat{u}^{n-1})}{\Delta t},\partial_t\xi^{n+1}\Big)+A(Eu_h^n : \eta^{n+1},\partial_t\xi^{n+1})\\
&+B(Eu_h^n : \frac{\frac{3}{2}\eta^{n+1}-2\eta^n+\frac{1}{2}\eta^{n-1}}{\Delta t},\partial_t\xi^{n+1})\\
&-\Big((c(Eu_h^n)-c(u^{n+1}))\frac{\frac{3}{2}u^{n+1}-2\check{u}^n+\frac{1}{2}\hat{u}^{n-1}}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&-\Big((a(Eu_h^n)-a(u^{n+1}))\nabla u^{n+1},\nabla\partial_t\xi^{n+1}\Big)\\
&-\Big((b(Eu_h^n)-b(u^{n+1}))\frac{\frac{3}{2}\nabla u^{n+1}-2\nabla u^n+\frac{1}{2}\nabla u^{n-1}}{\Delta t},\nabla\partial_t\xi^{n+1})\\
&-(f(\boldsymbol{x},t^{n+1},Eu_h^n)-f(\boldsymbol{x},t^{n+1},u^{n+1}),\partial_t\xi^{n+1})-P_1-P_2.
\end{aligned}
\tag{4.10}
$$

Notice that

$$
\begin{aligned}
\frac{3}{2}\xi^{n+1}-2\check{\xi}^n+\frac{1}{2}\hat{\xi}^{n-1}=&\frac{3}{2}(\xi^{n+1}-\xi^n)-\frac{1}{2}(\xi^n-\xi^{n-1})\\
&-\frac{1}{2}(\xi^{n-1}-\hat{\xi}^{n-1})-2(\check{\xi}^n-\xi^n),
\end{aligned}
\tag{4.11}
$$

$$\frac{3}{2}\xi^{n+1}-2\xi^n+\frac{1}{2}\xi^{n-1} = (\xi^{n+1}-\xi^n)+\frac{1}{2}[(\xi^{n+1}-\xi^n)-(\xi^n-\xi^{n-1})]. \tag{4.12}$$

Therefore, from (4.10), (4.11), and (4.12), we obtain

$$\begin{aligned}
&\Big(c(Eu_h^n)\frac{\frac{3}{2}(\xi^{n+1}-\xi^n)-\frac{1}{2}(\xi^n-\xi^{n-1})}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&+A(Eu_h^n:\xi^{n+1},\partial_t\xi^{n+1})\\
&+B(Eu_h^n:\frac{(\xi^{n+1}-\xi^n)+\frac{1}{2}[(\xi^{n+1}-\xi^n)-(\xi^n-\xi^{n-1})]}{\Delta t},\partial_t\xi^{n+1})\\
=&\frac{1}{2}\Big(c(Eu_h^n)\frac{(\xi^{n-1}-\hat{\xi}_h^{n-1})}{\Delta t},\partial_t\xi^{n+1}\Big)+2\Big(c(Eu_h^n)\frac{(\check{\xi}^n-\xi^n)}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&+\Big(c(Eu_h^n)\frac{\frac{3}{2}(\eta^{n+1}-\eta^n)-\frac{1}{2}(\eta^n-\eta^{n-1})}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&-\Big(c(Eu_h^n)\frac{\frac{1}{2}(\eta^{n-1}-\hat{\eta}^{n-1})+2(\check{\eta}^n-\eta^n)}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&-2\Big(c(Eu_h^n)\frac{(\check{\check{u}}^n-\check{u}^n)}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&+\frac{1}{2}\Big(c(Eu_h^n)\frac{(\hat{\hat{u}}^{n-1}-\hat{u}^{n-1})}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&+A(Eu_h^n:\eta^{n+1},\partial_t\xi^{n+1})-A(u^{n+1}\ :\ \eta^{n+1},\partial_t\xi^{n+1})\\
&+B(Eu_h^n:\frac{\frac{3}{2}\eta^{n+1}-2\eta^n+\frac{1}{2}\eta^{n-1}}{\Delta t}-\eta_t^{n+1},\partial_t\xi^{n+1})\\
&+B(Eu_h^n:\eta_t^{n+1},\partial_t\xi^{n+1})-B(u^{n+1}:\eta_t^{n+1},\partial_t\xi^{n+1})\\
&-\Big((c(Eu_h^n)-c(u^{n+1}))\frac{\frac{3}{2}u^{n+1}-2\check{u}^n+\frac{1}{2}\hat{\hat{u}}^{n-1}}{\Delta t},\partial_t\xi^{n+1}\Big)\\
&-\Big((a(Eu_h^n)-a(u^{n+1}))\nabla u^{n+1},\nabla\partial_t\xi^{n+1}\Big)\\
&-\Big((b(Eu_h^n)-b(u^{n+1}))\frac{\frac{3}{2}\nabla u^{n+1}-2\nabla u^n+\frac{1}{2}\nabla u^{n-1}}{\Delta t},\nabla\partial_t\xi^{n+1})\\
&-(f(\boldsymbol{x},t^{n+1},Eu_h^n)-f(\boldsymbol{x},t^{n+1},u^{n+1}),\partial_t\xi^{n+1})-Q_1-Q_2=\sum_{i=1}^{17}R_i.
\end{aligned} \tag{4.13}$$

