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NON-HOPFIAN SQ-UNIVERSAL GROUPS

Donghi Lee

Abstract. In [9], Lee and Sakuma constructed 2-generator non-Hopfian groups each of which has a specific presentation $\langle a, b \,|\, R\rangle$ satisfying small cancellation conditions $C(4)$ and $T(4)$. In this paper, we prove the SQ-universality of those non-Hopfian groups.

1. Introduction

Recall that a group G is called *SQ-universal* if every countable group can be embedded in a quotient of G. Being SQ-universal is a group-theoretic property that is traditionally thought as measuring "largeness" of a group, since any SQ-universal group contains an infinitely generated free subgroup and has uncountably many pairwise non-isomorphic quotients.

Examples of SQ-universal groups include the free group of rank 2 [6], various HNN-extensions and amalgamated free products [3, 10, 13], groups of deficiency 2 [2], non-elementary hyperbolic groups [12], non-elementary relatively hyperbolic groups [1], etc. For finitely presented small cancellation groups, most $C(3) - T(6)$ groups [7], and all $C(p) - T(q)$ groups [4] with (p, q) being positive integers such that $1/p + 1/q < 1/2$ are SQ-universal. On the other hand, for infinitely presented small cancellation groups, Gruber [5] proved the SQ-universality of $C(6)$ groups.

Motivated by Gruber's direct proof of the SQ-universality of $C(6)$ groups, we prove the SQ-universality of the non-Hopfian groups constructed in [9]. Recall that the simplest non-Hopfian group G in [9] has the presentation

$$G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$$

which satisfies small cancellation conditions $C(4) - T(4)$. Here, u_{r_i} is the single relator of the upper presentation $\langle a, b \,|\, u_{r_i}\rangle$ of the 2-bridge link group of slope r_i, where $r_0 = [4, 3, 3]$ and $r_i = [4, 2, (i-1)\langle 3\rangle, 4, 3]$ in continued fraction expansion for every integer $i \geq 1$. Recall that for $(m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k$,

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$$[m_1, m_2, \ldots, m_k] := \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_k}}}.$$

Recall also that the symbol "$(i-1)\langle 3\rangle$" represents $i-1$ successive 3's if $i-1 \geq 1$, whereas "$0\langle 3\rangle$" means that 3 does not occur in that place, so that $r_1 = [4, 2, 0\langle 3\rangle, 4, 3] = [4, 2, 4, 3]$.

The following is the main result of this paper.

Theorem 1.1. *Let $r_0 = [4, 3, 3]$, and let $r_i = [4, 2, (i-1)\langle 3\rangle, 4, 3]$ for every integer $i \geq 1$. Then the group $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$ is SQ-universal.*

In the proof of Theorem 1.1, the following definition and result from [12] play an important role.

Definition 1.2 ([12])**.** Let G be a group and H a subgroup of G. Then H has the *congruence extension property* (*CEP*) if for every normal subgroup N of H (i.e., N is normal in H), we have $\langle N\rangle^G \cap H = N$, where $\langle N\rangle^G$ denotes the normal closure of N in G. The group G has *property* $F(2)$ if there exists a subgroup H of G that is a free group of rank 2 and that has the CEP.

Proposition 1.3 ([12])**.** *If a group G has property $F(2)$, then G is SQ-universal.*

In the viewpoint of Proposition 1.3, we will show that the group $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$ has property $F(2)$ to prove Theorem 1.1. To be precise, putting $s_0 = [5, 4, 4]$ and $s_1 = [5, 3, 5, 4]$, we will show that the subgroup $H = \langle u_{s_0}, u_{s_1}\rangle$ of G is a free group of rank 2 and has the CEP. By looking at the proof of Theorem 1.1 in Section 3, it is not hard to see that a similar result holds not only for $r_0 = [4, 3, 3]$ but also for $r_0 = [m+1, m, m]$ with m being any integer greater than 3. Thus we only state its general form without a detailed proof.

