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# FURTHER EXTENSION OF TWO RESULTS INVOLVING ${}_0F_1$ DUE TO BAILEY

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Abstract. Bailey presented two interesting identities involving ${}_0F_1$, which have been generalized by Choi and Rathie who used two hypergeometric summation formulas due to Qureshi et al. In this note, we aim to show how one can establish, in an elementary way, two generalized formulas involving ${}_0F_1$ which include the above-mentioned identities as special cases.

## 1. Introduction

We begin by recalling two interesting results (see [1, Eqs. (3.2) and (3.3)])

$$e^x \ {}_0F_1\left[\begin{matrix} -; \\ \frac{1}{2}; \end{matrix} \ -\frac{x^2}{4}\right] = \sum_{m=0}^{\infty} \frac{x^m}{m!} \, 2^{\frac{m}{2}} \cos\left(\frac{m\pi}{4}\right) \tag{1.1}$$

and

$$e^x \ {}_0F_1\left[\begin{matrix} -; \\ \frac{3}{2}; \end{matrix} \ -\frac{x^2}{4}\right] = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!} \, 2^{\frac{m}{2}} \sin\left(\frac{m\pi}{4}\right), \tag{1.2}$$

where ${}_pF_q$ denotes the familiar generalized hypergeometric function (see, e.g., [5, Section 1.5]).

Choi and Rathie [2, Eqs. (3.1) and (3.2)] generalized Bailey's results (1.1) and (1.2) to present

$$e^x \ {}_0F_1\left[\begin{matrix} -; \\ \frac{1}{2}; \end{matrix} \ -\frac{b^2x^2}{4a^2}\right] = \sum_{m=0}^{\infty} \frac{x^m}{m!} \, \frac{(a^2+b^2)^{\frac{m}{2}}}{a^m} \cos(m\theta) \tag{1.3}$$

and

$$e^x \ {}_0F_1\left[\begin{matrix} -; \\ \frac{3}{2}; \end{matrix} \ -\frac{b^2x^2}{4a^2}\right] = \sum_{m=1}^{\infty} \frac{x^{m-1}}{m!} \, \frac{(a^2+b^2)^{\frac{m}{2}}}{a^{m-1}b} \sin(m\theta), \tag{1.4}$$

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where

$$\theta := \begin{cases} \arctan\left(\frac{b}{a}\right) & (a,\, b \in \mathbb{R}^+), \\ \pi - \arctan\left(\frac{b}{|a|}\right) & (a \in \mathbb{R}^-;\, b \in \mathbb{R}^+), \\ \arctan\left(\frac{b}{a}\right) - \pi & (a,\, b \in \mathbb{R}^-), \\ -\arctan\left(\frac{|b|}{a}\right) & (a \in \mathbb{R}^+;\, b \in \mathbb{R}^-). \end{cases} \tag{1.5}$$

Here and in the following, let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$, and $\mathbb{N}$ be the sets of real numbers, positive real numbers, negative real numbers, and positive integers, respectively. They [2] proved (1.3) and (1.4) by using two hypergeometric summation formulas [4, Eqs (18) and (19)]. Choi and Rathie [3] also derived (1.3) and (1.4) in an elementary way.

In this note, we aim to present two formulas which generalize the abovementioned results (1.1), (1.2), (1.3) and (1.4).

## 2. Main results

Here we establish two identities which include (1.1), (1.2), (1.3) and (1.4) as special cases, asserted by the following theorem.

**Theorem 2.1.** *Let $a$, $b$, $\theta$ be given in* (1.5)*. Also let $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Then*

$$e^x \; {}_0F_1\left[\begin{array}{c} -; \\ \frac{1}{2}; \end{array} \; -\left\{\frac{\Im(a+ib)^k}{\Re(a+ib)^k}\,\frac{x}{2}\right\}^2\right] = \sum_{m=0}^{\infty} \frac{x^m}{m!}\,\frac{(a^2+b^2)^{\frac{km}{2}}}{\{\Re(a+ib)^k\}^m}\,\cos(k\,m\theta) \tag{2.1}$$

*and*

$$\begin{aligned} &e^x \; {}_0F_1\left[\begin{array}{c} -; \\ \frac{3}{2}; \end{array} \; -\left\{\frac{\Im(a+ib)^k}{\Re(a+ib)^k}\,\frac{x}{2}\right\}^2\right] \\ &\quad = \sum_{m=0}^{\infty} \frac{x^m}{(m+1)!}\,\frac{(a^2+b^2)^{\frac{k(m+1)}{2}}}{\{\Re(a+ib)^k\}^m\,\Im(a+ib)^k}\,\sin(k(m+1)\theta). \end{aligned} \tag{2.2}$$

*Proof.* Let $\mathcal{L}$ be the left side of (2.1). By using (see, e.g., [5, p. 73])

$$\cos z = {}_0F_1\left[\begin{array}{c} -; \\ \frac{1}{2}; \end{array} \; -\frac{z^2}{4}\right],$$

we obtain

$$\mathcal{L} = e^x \, \cos\left[\frac{\Im(a+ib)^k}{\Re(a+ib)^k}x\right].$$

We have

$$\begin{aligned}\mathcal{L} &= e^x\, \Re\left\{\exp\left[i\frac{\Im(a+ib)^k}{\Re(a+ib)^k}x\right]\right\}\\ &= \Re\left\{\exp\left[\frac{\Re(a+ib)^k + i\,\Im(a+ib)^k}{\Re(a+ib)^k}x\right]\right\}\\ &= \Re\left\{\exp\left[\frac{(a+ib)^k}{\Re(a+ib)^k}x\right]\right\}.\end{aligned} \tag{2.3}$$

Using $a + ib = \sqrt{a^2+b^2}\, e^{i\theta}$ in (2.3) and expanding the resulting exponential, we have

$$\begin{aligned}\mathcal{L} &= \Re\left\{\exp\left[\frac{\left(a^2+b^2\right)^{\frac{k}{2}}}{\Re(a+ib)^k}e^{ik\theta}\,x\right]\right\}\\ &= \Re\left\{\sum_{m=0}^{\infty}\frac{\left(a^2+b^2\right)^{\frac{km}{2}}}{m!\,\left\{\Re(a+ib)^k\right\}^m}\,x^m\,e^{ikm\theta}\right\},\end{aligned}$$

which, upon considering $e^{i\alpha} = \cos\alpha + i\sin\alpha$, leads to the right side of (2.1).

The proof of (2.2) would run parallel to that of (2.1), by considering (see, e.g., [5, p. 73])

$$\sin z = z\,{}_0F_1\left[\begin{matrix}-; \\ \frac{3}{2};\end{matrix}\quad -\frac{z^2}{4}\right].$$

We omit the details. $\square$

## 3. Concluding remarks

The method used here is very elementary. Setting $k = 1$ in (2.1) and (2.2) yield, respectively, (1.3) and (1.4). Further, Setting $k = 1$ and $a = b$ in (2.1) and (2.2) yield, respectively, (1.1) and (1.2).

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