Now denoting three terms of the left-hand side of (4.13) by L_1, L_2 and L_3, respectively, the lower bounds of L_1, L_2 and L_3 are given as follows:

$$\begin{aligned}
L_1 \geq c_*\|\partial_t\xi^{n+1}\|^2 &+\frac{1}{4}(\|\sqrt{c(Eu_h^n)}\partial_t\xi^{n+1}\|^2-\|\sqrt{c(Eu_h^{n-1})}\partial_t\xi^n\|^2)\\
&+\frac{1}{4}(\|\sqrt{c(Eu_h^{n-1})}\partial_t\xi^n\|^2-\|\sqrt{c(Eu_h^n)}\partial_t\xi^n\|^2),
\end{aligned}$$

$$
\begin{aligned}
L_2 \geq & \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla\xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^n\|^2) \\
& + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^{n-1})}\nabla\xi^n\|^2 - \|\sqrt{a(Eu_h^n)}\nabla\xi^n\|^2), \\
L_3 \geq & b_*\|\nabla\partial_t\xi^{n+1}\|^2 + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla\partial_t\xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\partial_t\xi^n\|^2) \\
& \qquad + \frac{1}{4}(\|\sqrt{b(Eu_h^{n-1})}\nabla\partial_t\xi^n\|^2 - \|\sqrt{b(Eu_h^n)}\nabla\partial_t\xi^n\|^2),
\end{aligned}
$$

By applying these lower bounds of $L_1 \sim L_3$ to (4.13), we get

$$
\begin{aligned}
& c_*\|\partial_t\xi^{n+1}\|^2 + b_*\|\nabla\partial_t\xi^{n+1}\|^2 \\
& \quad + \frac{1}{4}(\|\sqrt{c(Eu_h^n)}\partial_t\xi^{n+1}\|^2 - \|\sqrt{c(Eu_h^{n-1})}\partial_t\xi^n\|^2) \\
& \quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla\xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^n\|^2) \\
& \quad + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla\partial_t\xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\partial_t\xi^n\|^2) \\
& \leq \frac{1}{4}(\|\sqrt{c(Eu_h^n)}\partial_t\xi^n\|^2 - \|\sqrt{c(Eu_h^{n-1})}\partial_t\xi^n\|^2) \\
& \quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla\xi^n\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^n\|^2) \\
& \quad + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla\partial_t\xi^n\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\partial_t\xi^n\|^2) + \sum_{i=1}^{17} R_i.
\end{aligned}
\tag{4.14}
$$