Theorem 1.4. *Suppose that m is an integer with $m \geq 3$. Let $r_0 = [m+1, m, m]$, and let $r_i = [m+1, m-1, (i-1)\langle m\rangle, m+1, m]$ for every integer $i \geq 1$. Then the group $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$ is SQ-universal.*

The present paper is organized as follows. In Section 2, we recall the upper presentation of a 2-bridge link group, and a basic fact established in [8] concerning the single relator u_r of the upper presentation. We also recall key facts from [9] obtained by applying small cancellation theory to $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

2.1. Upper presentations of 2-bridge link groups

Consider the discrete group, H, of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in the lattice $\mathbb{Z}^2$. Set $(\boldsymbol{S}^2, \boldsymbol{P}) := (\mathbb{R}^2, \mathbb{Z}^2)/H$ and call it the *Conway sphere.* Then $\boldsymbol{S}^2$ is homeomorphic to the 2-sphere, and $\boldsymbol{P}$ consists of four points in $\boldsymbol{S}^2$. We also call $\boldsymbol{S}^2$ the Conway sphere. Let $\boldsymbol{S} := \boldsymbol{S}^2 - \boldsymbol{P}$ be the complementary 4-times punctured sphere. For each $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let α_r be the unoriented simple loop in $\boldsymbol{S}$ obtained as the projection of any straight line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope r. Then α_r is *essential* in $\boldsymbol{S}$, i.e., it does not bound a disk nor a once-punctured disk in $\boldsymbol{S}$. Conversely, any essential simple loop in $\boldsymbol{S}$ is isotopic to α_r for a unique $r \in \hat{\mathbb{Q}}$. Then r is called the *slope* of the simple loop. Similarly, any simple arc δ in $\boldsymbol{S}^2$ joining two different points in $\boldsymbol{P}$ such that $\delta \cap \boldsymbol{P} = \partial\delta$ is isotopic to the image of a line in $\mathbb{R}^2$ of some slope $r \in \hat{\mathbb{Q}}$ which intersects $\mathbb{Z}^2$. We call r the *slope* of δ. Thus, for every slope $r \in \hat{\mathbb{Q}}$, there exist two arcs and one loop of slope r in $(\boldsymbol{S}^2, \boldsymbol{P})$ (all unoriented).

A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs properly embedded in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3. By a *rational tangle*, we mean a trivial tangle (B^3, t) which is endowed with a homeomorphism from $(\partial B^3, \partial t)$ to $(\boldsymbol{S}^2, \boldsymbol{P})$. Through the homeomorphism we identify the boundary of a rational tangle with the Conway sphere. Thus the slope of an essential simple loop in $\partial B^3 - t$ is defined. We define the *slope* of a rational tangle to be the slope of an essential loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. We denote a rational tangle of slope r by $(B^3, t(r))$.

For each $r \in \hat{\mathbb{Q}}$, the *2-bridge link $K(r)$ of slope r* is the sum of the rational tangle $(B^3, t(\infty))$ of slope ∞ and the rational tangle $(B^3, t(r))$ of slope r. Recall that $\partial(B^3 - t(\infty))$ and $\partial(B^3 - t(r))$ are identified with $\boldsymbol{S}$ so that α_∞ and α_r bound disks in $B^3 - t(\infty)$ and $B^3 - t(r)$, respectively. By van-Kampen's theorem, the link group $G(K(r)) := \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle.$$

Let $\{a, b\}$ be the standard meridian generator pair of $\pi_1(B^3 - t(\infty), x_0)$ as described in [8, Section 3]. Then $\pi_1(B^3 - t(\infty))$ is identified with the free group $F(a, b)$ with basis $\{a, b\}$. For a positive rational number $r = q/p$, where p and q are relatively prime positive integers, let u_r be the word in $\{a, b\}$ obtained as follows. Set $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(1) If p is odd, then

$$u_{q/p} = a\hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \cdots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}}$.

(2) If p is even, then

$$u_{q/p} = a\hat{u}_{q/p}a^{-1}\hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\epsilon_1}a^{\epsilon_2}\cdots a^{\epsilon_{p-2}}b^{\epsilon_{p-1}}$.

Then $u_r \in F(a,b) \cong \pi_1(B^3 - t(\infty))$ is represented by the simple loop α_r, and we obtain the following two-generator and one-relator presentation of a 2-bridge link group:

$$G(K(r)) \cong \pi_1(B^3 - t(\infty))/\langle\langle\alpha_r\rangle\rangle \cong \langle a, b \,|\, u_r\rangle.$$

This presentation is called the *upper presentation* of the 2-bridge link group.

2.2. A basic fact concerning the relator u_r of the upper presentation

Throughout this paper, a *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v. Also the symbol "$\equiv$" denotes the *letter-by-letter equality* between two words or between two cyclic words. Now we recall definitions and basic facts from [8] which are needed in the proof of Theorem 1.1 in Section 3.

A word v is called a *positive* (or *negative*) word, if all letters in v have positive (or negative, respectively) exponents.

Definition 2.1. Let v be a reduced word in $\{a,b\}$. Decompose v into

$$v \equiv v_1v_2\cdots v_t,$$

where, for each $i = 1,\ldots,t-1$, v_i is a positive (or negative) subword, and v_{i+1} is a negative (or positive, respectively) subword. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \ldots, |v_t|)$ is called the *S-sequence of v*.

A reduced word w in $\{a,b\}$ is said to be *alternating* if $a^{\pm1}$ and $b^{\pm1}$ appear in w alternately, to be precise, neither $a^{\pm2}$ nor $b^{\pm2}$ appears in w. Also a cyclically reduced word w in $\{a,b\}$ is said to be *cyclically alternating*, i.e., all the cyclic permutations of w are alternating. In particular, u_r is a cyclically alternating word in $\{a,b\}$.

Lemma 2.2 ([8, Propositions 4.3 and 4.4])**.** *For a rational number $r = [m_1, m_2, \ldots, m_k]$ with $k \geq 2$ and $m_2 \geq 2$, putting $m = m_1$, we have*

$$S(u_r) = (m+1, (m_2-1)\langle m\rangle, m+1, \ldots, m+1, m_2\langle m\rangle),$$

where the symbol "$t\langle m\rangle$" represents t successive m's.

2.3. Small cancellation theory applied to $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$

A subset R of the free group $F(a,b)$ is called *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R.

Definition 2.3. Suppose that R is a symmetrized subset of $F(a,b)$. A nonempty word v is called a *piece* (*with respect to* R) if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv vc_1$ and $w_2 \equiv vc_2$. The small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [11]).

(1) Condition $C(p)$: If $w \in R$ is a product of n pieces, then $n \geq p$.
(2) Condition $T(q)$: For $w_1, \dots, w_n \in R$ with no successive elements w_i, w_{i+1} an inverse pair (i mod n), if $n < q$, then at least one of the products $w_1 w_2, \dots, w_{n-1}w_n$, $w_n w_1$ is freely reduced without cancellation.

The following proposition enables us to apply small cancellation theory to the presentation $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$ in Theorem 1.1.

Lemma 2.4 ([9, Lemma 3.8]). *Let R be the symmetrized subset of $F(a,b)$ generated by the set of relators $\{u_{r_i} \,|\, i \geq 0\}$ of the presentation $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$ in Theorem 1.1. Then R satisfies $C(4)$ and $T(4)$.*

We may interpret [9, Claim 2 in the proof of Lemma 3.8] as the following useful format.