By using the assumption on (4.8) and $\Delta t = O(h)$, we get

$$
\begin{aligned}
& \|Eu_h^n - Eu_h^{n-1}\|_\infty \\
& = \|E(u_h^n - \tilde{u}^n) - E(u_h^{n-1} - \tilde{u}^{n-1}) + E\tilde{u}^n - E\tilde{u}^{n-1}\|_\infty \\
& \leq \Delta t\Big(2\|\partial_t\xi^n\|_\infty + \|\partial_t\xi^{n-1}\|_\infty\Big) + K\Delta t \\
& \leq K\Delta t.
\end{aligned}
\tag{4.15}
$$

Hence, by the assumption (A2) and (4.15), (4.14) can be estimated as follows: By the assumption (A1) and Cauchy-Schwartz inequality, we can estimate $R_1 \sim R_3$ as follows: for an $\epsilon > 0$

$$
\begin{aligned}
R_1 & \leq \epsilon\|\partial_t\xi^{n+1}\|^2 + K\|\nabla\xi^{n-1}\|^2, \\
R_2 & \leq \epsilon\|\partial_t\xi^{n+1}\|^2 + K\|\nabla\xi^n\|^2, \\
R_3 & = \frac{3}{2}\Big(c(Eu_h^n)\partial_t\eta^{n+1}, \partial_t\xi^{n+1}\Big) - \frac{1}{2}\Big(c(Eu_h^n)\partial_t\eta^n, \partial_t\xi^{n+1}\Big) \\
& \leq \epsilon\|\partial_t\xi^{n+1}\|^2 + K(\|\partial_t\eta^{n+1}\|^2 + \|\partial_t\eta^n\|^2).
\end{aligned}
$$

Since

$$
\eta^{n-1} - \hat{\eta}^{n-1} = 2\Delta t\nabla\eta(\tilde{\boldsymbol{x}}_1, t^{n-1}) \cdot \tilde{\boldsymbol{d}}(Eu_h^n)
$$

and

$$\eta^n - \check{\eta}^n = \Delta t \nabla \eta(\tilde{\boldsymbol{x}}_2, t^n) \cdot \tilde{\boldsymbol{d}}(Eu_h^n)$$

for some $\tilde{\boldsymbol{x}}_1 \in (\hat{\boldsymbol{x}}, \boldsymbol{x})$ and $\tilde{\boldsymbol{x}}_2 \in (\check{\boldsymbol{x}}, \boldsymbol{x})$, by integration by parts, we have

$$\begin{aligned}
R_4 = &\Big(\eta(\tilde{\boldsymbol{x}}_1, t^{n-1})\tilde{\boldsymbol{d}}(Eu_h^n), \nabla(c(Eu_h^n)\partial_t \xi^{n+1})\Big) \\
&+ \Big(\eta(\tilde{\boldsymbol{x}}_1, t^{n-1})\nabla \cdot \tilde{\boldsymbol{d}}(Eu_h^n), c(Eu_h^n)\partial_t \xi^{n+1}\Big) \\
&- 2\Big(\eta(\tilde{\boldsymbol{x}}_2, t^n)\tilde{\boldsymbol{d}}(Eu_h^n), \nabla(c(Eu_h^n)\partial_t \xi^{n+1})\Big) \\
&- 2\Big(\eta(\tilde{\boldsymbol{x}}_2, t^n)\nabla \cdot \tilde{\boldsymbol{d}}(Eu_h^n), c(Eu_h^n)\partial_t \xi^{n+1}\Big) \\
&\le \epsilon\|\nabla\partial_t \xi^{n+1}\|^2 + \epsilon\|\partial_t \xi^{n+1}\|^2 + K\|\eta^n\|^2 + K\|\eta^{n-1}\|^2
\end{aligned}$$

By Taylor expansion, we have

$$\begin{aligned}
&2(\check{u}^n - \check{\check{u}}^n) + \frac{1}{2}(\hat{\hat{u}}^{n-1} - \hat{u}^{n-1}) \\
=&2u(\boldsymbol{x} - \tilde{\boldsymbol{d}}(Eu_h^n)\Delta t, t^n) - 2u(\boldsymbol{x} - \tilde{\boldsymbol{d}}(u^{n+1})\Delta t, t^n) \\
&+ \frac{1}{2}u(\boldsymbol{x} - 2\tilde{\boldsymbol{d}}(u^{n+1})\Delta t, t^{n-1}) - \frac{1}{2}u(\boldsymbol{x} - 2\tilde{\boldsymbol{d}}(Eu_h^n)\Delta t, t^{n-1}) \\
=&(\tilde{\boldsymbol{d}}(u^{n+1}) - \tilde{\boldsymbol{d}}(Eu_h^n))\Delta t \,\nabla u^n + (\Delta t)^2(\tilde{\boldsymbol{d}}(u^{n+1}) - \tilde{\boldsymbol{d}}(Eu_h^n))\nabla u_t^n \\
&- \frac{(\Delta t)^3}{3}(\tilde{\boldsymbol{d}}(Eu_h^n))^3\nabla^3 u^n(\tilde{\boldsymbol{x}}_1) + \frac{(\Delta t)^3}{3}(\tilde{\boldsymbol{d}}(u^{n+1}))^3\nabla^3 u^n(\tilde{\boldsymbol{x}}_2) \\
&- \frac{2(\Delta t)^3}{3}(\tilde{\boldsymbol{d}}(u^{n+1}))^3\nabla^3 u^n(\tilde{\boldsymbol{x}}_3) - (\Delta t)^3(\tilde{\boldsymbol{d}}(u^{n+1}))^2\nabla^2 u_t^n(\tilde{\boldsymbol{x}}_4) \\
&- \frac{(\Delta t)^3}{2}\tilde{\boldsymbol{d}}(u^{n+1}) \,\nabla u_{tt}^n(\tilde{\boldsymbol{x}}_5) - \frac{1}{12}(\Delta t)^3 u_{ttt}(\boldsymbol{x} - 2\tilde{\boldsymbol{d}}(u^{n+1})\Delta t, t_1^n) \\
&+ \frac{2(\Delta t)^3}{3}(\tilde{\boldsymbol{d}}(Eu_h^n))^3\nabla^3 u^n(\tilde{\boldsymbol{x}}_6) + (\Delta t)^3(\tilde{\boldsymbol{d}}(Eu_h^n))^2\nabla^2 u_t^n(\tilde{\boldsymbol{x}}_7) \\
&+ \frac{(\Delta t)^3}{2}\tilde{\boldsymbol{d}}(Eu_h^n) \,\nabla u_{tt}^n(\tilde{\boldsymbol{x}}_8) + \frac{1}{12}(\Delta t)^3 u_{ttt}(\boldsymbol{x} - 2\tilde{\boldsymbol{d}}(Eu_h^n)\Delta t, t_2^n)
\end{aligned}$$

where $t_1^n, t_2^n \in (t^{n-1}, t^n)$, $\tilde{\boldsymbol{x}}_1, \tilde{\boldsymbol{x}}_3, \tilde{\boldsymbol{x}}_4, \tilde{\boldsymbol{x}}_5 \in (\hat{\hat{\boldsymbol{x}}}, \boldsymbol{x})$, and $\tilde{\boldsymbol{x}}_2, \tilde{\boldsymbol{x}}_6, \tilde{\boldsymbol{x}}_7, \tilde{\boldsymbol{x}}_8 \in (\hat{\boldsymbol{x}}, \boldsymbol{x})$. Notice that

$$\begin{aligned}
&\tilde{\boldsymbol{d}}(u^{n+1}) - \tilde{\boldsymbol{d}}(Eu_h^n) \\
&= \frac{c(u^{n+1})[\boldsymbol{d}(u^{n+1}) - \boldsymbol{d}(Eu_h^n)] - \boldsymbol{d}(u^{n+1})[c(u^{n+1}) - c(Eu_h^n)]}{c(u^{n+1})c(Eu_h^n)}
\end{aligned}$$

and

$$\|u^{n+1} - Eu_h^n\| \le \|\eta^{n+1}\| + K(\Delta t)^2 + 2\|\xi^n\| + \|\xi^{n-1}\|. \tag{4.16}$$