Lemma 2.5. *Let r_i and R be as in Lemma 2.4. If a subword w of the cyclic word $(u_{r_i}^{\pm 1})$ is a product of no less than 2 pieces with respect to R, then $S(w)$ contains a term 4.*

3. Proof of Theorem 1.1

Let $s_0 := [5,4,4]$ and $s_1 := [5,3,5,4]$ be rational numbers. Then both u_{s_0} and u_{s_1} are cyclically alternating words in $\{a,b\}$ which begin with a and end with b^{-1}. Also by Lemma 2.2,

$$S(u_{s_0}) = (6,5,5,5,6,\dots,6,5,5,5,5),$$
$$S(u_{s_1}) = (6,5,5,6,\dots,6,5,5,5).$$

So we can see that for any product p of elements in $\{u_{s_0}, u_{s_1}\}^{\pm 1}$, the cyclic word (p) has the form

$$(p) \equiv (w_1 b^{\pm 2} w_2 b^{\pm 2} \cdots w_n b^{\pm 2}), \qquad (\dagger)$$

where w_i is an alternating word in $\{a,b\}$ such that w_i begins and ends with $a^{\pm 1}$ and such that $S(w_i)$ consists of 5 and 6, for every $i = 1, 2, \dots, n$.

Let $H := \langle u_{s_0}, u_{s_1}\rangle$ be a subgroup of $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$. We will show that G has property $F(2)$ by showing that H is a free group of rank 2 and has the CEP.

Lemma 3.1. *The subgroup $H = \langle u_{s_0}, u_{s_1}\rangle$ of $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$ is a free group of rank 2.*

Proof. Suppose that there exists some nontrivial product p of elements in $\{u_{s_0}, u_{s_1}\}^{\pm 1}$ equal to the identity in G. Then there is a reduced van Kampen diagram Δ over $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = 1\rangle$ such that $(\phi(\partial\Delta)) \equiv (p)$ (see [11]). Since

Δ is a $[4,4]$-map by Lemma 2.4, we have by the Curvature Formula of Lyndon and Schupp (see [11, Corollary V.3.4])

$$\sum_{v\in\partial\Delta}(3-d(v))\geq 4.$$

This implies that there exists a vertex of degree 2 on $\partial\Delta$, so that $(\phi(\partial\Delta))$ contains a subword of some $(u_{r_i}^{\pm 1})$ which cannot be expressed as a product of less than 2 pieces with respect to the symmetrized subset R in Lemma 2.4 (see [8, Section 6]). Then, since $(\phi(\partial\Delta))\equiv(p)$, the cyclic word (p) contains a subword w of the cyclic word $(u_{r_i}^{\pm 1})$ such that $S(w)$ contains a term 4 by Lemma 2.5. But this is obviously a contradiction to (†). □

Now to prove that H has the CEP, let c and d be symbols not in $F(a,b)$. Put

$$R=\{u_{r_0},u_{r_1},u_{r_2},\dots\}\subseteq F(a,b),$$
$$W=\{c^{-1}u_{s_0},d^{-1}u_{s_1}\}\subseteq F(a,b,c,d).$$

Here, $F(a,b,c,d)$ denotes the free group with basis $\{a,b,c,d\}$. Then clearly

$$G=\langle a,b\,|\,R\rangle\cong\langle a,b,c,d\,|\,W\cup R\rangle.$$

Under this isomorphism, the subgroup $H=\langle u_{s_0},u_{s_1}\rangle$ of G maps to $\langle c,d\rangle$ which is a subgroup of $\langle a,b,c,d\,|\,R\cup W\rangle$. From now on, we consider the presentation $\langle a,b,c,d\,|\,W\cup R\rangle$ for G and $\langle c,d\rangle$ for H.

Lemma 3.2. *Let G and H be as above, and let N be a normal subgroup of H. Then $\langle N\rangle^G\cap H=N$.*

Proof. Suppose on the contrary that there exists $g\in(\langle N\rangle^G\cap H)\setminus N$. Let L be the set of words in $\{c,d\}$ representing elements of N, and consider the presentation

$$\langle a,b,c,d\,|\,L\cup W\cup R\rangle.$$

Let w be a word in $\{c,d\}$ representing g. Since $g\in\langle N\rangle^G\cap H$, w is equal to the identity in the group $\langle a,b,c,d\,|\,L\cup W\cup R\rangle$. Then there is a reduced van Kampen diagram Δ over $\langle a,b,c,d\,|\,L\cup W\cup R\rangle$ such that $(\phi(\partial\Delta))\equiv(w)$. Assume that g, w and Δ are chosen such that the $(L;W;R)$-lexicographic area of Δ is minimal for all possible choices (i.e., we first minimize the number of faces labelled by elements of L, then the number of faces labelled by elements of W and then the number of faces labelled by elements of R), and among these choices, the number of edges of Δ is minimal.

The following claim may be immediately adopted from [5, Claim 1 in the proof of Proposition 2.15], since $C(6)$-condition was used nowhere in its proof.

Claim 1. Δ has the following properties:

a) Δ is a simple disk diagram, and w is cyclically reduced.

b) No L-face intersects $\partial\Delta$. Therefore, every edge of $\partial\Delta$ is contained in a W-face.

c) Every L-face is simply connected, and no two L-faces intersect. Therefore, every L-face shares all its boundary edges with W-faces. We say it is surrounded by W-faces.

d) The intersection of two W-faces does not contain a $\{c, d\}$-edge.

Let $\pi_1, \dots, \pi_t$ denote the L-faces in Δ. By Claim 1c), each π_i is surrounded by W-faces, say $\sigma_{i,1}, \dots, \sigma_{i,h_i}$. Since every W-face has only one $\{c, d\}$-edge, if $i \neq i'$ then $\sigma_{i,j} \neq \sigma_{i',j'}$ for every j and j'.

Put

$$S = \{u_{s_0}, u_{s_1}\} \subseteq F(a, b).$$

As illustrated in Figure 1, for each $i = 1, \dots, t$, we may replace a subdiagram $D_i = \pi_i \cup \sigma_{i,1} \cup \dots \cup \sigma_{i,h_i}$ with $D'_i = \tau_{i,1} \cup \dots \cup \tau_{i,h_i}$ consisting of S-faces $\tau_{i,1}, \dots, \tau_{i,h_i}$ such that D_i and D'_i have the same boundary label. Here, an S-face $\tau_{i,j}$ is chosen in such a way that if $(\phi(\partial\sigma_{i,j})) \equiv (c^{\mp 1} u_{s_0}^{\pm 1})$ then $(\phi(\partial\tau_{i,j})) \equiv (u_{s_0}^{\pm 1})$; if $(\phi(\partial\sigma_{i,j})) \equiv (d^{\mp 1} u_{s_1}^{\pm 1})$ then $(\phi(\partial\tau_{i,j})) \equiv (u_{s_1}^{\pm 1})$. In this way, we may remove all L-faces from Δ to obtain a new diagram Δ'. Then Δ' is regarded as a reduced van Kampen diagram over the presentation

$$\langle a, b, c, d \mid S \cup W \cup R \rangle$$

and has the same boundary label as Δ. So $(\phi(\partial\Delta')) \equiv (w)$. Let $\mathcal{R}$ be the symmetrized subset of the free group $F(a, b, c, d)$ generated by $S \cup R$. As mentioned in [9, Introduction], a similar statement as Lemma 2.4 holds for $r_0 = [5, 4, 4]$. So $\mathcal{R}$ satisfies small cancellation condition $C(4) - T(4)$ due to Lemma 2.4 together with the fact that $S(u_{r_i})$ consists of 4 and 5, while $S(u_{s_j})$ consists of 5 and 6.