So we have

$$R_5 + R_6 = -2\Big(c(Eu_h^n)\frac{(\check{\check{u}}^n - \check{u}^n)}{\Delta t}, \partial_t \xi^{n+1}\Big) + \frac{1}{2}\Big(c(Eu_h^n)\frac{(\hat{\hat{u}}^{n-1} - \hat{u}^{n-1})}{\Delta t}, \partial_t \xi^{n+1}\Big)$$
$$\leq \epsilon\|\partial_t \xi^{n+1}\|^2 + K[\|\eta^{n+1}\|^2 + (\Delta t)^4 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2].$$

And we obtain the bound for $R_7 + R_8$ as follows:

$$R_7 + R_8 = ((a(Eu_h^n) - a(u^{n+1}))\nabla\eta^{n+1}, \nabla\partial_t\xi^{n+1})$$
$$\leq \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4).$$

Since

$$\frac{1}{\Delta t}\Big[\frac{3}{2}\eta^{n+1} - 2\eta^n + \frac{1}{2}\eta^{n-1}\Big] - \eta_t^{n+1} = O((\Delta t)^2),$$

we get

$$R_9 \leq \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\Delta t)^4.$$

And for $R_{10} + R_{11}$, by (4.17) and Lemma 3.2, we have

$$R_{10} + R_{11} = ((b(Eu_h^n) - b(u^{n+1}))\nabla\eta_t^{n+1}, \nabla\partial_t\xi^{n+1})$$
$$\leq \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4).$$

Since

$$\frac{3}{2}u^{n+1} - 2\check{\check{u}}^n + \frac{1}{2}\hat{\hat{u}}^{n-1} = O(\Delta t),$$

we get

$$R_{12} \leq \epsilon\|\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4).$$

Similarly, the bounds for $R_{13} \sim R_{17}$ are obtained as follows:

$$R_{13} \leq \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4),$$
$$R_{14} \leq \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4),$$
$$R_{15} \leq \epsilon\|\partial_t\xi^{n+1}\|^2 + K(\|\eta^{n+1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4).$$

Notice that by Taylor expansion

$$c(u^{n+1})\frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t} - \psi(u^{n+1})\frac{\partial u^{n+1}}{\partial\nu} \approx O((\Delta t)^2),$$
$$\frac{\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1}}{\Delta t} - u_t^{n+1} \approx O((\Delta t)^2).$$

Therefore, we get

$$R_{16} \leq \epsilon\|\partial_t\xi^{n+1}\|^2 + K(\Delta t)^4,$$
$$R_{17} \leq \epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + K(\Delta t)^4.$$

Now, using the estimates for $R_1 \sim R_{17}$, we obtain from (4.16)

$$\begin{aligned} & c_*\|\partial_t\xi^{n+1}\|^2 + b_*\|\nabla\partial_t\xi^{n+1}\|^2 \\ & \quad + \frac{1}{4}(\|\sqrt{c(Eu_h^n)}\partial_t\xi^{n+1}\|^2 - \|\sqrt{c(Eu_h^{n-1})}\partial_t\xi^n\|^2) \\ & \quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla\xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^n\|^2) \\ & \quad + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla\partial_t\xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\partial_t\xi^n\|^2) \\ & \le K[\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + \|\nabla\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2] \\ & \quad + K\Delta t[\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2] + 7\epsilon\|\nabla\partial_t\xi^{n+1}\|^2 + 8\epsilon\|\partial_t\xi^{n+1}\|^2 \\ & \quad + K[\|\eta^n\|^2 + \|\eta^{n+1}\|^2 + \|\partial_t\eta^n\|^2 + \|\partial_t\eta^{n+1}\|^2 + (\Delta t)^4] \end{aligned} \tag{4.17}$$

Hence, by using Poincare's inequality and Lemma 3.1, (4.18) can be estimated as follows:

$$\begin{aligned} & \alpha\|\partial_t\xi^{n+1}\|^2 + \beta\|\nabla\partial_t\xi^{n+1}\|^2 \\ & \quad + \frac{1}{4}(\|\sqrt{c(Eu_h^n)}\partial_t\xi^{n+1}\|^2 - \|\sqrt{c(Eu_h^{n-1})}\partial_t\xi^n\|^2) \\ & \quad + \frac{1}{2\Delta t}(\|\sqrt{a(Eu_h^n)}\nabla\xi^{n+1}\|^2 - \|\sqrt{a(Eu_h^{n-1})}\nabla\xi^n\|^2) \\ & \quad + \frac{1}{4}(\|\sqrt{b(Eu_h^n)}\nabla\partial_t\xi^{n+1}\|^2 - \|\sqrt{b(Eu_h^{n-1})}\nabla\partial_t\xi^n\|^2) \\ & \le K[\|\nabla\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2 + \Delta t(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2) + h^{2\mu} + (\Delta t)^4]. \end{aligned} \tag{4.18}$$

for sufficiently small ϵ. Now, we add both sides of (4.19) from $n = 1$ to $l-1$ to get

$$\begin{aligned} & \Delta t\sum_{n=1}^{l-1}[\alpha\|\partial_t\xi^{n+1}\|^2 + \beta\|\nabla\partial_t\xi^{n+1}\|^2] + \|\sqrt{a(Eu_h^{l-1})}\nabla\xi^l\|^2 \\ & \qquad + \Delta t[\|\sqrt{c(Eu_h^{l-1})}\partial_t\xi^l\|^2 + \|\sqrt{b(Eu_h^{l-1})}\nabla\partial_t\xi^l\|^2] \\ \le & K\Delta t\sum_{n=1}^{l-1}[\|\nabla\xi^n\|^2 + \|\nabla\xi^{n-1}\|^2 + \Delta t(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2) + h^{2\mu} + (\Delta t)^4], \end{aligned}$$

which yields

$$\begin{aligned} & \|\nabla\xi^l\|^2 + \Delta t\{\|\partial_t\xi^l\|^2 + \|\nabla\partial_t\xi^l\|^2\} \\ \le & K\Big[\Delta t\sum_{n=1}^{l-1}\{\|\nabla\xi^n\|^2 + \Delta t(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2)\} + K\Delta t\sum_{n=1}^{l-1}\{h^{2\mu} + (\Delta t)^4\}\Big], \end{aligned}$$

for sufficiently small Δt. Using Gronwall's inequality, we get

$$\|\nabla\xi^l\|^2 + \Delta t\{\|\partial_t\xi^l\|^2 + \|\nabla\partial_t\xi^l\|^2\} \le K[h^{2\mu} + (\Delta t)^4],$$

and so, we complete the proof of the statement (4.8). Using the Poincare's inequality and the triangle inequality, it can be easily proved that $\|u^l - u_h^l\| \leq K(h^\mu + (\Delta t)^2)$ and $\|\nabla(u^l - u_h^l)\| \leq K(h^{\mu-1} + (\Delta t)^2)$. Thus the result of this theorem holds. □

Remark 4.1. The result of Theorem 4.2 is the same as one in [16, 19]. Even though two techniques give us the same result, the backward three point technique is quite different from the Crank-Nicolson technique. Both techniques are widely used to obtain higher order of convergence in the time direction or in the characteristic direction. And an extrapolation technique can be used to avoid the difficulties of solving a system of nonlinear equations.

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