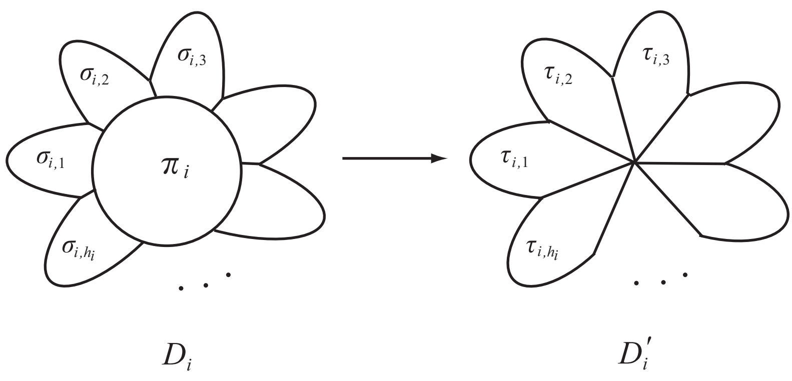

FIGURE 1. Replacing a subdiagram D_i which consists of an L-face π_i and W-faces $\sigma_{i,1}, \dots, \sigma_{i,h_i}$ surrounding π_i with D'_i which consists of S faces $\tau_{i,1}, \dots, \tau_{i,h_i}$ so that D_i and D'_i have the same boundary label.

Claim 2. Δ' is a $[4,4]$-map (for the definition and convention, see [8, Section 6]).

Proof of Claim 2. Clearly, every interior vertex of Δ' has degree at least 4. Now we show that every face in Δ' has at least 4 edges in its boundary, by showing that a path in the intersection of any two faces in Δ' is a piece with respect to $\mathcal{R}$. Clearly a path in the intersection of two R-faces, two S-faces, an R-face and an S-face, or an R-face and a W-face in Δ' is a piece with respect to $\mathcal{R}$. By Claim 1d), the intersection of two W-faces in Δ, so in Δ', does not contain a $\{c,d\}$-edge, and hence a path in the intersection of two W-faces is a piece with respect to $\mathcal{R}$.

It remains to consider the intersection of an S-face and a W-face in Δ'. Note the intersection of an S-face and a W-face in Δ' corresponds to that of two W-faces in Δ, since every S-face was obtained by replacing a W-face surrounding an L-face in Δ. So if a path in the intersection of an S-face and a W-face in Δ' is a product of no less than 2 pieces, then a path in the corresponding intersection of two W-faces in Δ is a product of no less than 2 pieces. But then those two W-faces form a reducible pair in Δ, which is a contradiction to the assumption that Δ is reduced. Therefore a path in the intersection of an S-face and a W-face in Δ' is a piece with respect to $\mathcal{R}$. Since $\mathcal{R}$ satisfies $C(4)$, Δ' is a $[4,4]$-map. □

By Claim 2, we obtain that by the Curvature Formula of Lyndon and Schupp,

$$\sum_{v\in\partial\Delta'}(3-d(v))\geq 4,$$

so that there exists a vertex of degree 2 on $\partial\Delta'$. This together with Claim 1b) implies that $(\phi(\partial\Delta'))$ contains a subword of the cyclic word $(c^{\mp1}u_{s_0}^{\pm1})$ or the cyclic word $(d^{\mp1}u_{s_1}^{\pm1})$ which cannot be expressed as a product of less than 2 pieces. Then, since $(\phi(\partial\Delta'))\equiv(w)$, the cyclic word (w) contains a nontrivial subword of $(u_{s_0}^{\pm1})$ or $(u_{s_1}^{\pm1})$. But since $u_{s_0}^{\pm1}$ and $u_{s_1}^{\pm1}$ are reduced words in $\{a,b\}$ while w is a cyclically reduced word in $\{c,d\}$ by Claim 1a), this is obviously a contradiction, completing the proof of Lemma 3.2. □

By Lemmas 3.1 and 3.2, $G=\langle a,b\,|\,u_{r_0}=u_{r_1}=\cdots=1\rangle$ has property $F(2)$, which completes the proof of Theorem 1.1 due to Proposition 1.3. □

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Donghi Le
Department of Mathematics
Pusan National University
San-30 Jangjeon-Dong, Geumjung-Gu, Pusan, 609-735, Korea
E-mail address: donghi@pusan.ac.kr